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First Boundary Value Problem For A Class Of Parabolic Equations With Discontinuous Coefficients, The Basic Coercive Estimation In Sobolev Weight Space

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Abstract. *It is known that if the coefficients of a uniform parabolic operator are discontinuous, then the appropriate first boundary value problem is not solvable in space* $\overset{0}{W}{}_2^{1,0}$ *. In the paper in Sobolev weight spaces we obtain coercive estimation of a class of parabolic equations of second order with discontinuous coefficients given in* P*-domains. The coefficients of these equations bear discontinuity at the vertex of* P*-domain. The obtained coercive estimation allows to prove weak solvability of the first boundary value problem in Sobolev weight spaces. The weight is found in the explicit form.*

Keywords. P-domain · parabolic operator · Sobolev space · Laplace operator · coercive estimation.

Introduction. The paper is devoted to obtaining coercive estimation in Sobolev weight spaces of second order parabolic operators of the form

$$\mathcal{L} = \Delta u + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

with discontinuous coefficients given in P-domain. The obtained coercive estimation allows to prove weak solvability of the appropriate first boundary value problem in Sobolev weight spaces.

Such parabolic equations find wide application in theory of diffusive processes and in theory of heat-conductivity. Such operators were first investigated in detail by Gilbary and Serrin.

It is known that in a cylindrical domain with rather smooth boundary of foundation the first boundary value problem under homogeneous boundary conditions is uniquely solvable in the space $W_{2,0}^{2,1}$ if the coefficients are continuous in a closed domain.

Note that in the case of discontinuous coefficients, the mentioned facts, generally speaking , don't hold.

Let E_n and R_{n+1} be n-dimensional and $n(n+1)$-dimensional Euclidean spaces of the points $x = (x_1, ..., x_n)$, and $(x,t) = (x_1, ..., x_n, t)$ respectively, D be a bounded domain in E_n with the boundary ∂D, $0 \in D$, and $R_{n+1}^{-} = R_{n+1} \cap \{(x,t) : t < 0\}$.

Call the domain $Q \subset R_{n+1}^{-}$ a domain of parabolic type (or P-domain) if its cross-section of each hyperplane $t = \tau (\tau < 0)$ has the form:

$$\left\{ x : \frac{x}{\sqrt{-\tau}} \in D \right\}.$$

The domain D is called the foundation of P-domain Q.

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Let further

$$Q_T = Q \cap \{(x;t) : -T < t < 0\},$$

$$S_T = \partial Q \cap \{(x;t) : -T < t < 0\},\ D_T = Q \cap \{(x;t) : t = -T\},$$

$\Gamma(Q_T)$ is a parabolic boundary of the domain Q_T.

Let's consider in Q_T the following operator

$$\mathcal{L} = \Delta u + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

where Δ is the Laplace operator, the numerical parameter satisfies the condition

$$\frac{1}{d^2} < \lambda < \infty. \tag{1}$$

Here $d = \sup\limits_{y \in D} |y|$.

Let's consider the quadratic form in R^n

$$\sum_{i,j=1}^{n} \left(\delta_{ij} + \lambda \frac{x_i x_j}{4(-t)}\right) \eta_i \eta_j,$$

where $\eta \in R^n,\ (x,t) \in Q_T$.

Repersent it in the form

$$\sum_{i,j=1}^{n} \left(\delta_{ij} + \lambda \frac{x_i x_j}{4(-t)}\right) \eta_i \eta_j = |\eta|^2 + \lambda \sum_{i,j=1}^{n} \xi_i \xi_j \eta_i \eta_j = |\eta|^2 + \lambda \left(\sum_{i=1}^{n} \xi_i \eta_i\right)^2,$$

where $\xi_i = \frac{x_i}{2\sqrt{-t}}, \xi \in D$.

1) Let $\lambda > 0$. Then

$$|\eta|^2 + \lambda \left(\sum_{i=1}^{n} \xi_i \eta_i\right) < |\eta|^2 + \lambda |\xi|^2 \cdot |\eta|^2 = |\eta|^2 \left(1 + \lambda \cdot |\xi|^2\right) \le |\eta|^2 \left(1 + \lambda \cdot d^2\right),$$

where $d = \sup\limits_{\xi \in D} |\xi|$.

$$|\eta|^2 + \lambda \left(\sum_{i=1}^{n} \xi_i \eta_i\right)^2 > |\eta|^2.$$

Thus,

$$|\eta|^2 < \sum_{i,j=1}^{n} \left(\delta_{i,j} + \lambda \frac{x_i x_j}{4(-t)}\right) \eta_i \eta_j, (1 + \lambda d^2) \cdot |\eta|^2$$

2) Let $\lambda < 0$. Then

$$|\eta|^2 + \lambda \left(\sum_{i=1}^{n} \xi_i \eta_i\right)^2 > |\eta|^2$$

$$|\eta|^2 + \lambda \left(\sum_{i=1}^{n} \xi_i \eta_i\right)^2 > |\eta|^2 + \lambda |\eta|^2 \cdot |\xi|^2 = |\eta|^2 \left(1 + \lambda |\xi|^2\right) \ge |\eta|^2 \left(1 + \lambda \cdot d^2\right).$$

Hence

$$1 + \lambda d^2 > \Longrightarrow \lambda > -\frac{1}{d^2}.$$

Give the following denotation:

u_i and u_{ij} are the derivatives of $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$, respectively, $u_x = (u_1, ..., u_n)$,

$$u_{xx} = (u_{ij}),\ u_x^2 = (u_{ij}),\ u_x^2 = \sum_{i=1}^{n} u_i^2,\ u_{xx}^2 = \sum_{i,i=1}^{n} u_{ij}^2;\ i,j = \overline{1,n}.$$

Let the numerical parameter γ satisfy the condition

$$\gamma \in \left(\frac{n^2\left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}, \infty \right). \tag{2}$$

Let $A_0^\infty(Q_T)$ be a space of infinitely differentiable and finite in Q_T functions for which the integral

$$\int_{Q_T} (-t)^\gamma u^2 dxdt$$

is finite, $L_{2,\gamma}(Q_T)$ be a Banach space of measurable functions $u(x,t)$ given on Q_T, with the finite norm

$$\|u\|_{L_{2,\gamma}}(Q_T) = \left(\int_{Q_T} (-t)^\gamma u^2 dxdt \right)^{\frac{1}{2}}.$$

Let $W_{2,\gamma}^{1,0}(\dot{Q}_T)$ be a Banach space of measurable functions $u(x,t)$ given on Q_T, with the finite norm

$$\|u\|_{W_{2,\gamma}^{1,0}}(Q_T) = \left(\int_{Q_T} (-t)^\gamma (u^2 + u_x^2) dxdt \right)^{\frac{1}{2}},$$

$\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$ be the subspace of $W_{2,\gamma}^{1,0}(\dot{Q}_T)$ whose dense set is $A_0^\infty(Q_T)$.

Basic result:

Theorem. *It with respect to parameters α and γ conditions 1) and 2) are fulfilled, then for any function*

$$u(x,t) \in \overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$$

the estimation

$$\|u\|_{W_{2,\gamma}^{1,0}(\dot{Q}_T)} \le C_2(\gamma,\lambda,n,d) \|\mathcal{L}u\|_{L_{2,\gamma}(Q_T)}.$$

is valid.

At first we prove the following auxiliary statements.

Lemma 1. (*Analogue of the Fridrichs inequality*). *For any function $u(x,t) \in A_0^\infty(Q_T)$ the estimation*

$$\int_{Q_T} (-t)^\gamma u^2 dxdt \le C_1(Q_T) \int_{Q_T} (-t)^\gamma u_x^2 dxdt \tag{3}$$

is valid.

Proof. As Q_T is a bounded domain, then there exists a parallelepiped

$$K = \left\{(x,t) : -R \le x_i \le R,\ i = \overline{1,n}; -T \le t \le 0\right\},$$

containing Q_T in itself. Continue the function $u(x,t)$ by zero in $K \backslash Q_T$ and denote the continued function again by $u(x,t)$. Let $x' = (x_2, ..., x_n)$. We have

$$u(x_1, x', t) = u(-R, x', t) + \int\limits_{-R}^{x_1} u_1(y, x', t)dy = \int\limits_{-R}^{x_1} u_1(y, x', t)dy,$$

where $u_i = \dfrac{\partial u}{\partial x_i},\ i = \overline{1,n}$.

Therefore

$$u^2\left(x_j, x', t\right) = \left(\int\limits_{-R}^{x_1} u_1(y, x', t)dy\right)^2 \le 2R \int\limits_{-R}^{R} u_1^2 dy.$$

Multiply the both sides of the last inequality by $(-t)^\gamma$ and integrate in domain K. We get

$$\int\limits_K (-t)^\gamma u^2 dxdt \le 4R^2 \int\limits_K (-t)^\gamma u_1^2 dxdt \le 4R^2 \int\limits_K (-t)^\gamma u_x^2 dxdt.$$

Now it suffices to take into account that $u(x,t) = 0$ in $K \backslash Q_T$, and the required inequality (3) is proved.

Lemma 2. *If with respect to parameters λ and γ conditions (1) and (2) are fulfilled, then for any function $u(x,t) \in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$ the estimation*

$$\varepsilon_1 \int\limits_{Q_T} (-t)^\gamma u^2 dxdt \le -\int\limits_{Q_T} (-t)^\gamma u \mathcal{L} u dxdt, \tag{4}$$

is valid, where the positive constant ε_1 depends only on λ, γ, d and n.

Proof. It suffices to prove estimation (4) for the functions $u(x,t) \in A_0^\infty(Q_T)$. Let for $\sigma \in (0,T) Q_{T,\sigma} = Q_T \backslash \overline{Q}_\sigma$.

We have

$$\mathcal{L}u = \Delta u + \lambda \sum_{i,j=1}^n \left(\frac{x_i x_j}{4(-t)} u_j\right)_i - \lambda(n+1) \sum_{i=1}^n \frac{x_i}{4(-t)} u_i - u_t.$$

For any function $u(x,t) \in A_0^\infty(Q_T)$

$$-\int\limits_{Q_{T,\sigma}} (-t)^\gamma u \mathcal{L} u dxdt = -\int\limits_{Q_{T,\sigma}} (-t)^\gamma u \Delta u dxdt$$

$$-\lambda \sum_{i,j=1}^n \int\limits_{Q_{T,\sigma}} (-t)^\gamma u \left(\frac{x_i x_j}{4(-t)} u_j\right) dxdt + \lambda(n+1) \sum_{j=1}^n (-t)^\gamma \frac{x_j}{4(-t)} u_j u dxdt$$

$$+ \int\limits_{Q_{T,\sigma}} (-t)^\gamma u \cdot u_t dxdt = J_{1,\sigma} + J_{2,\sigma} + J_{3,\sigma} + J_{4,\sigma}. \tag{5}$$

But on the other hand

$$J_{1,\delta} = -\int\limits_{Q_{T,\sigma}} (-t)^\gamma u \Delta u dxdt = -\int\limits_{Q_{T,\sigma}} (-t)^\gamma \sum_{j=1}^n u \cdot u_{ii} dxdt$$

$$= \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \sum_{j=1}^{n} u_i^2 dxdt = \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u_x^2 dxdt; \tag{6}$$

$$J_{2,\delta} = -\lambda \sum_{i,j=1}^{n} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u \left(\frac{x_i x_j}{4(-t)} u_j \right) dxdt = \lambda \sum_{i,j=1}^{n} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{x_i x_j}{4(-t)} u_i u_j dxdt; \tag{7}$$

$$J_{3,\delta} = \lambda(n+1) \cdot \sum_{i=1}^{n} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{x_i}{4(-t)} u_i dxdt$$

$$= \frac{\lambda(n+1)}{2} \cdot \sum_{i=1}^{n} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{x_i}{4(-t)} (u^2) dxdt = -\frac{\lambda(n+1)}{2} \cdot \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{u^2}{4(-t)} dxdt. \tag{8}$$

Besides, continuing the function $u(x,t)$ by zero in $K_\sigma \backslash Q_{T,\sigma}$, and denoting the obtained continuation again by $u(x,t)$ we get

$$I_{4,\sigma} = -\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u u_t dxdt = \frac{1}{2} \int\limits_{K_\sigma} (-t)^{\gamma} (u^2)_t dxdt$$

$$= \frac{\gamma}{2} \int\limits_{K_\sigma} (-t)^{\gamma-1} u^2 dxdt + \frac{\sigma^{\gamma}}{2} \int\limits_{\Pi_R} u^2(x,-\sigma) dx - \frac{T^{\gamma}}{2} \int\limits_{\Pi_R} u^2(x,-T) dx$$

$$= \frac{\gamma}{2} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma-1} u^2 dxdt + \frac{\sigma^{\gamma}}{2} \int\limits_{\Pi_R} u^2(x,-\sigma) dx \geq \frac{\gamma}{2} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma-1} u^2 dxdt. \tag{9}$$

Here $K_\sigma = \Pi_R \times (-T,-\sigma)$, $\Pi_R = \left\{ x : |x_i| < R, i = \overline{1,n} \right\}$. Taking into account (6)-(9) in (5), we deduce

$$-\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u \mathcal{L} u dxdt \geq \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u_x^2 dxdt + \gamma \int\limits_{Q_{T,\sigma}} \sum_{i,j=1}^{n} (-t)^{\gamma} \frac{x_i x_j}{4(-t)} u_i u_j dxdt$$

$$-\frac{\lambda n(n+1)}{2} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{u^2}{4(-t)} dxdt + \frac{\gamma}{2} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma-1} u^2 dxdt.$$

Hence it follows that

$$\int\limits_{Q_{T,\sigma}} (-t)^{\gamma-1} \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \right) dxdt$$

$$\leq \frac{\lambda n(n+1) - 4\gamma}{2} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{u^2}{4(-t)} dxdt - \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u \mathcal{L} u dxdt. \tag{10}$$

From (10) it is seen that if

$$\gamma \geq \frac{\lambda n(n+1)}{4} \tag{11}$$

then

$$\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \left(u_x^2 + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} u_i u_j \right) dxdt \leq -\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u \mathcal{L} u dxdt \tag{12}$$

For any $\varepsilon_2 > 0$ we have

$$\sum_{i,j=1}^{n} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{1}{2\sqrt{-t}} \cdot \frac{x_i}{2\sqrt{-t}} u u_i dxdt \leq \frac{\varepsilon^2}{2} \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} \frac{u^2}{4(-t)} dxdt$$

$$+\frac{1}{2\varepsilon_2}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt. \tag{13}$$

But on the other hand

$$\sum_{i,j=1}^{n}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\frac{1}{2\sqrt{-t}}\cdot\frac{x_i}{2\sqrt{-t}}uu_idxdt$$

$$=-\frac{1}{2}\sum_{i=1}^{n}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\frac{x_i}{4(-t)}(u^2)u_jdxdt=\frac{n}{2}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt. \tag{14}$$

From (13) and (14) it follows that

$$\frac{n-\varepsilon^2}{2}\int\limits_{Q_{T,\sigma}}(t)^{\gamma}\frac{u^2}{4(-t)}dxdt\le\frac{1}{2\varepsilon_2}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt.$$

Now assuming $\varepsilon_2=\frac{n}{2}$, from the last inequality we deduce

$$\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\frac{u^2}{4(-t)}dxdt\le\frac{4}{n^2}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt. \tag{15}$$

Taking into account (15) in (10), we get that if condition (11) is not fulfilled, then

$$\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\left(u_x^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j\right)dxdt$$

$$\le\frac{2\lambda n(n+1)-8\gamma}{n^2}\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_jdxdt-\int\limits_{Q_{T,\sigma}}(-t)^{\gamma}u\mathcal{L}udxdt. \tag{16}$$

Solving the inequality

$$\frac{2\lambda nn+1)-8\gamma}{n^2}<\frac{1}{d^2}+\lambda,$$

we get

$$\gamma>\frac{n^2\left(\lambda-\frac{1}{d^2}\right)+2\lambda n}{8}=\gamma_1. \tag{17}$$

But on the other hand, as the condition (11) is not fulfilled, then

$$\gamma<\frac{\lambda n(n+1)}{4}=\gamma_2.$$

Further, from condition (1) $\gamma_1<\gamma_2$. So, if condition (11) is fulfilled, then estimation (12) is valid. However

$$\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_iu_j=\lambda\left(\sum_{i=1}^{n}\frac{1}{2\sqrt{-t}}x_iu_i\right)^2.$$

If $\lambda\ge0$, then

$$\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2\ge0. \tag{18}$$

But if $-\dfrac{1}{d^2} < \lambda < 0$, then

$$\lambda\left(\sum_{i=1}^{n}\frac{x_i}{2\sqrt{-t}}u_i\right)^2 \geq \lambda d^2 u_x^2. \tag{19}$$

Thus, from (18) and (19) we deduce that subject to condition (11) it is valid the estimation

$$\varepsilon_2 \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u_x^2 dxdt \leq -\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u\mathcal{L}udxdt, \tag{20}$$

where

$$\varepsilon_3 = \begin{cases} 1+\lambda d^2, & \lambda < 0 \\ 1, & \lambda < 0. \end{cases}$$

Now we consider the case when

$$\gamma \in \left\{\frac{n^2\left(\lambda - \dfrac{1}{d^2}\right) + 2\lambda n}{8}, \frac{\lambda n(n+1)}{4}\right\}.$$

Taking into account estimation (16), we get that if $\varepsilon_4 \in (0,1)$ is such that

$$\frac{2\lambda n(n+1) - 8\lambda}{n^2} < \frac{1}{d^2} + \lambda - \frac{\varepsilon_4}{d^2},$$

then

$$\int\limits_{Q_{T,\sigma}} (-t)^{\gamma}\left(u_x^2 + \left(\frac{\varepsilon^4}{d^2} - \frac{1}{d^2}\right)\sum_{i,j=1}^{n}\frac{x_i x_j}{4(-t)}u_i u_j\right) dxdt \leq -\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u\mathcal{L}udxdt.$$

Thus, in this case it holds the inequality

$$\varepsilon_5 \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u_x^2 dxdt \leq -\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u\mathcal{L}udxdt \tag{21}$$

with positive constant ε_5 dependent only on γ, λ, n and d.

From (20), (21) and lemma 1 we deduce

$$\varepsilon_6 \int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u^2 dxdt \leq -\int\limits_{Q_{T,\sigma}} (-t)^{\gamma} u\mathcal{L}udxdt,$$

where

$$\varepsilon_6 = \frac{1}{C_1}\min\{\varepsilon_3, \varepsilon_5\}.$$

Now it suffices to tend σ to zero and the required estimation (4) is proved.

Now prove that for any function $u(x,t) \in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$ it is valid the estimation

$$\|u\|_{W^{1,0}_{2,\gamma}(Q_T)} \leq C_2(\gamma, \lambda, n, d)\,\|\mathcal{L}u\|_{L_{2,\gamma}(Q_T)}. \tag{22}$$

From estimations (20), (21) and lemma 1 we deduce

$$\varepsilon_7 \int\limits_{Q_T} (-t)^{\gamma}\left(u^2 + u_x^2\right) dxdt \leq -\int\limits_{Q_T} (-t)^{\gamma} u\mathcal{L}udxdt \tag{23}$$

where the positive constant ε_7 depends only on γ, λ, n and d. On the other hand, for any $\varepsilon_8 > 0$

$$\int\limits_{Q_T} (-t)^{\gamma} u\mathcal{L}udxdt \leq \frac{\varepsilon_8}{2}\int\limits_{Q_T} (-t)^{\gamma} u^2 dxdt + \frac{1}{2\varepsilon_8}\int\limits_{Q_T} (-t)^{\gamma} (\mathcal{L}u)^2 dxdt.$$

Assuming $\varepsilon_8 = \varepsilon_7$ and using the last inequality (23), we get

$$\frac{\varepsilon_7}{2}\int\limits_{Q_T}(-t)^{\gamma}(u^2+u_x^2)dxdt \le \frac{1}{2\varepsilon_7}\int\limits_{Q_T}(-t)^{\gamma}(\mathcal{L}u)^2dxdt.$$

Hence it follows the required estimation (22) with $c_2 = \dfrac{1}{\varepsilon_7}$. The theorem is proved.

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