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Investigation Of The Spectrum And Resolvent Of A Differential Bundle On A Semi-Axis

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Abstract. *In the paper, in $L_2(0,\infty)$ we consider a bundle of fourth order differential operators when the principal characteristic polynomial has a unique multiple root. We establish double asymptotics of fundamental systems of solutions of the appropriate differential equation, study the structure of the spectrum, construct the kernel of the bundle. It is defined that for correct formulation of the problem it is necessary to give four conditions at the left end.*

In the space $L_2(0,\infty)$ consider a bundle of differential operators L_λ^α generated by the differential expression

$$l_\lambda(Y) \equiv Y^{IV} - 4iY^{III} - 6\lambda^2 Y^I + \left(4i\lambda^3 + P_{30}(x)\right) Y^I + \left(\lambda^4 + P_{41}(x)\lambda + P_{40}(x)\right) Y = 0, \tag{1}$$

and boundary condition

$$U_\nu(Y) = \alpha_{\nu 0} Y(0,\lambda) + \alpha_{\nu 1} Y^I(0,\lambda) + \alpha_{\nu 2} Y^{II}(0,\lambda) + \alpha_{\nu 3} Y^{III}(0,\lambda) = 0, \ \nu = \overline{1,4}, \tag{2}$$

where λ is a spectral parameter, the functions $P_{30}(x), P_{41}(x), P_{40}(x)$ are complex-valued summable on $[0,\infty)$, they and their fourth order derivatives are rather quickly stabilized at infinity, $\alpha_{ik}, \nu = \overline{1,4}, k = \overline{0,3}$ are fixed complex numbers such that $U_\nu(Y)$ are linear independent.

Specifity of the bundle L_λ^α is that the principal characteristic polynomial of the equation $l_\lambda(Y) = 0$ has a unique root i with multiplicity 4. In the case of multiple roots of this polynomial the formal solutions of equation (1) may contain fractional degrees of parameter both in the index of the exponent and at the multiplier of the exponent, and the structure of asymptotic representations depends not only on higher coefficients but also on algebraic combinations of coefficients at lower degrees of parameter [1]. Here these properties are taken into account so that the formal solutions do not contain fractional degrees of parameter.

Direct spectral aspects of ordinary differential operators on a finite segment in the case of different characteristic roots in Birkhoff -Tamarkin sense were studied rather well. The most complete investigations of different spectral aspects were carried out in the works of G.D.Brikhoff, Ya.D. Tamarkin, M.A. Naimark, M.V. Keldysh, A.G. Kostyunchenko, V.A. Il'in, M.G. Gasymov, M.L. Rasulov, A.A. Shkalikov

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and others. In particular, the issues of multiple completeness of the system of eigen and adjoint functions of such bundles were solved depending on lacation of these roots. Herewith, the essential condition of multiple completeness is the location of roots on different beams outgoing from the origin. When this condition is violated, the given system of adjoint functions has infinite deficiencies in the sense of multiple completeness [2,3,4].

Differential bundles given on infinite intervals were also studied well in the case of diferent characcteristic roots. And here it was revaled such an effect that the member of boundary conditions on the left end in the case of semi-axis also depends on the location of the parameter λ and is connected with arrangement of the roots of the characteristic equation, and the appropriate bundle is not an analytic function of the parameter λ on all complex plane [5,6].

As the bundle under of consideration has one multiple characteristic root, and this means that all of them lie on one beam outgoing from the origin, this beam should be specially studied. When there are characteristic multiple roots, but they are symmetrically arranged with respect to origin, the appropriate results on expansion in eigen function of continuous and discrete spectra were obtained in the papers [7,8,9].

Formal series of representations of fundamental systems of solutions $l_\lambda(Y) = 0$ contain only entire degree of parameter because some coefficients at lower degrees of parametric of a polynomial bundle don't participate at it and as $|\lambda| \to \infty$ they have ordinary exponential asymptotics as with the principal terms $1, x, x^2, x^3$ and the solutions are found by the method of the papers [1,10]. As not all the coefficients of the differential expression $l_\lambda(Y)$ decrease at infinity, the number composing the fundamental system of solutions (f.s.s.) belonging to $L_2(0, \infty)$ depends to which half-plane $\pm Jm\lambda \geq 0$ the spectral parameter enters.

In the present paper, a formula of double asymptotics of the solutions of equation (1) both with respect to the parameter and argument x is established. The structure of the spectrum is studied, the resolvent of the bundle is constructed.

Theorem 1. *Assume that the integrals* $\int\limits_a^\infty x^4 \left| P_{ks}^{(j)}(x) \right| dx,\ j = \overline{0, k+s}$ *converge for any* $a > 0$. *Then differential equation (1) at each of half-planes* $\pm Jm\lambda \geq 0$ *has f.s.s.* $Y_k(x, \lambda)$, $k = \overline{1,4}$ *that allow as* $|\lambda| \to \infty$ *and* $x \to \infty$ *the asymptotic representations as*

$$Y_k(x, \lambda) = \left[g_k^{(0)}(x) + \frac{1}{\lambda} g_k^{(1)}(x) + \frac{1}{\lambda^2} g_k^{(2)}(x) + \frac{1}{\lambda^3} g_k^{(3)}(x) + \frac{E_k(x, \lambda)}{\lambda^4} \right] e^{i\lambda x},\ k = \overline{1,4} \qquad (3)$$

as $|\lambda| \to \infty$ and for *each fixed* x.

As $x \to \infty$ *uniformly with respect to* $\lambda : \{ \pm Jm\lambda \geq 0, |\lambda| \geq R,\ R$ *is a rather large number* $\}$ *the following asymptotic representations hold:*

$$Y_k(x, \lambda) = \left[g_k^{(0)}(x) + o(1) \right] e^{i\lambda x},\ x \to \infty,\ k = \overline{1,4} \qquad (4)$$

The derivatives of these solutions allow similar asymptotic representations.

Here $g_k^{(0)}(x) = x^{i-1}, i = \overline{1,4}$ are *f.s.s. of the fourth order equation* $\frac{d^4}{dx^4} g_k^{(0)}(x) = 0,\ g_i^{(k)}(x), k = \overline{1,3}$ *are the particular solutions of the fourth order inhomogeneous differential equation with the left side* $\frac{d^4 g_k^{(0)}(x)}{dx^4}$ *, and the right side contains the coefficients of functions and their derivatives to third order inclusively, the functions* $E_i(x, \lambda)$ *are bounded in the domain* $\{a \leq x < \infty, |\lambda| \geq R\}$.

Proof. For obtaining asymptotic representation (3) we consider the formal solutions $Y_k(x, \lambda) = \left[\sum\limits_{\nu=0}^{\infty} \lambda^{-\nu} g_k^{(0)}(x) \right] e^{i\lambda x},\ k = \overline{1,4},$ that were studied in [10] for a finite segment. Substitute these formal series in equation (1), then equate between themselves the coefficients at the identical degree λ and get a recurrent system of fourth order differential equations and a system of algebraic relations. Under the theorem conditions from integrability of these differntial systems from x to ∞ we find an algorithm for determining the coefficients of Brikhoff expansion in λ. Hence, applying lemma1.1.of the paper (5) we get the statement of the theorem for the set $\{a \leq x < \infty,\ \pm Jm\lambda \geq 0, \forall a > 0\}$. Each of the obtained solutions $Y_k(x, \lambda)$, $k = \overline{1,4}$ may be continued also on the interval $[0, a]$ and we can construct on this interval a solution satisfying the conditions $Y^{(\nu)}(a,\lambda) = Y_k^{(\nu)}(a,\lambda), \nu = \overline{0,3}$ and the solutions determined in such a way on the interval $[0,a]$ satisfy the equation on the same interval. By immediate differentiation of formal solutions with respect to $[0,a]$ to third order, we get that f.s.s. $Y_k(x,\lambda)$, $k = \overline{1,4}$ together with derivatives allow such representations. From the same representation we can easily get formula (4), where smallness of the second summand is determined by the method of the paper [9].

If for some $\varepsilon > 0$ the coefficients of the function for all $x \in [0,\infty)$ satisfy the condition

$$e^{\varepsilon x}\left|P_{ks}^{(j)}(x)\right| \le c_{ks}, \tag{5}$$

then by using [5] it is easy to establish shat equation (1) has f.s.s. $Y_k(x,\lambda)$, $k = \overline{1,4}$ in the domain $\{(x,\lambda) : 0 \le x < \infty, 0 < |\lambda| \le r\}$, where r is a rather small positive number. For a fixed, $x \in [0,\infty)$ these solutions are holomorphic functions with respect to λ for $0 < |\lambda| \le r$. Asymptotic representations (3) hold uniformly with respect to λ for $0 < |\lambda| \le r$. And if $\lambda = 0$, then the equation obtained from $l_\lambda(Y) = 0$ has linearly independent solutions $Y_k(x,\lambda)$, $k = \overline{1,4}$ such that as $x \to \infty$ and at arbitrary positive $\varepsilon_1 < \varepsilon$ it holds

$$Y_k(x,\lambda) = \frac{x^{4-k}}{(4-k)!} + O\left(e^{-\varepsilon,x}\right), k = \overline{1,4}$$

that means holomorphy of solutions (3) and (4) with respect to λ and for $0 < |\lambda| \le R$.

Deonote by D the totality of all functions $Y(x,\lambda) \in L_2(0,\infty)$ such that:

1) the derivatives $Y^{(\nu)}(a,\lambda), \nu = \overline{0,3}$ exist and are absolutely continuous in each finite interval $[0,b], b > 0$ for each $\lambda : \pm Jm\lambda \ge 0$; 2) $l_\lambda(Y) \in L_2(0,\infty)$. Further, denote by D_α a totality of functions from D for which conditions (2) are fulfilled. Define L_λ^α in such a way: its domain of definition is D_α and $L_\lambda^\alpha = l_\lambda(Y)$ for $y \in D$. Denote $A(\lambda) = \det \|U_i(y_k)\|_{i,k=1}^4$ and consider the upper half-plane $\lambda : \pm Jm\lambda \ge 0$. In its open part all the solutions $Y_k(x,\lambda)$, $k = \overline{1,4}$ belong to the space $L_2(0,\infty)$. If λ is in the open lower half-plane, none of these solutions belong to this space. Then the eigen values of the operator L_λ^α are determined from the equation $A(\lambda) = 0$. On the real axis none on these solutions belong to $L_2(0,\infty)$, consequently, the operator L_λ^α has no eigen values arranged on this axis. Indeed, otervise, if λ_0 is an eigenvalue with $Jm\lambda_0 = 0$, then necessarily $Y_k(x,\lambda) = \sum\limits_{i=1}^{4} c_i\, Y_i(x,\lambda)$ and $Y(x,\lambda_0) \in L_2(0,\infty)$;herewith even if one of the numbers $c_i, k = \overline{1,4}$ should differ from zero, but as $x \to \infty$ we have

$$Y(x,\lambda_0) = \{c_1 + c_2 x + c_3 x^2 + c_4 x^3 + o(1)\}e^{i\lambda_0 x}.$$

Therefore, as $N \to \infty$

$$\int\limits_0^N |Y(x,\lambda_0)|^2\, dx = \int\limits_0^N \left|c_1 + c_2 x + c_3 x^2 + c_4 x^3\right|^2 dx = c_1^2 N + c_1^2\frac{N^3}{3} + c_3^2\frac{N^5}{5} + c_4^2\frac{N^7}{7} + o(1)$$

and if $Y(x,\lambda_0) \in L_2(0,\infty)$, then necessarily $c_k, k = \overline{1,4}$ i.e.$Y(x,\lambda_0) = 0$. There are also no eigen values+ in the open lower half-plane.

Theorem 2. *Subject to conditions (5) the bundle L_λ^α may have finitely many nonreal eigen values from the upper half-plane, finitely many real spectral properties, the real axis coincider with the continuous spectrum. Subject to the conditions of theorem 1, the spectrum of the bundle L_λ^α may form a denumerable set in $\lambda-$ plane, whose limit points may be only on the real axis and a continuous spectrum coinciding with the real axis.*

Let λ_k be a prime eigen value of the operator L_λ^α. Then the eigen function responding to λ_k will be $Y_k(x) = \sum\limits_{i=1}^{4} c_k\, Y_k$. Assume $c_4 = 1$. Form conditions (2) we have $\sum\limits_{k=1}^{3} c_k\, U_i(Y_k) + U_i(Y_4) = 0,\ i = \overline{1,4}$. For the existence of nonzero solutions with respect to $c_k, k = \overline{1,3}$ of these equations, the rank r of the system should be less than 4. Let $r = 3$. Then from the system

$$\sum_{k=1}^{3} c_k\, U_i(Y_k) = U_i(Y_4),\ i = \overline{1,4} \tag{6}$$

we find $c_i = \frac{\Delta_i}{\Delta_0}$, where $\Delta_0 = \det \|U_i(y_k)\|_{i,k=1}^3$ and Δ_i is obtained from Δ_0 replacing the i-th column by the column $\{-U_1(Y_4), U_2(Y_4) - U_3(Y_4)\}$. The appropriate eigen function is expressed by the formula

$$Y_k(x) = -\sum_{k=1}^{3} \frac{\Delta_i}{\Delta_0} Y_i\left(x, \lambda_k\right) + Y_4\left(x, \lambda_k\right). \tag{7}$$

Let's construct the resolvent R_λ^α of the operator $L_\lambda^\alpha : L_\lambda^\alpha Y = f \Longrightarrow Y = R_\lambda^\alpha f$. By the method of variation of constants, from the general solution

$$Y\left(x, \lambda\right) = \sum_{i=1}^{4} c_i(x) Y_i\left(x, \lambda\right) \tag{8}$$

we find

$$c_j'(x) = \frac{(-1)^j W(Y_1 ... Y_{j-1} Y_{j+1} ... Y_4)}{W(Y_1, Y_2, Y_3, Y_4)} f = \omega_j(x, \lambda) f,\ j = \overline{1,4} \tag{9}$$

where $\omega_j(x, \lambda),\ j = \overline{1,4}$ represent the solution of the equation adjoint to $L_\lambda^\alpha Y = f$ and is the ratio of the cofactor of the j-th element to the last row of the Woronskian determinant from f.s.s. to the Wronskian determinant itself , these functions are not the elements of the space $L_2\left(0, \infty\right)$.

Let us consider the upper half-plane . Integrating (9) for λ from this half-plane we find

$$c_j(x) = c_j^{(0)} + \int_0^x \omega_j(\xi, \lambda) f(\xi) d\xi. \tag{10}$$

Substituting (8) in (2), with regard to (10) we get

$$U_\nu(Y) = \sum_{i=1}^{4} c_j^{(0)} U_\nu(Y_i) + \sum_{i=1}^{4} U_\nu(Y_i) \int_0^x \omega_i(\xi, \lambda) f(\xi) d\xi = 0,\ \nu = \overline{1,4}. \tag{11}$$

Hence

$$c_j^{(0)} = -\sum_{k=1}^{4} \frac{1}{A(\lambda)} \int_0^x \begin{vmatrix} U_1(Y_1) ... U_1(Y_{j-1}) U_1(Y_k) U_1(Y_{j+1}) ... U_1(Y_4) \\ U_2(Y_1) ... U_2(Y_{j-1}) U_2(Y_k) U_2(Y_{j+1}) ... U_2(Y_4) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ U_4(Y_1) ... U_4(Y_{j-1}) U_4(Y_k) U_4(Y_{j+1}) ... U_4(Y_4) \end{vmatrix}$$

$$\times \omega_{ik}(\xi, \lambda) f(\xi) d\xi = -\sum_{k=1}^{4} \int_0^x \frac{A_{jk}(\lambda)}{A(\lambda)} \omega_k(\xi, \lambda) f(\xi) d\xi = \int_0^x h_j(\xi, \lambda) f(\xi) d\xi,$$

here

$$A_{jk}(\lambda) = \begin{vmatrix} U_1(Y_1) ... U_1(Y_{j-1}) U_1(Y_k) U_1(Y_{j+1}) ... U_1(Y_4) \\ U_2(Y_1) ... U_2(Y_{j-1}) U_2(Y_k) U_2(Y_{j+1}) ... U_2(Y_4) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ U_4(Y_1) ... U_4(Y_{j-1}) U_4(Y_k) U_4(Y_{j+1}) ... U_4(Y_4) \end{vmatrix},$$

$$h_j(\xi, \lambda) = \begin{cases} -\sum\limits_{k=1}^{4} \frac{A_{jk}(\lambda)}{A(\lambda)} \omega_k(\xi, \lambda), & if\ \ \xi \le x \\ 0 & if\ \ \ \xi > x. \end{cases}$$

Consequently, for the solution $Y\left(x, \lambda\right)$ we get the representation

$$Y\left(x, \lambda\right) = R_\lambda^\alpha f = \int_0^x K^+(x, \xi, \lambda) f(\xi) d\xi,\ \ Jm\lambda \ge 0 \tag{12}$$

where $f(x)$ is any function from the space $L_2(0,\infty)$,

$$K^{+}(x,\xi,\lambda)=\begin{cases}-\sum\limits_{k=1}^{4}Y_k(x,\lambda)h_k(\xi,\lambda), & if\ \ \xi\leq x\\ 0 & if\quad \xi>x.\end{cases},\ Jm\lambda\geq 0$$

Now we consider the half-plane $Jm\lambda\geq 0$. Here $Y_j(x,\lambda)\in L_2(0,\infty)$, $j=\overline{1,4}$, $\omega_j(\xi,\lambda)\in L_2(0,\infty)$, $j=\overline{1,4}$. Integrating (9) from x to ∞, we have

$$c_j(x)=c_j^{(\infty)}+\int\limits_0^x\omega_j(\xi,\lambda)f(\xi)d\xi,\ \ j=\overline{1,4}. \tag{13}$$

Taking into account (10) in (8) and from the fact that $Y(x,\lambda)$ should belong to the space $L_2(0,\infty)$, we have $c_j^{(\infty)}=0,\ j=\overline{1,4}$. Denoting

$$K^{-}(x,\xi,\lambda)=\begin{cases}-\sum\limits_{k=1}^{4}Y_j(x,\lambda)h_k(\xi,\lambda), & if\ \ \xi\leq x\\ 0 & if\ \ \xi>x.\end{cases},\ Jm\lambda\geq 0$$

we get

$$Y(x,\lambda)=R_\lambda^\alpha f=\int\limits_0^x K^{-}(x,\xi,\lambda)f(\xi)d\xi,\ \ Jm\lambda\geq 0. \tag{14}$$

Having the estimation for $Y_j(x,\lambda)$, $j=\overline{1,4}$, we get the kernel of the resolvent operator is the Hilbert-Schmidt kernel. Thus, we arrive at the following statement.

Theorem 3. *For all values of $Jm\lambda\geq 0$ not being the root of the equation $A(Y)=0$ and not lying on the real axis, the resolvent of the operator L_λ^α is an integral operator with a kernel satisfying the Carleman condition for which at rather large $|\lambda|$ the estimation $O(1)$ is valid uniformly with respect to x,λ at each finite interval. Herewith, by approximating of λ to the real axis the norm of the resolvent tends to infinity and the domain of definition of the resolvent operator is dense in $L_2(0,\infty)$ .If for $\lambda_0\in R(-\infty,\infty)$ it holds $A(\lambda_0)=0$, then this point is a point of spectral singularity of the operator L_λ^α.*

Thus, for determining the resolvent set of the bundle L_λ^α all four conditions of (2) should be given. The amount of conditions at the left end depends on how many solutions $Y_k(x,\lambda)$, $k=\overline{1,4}$, belong to $L_2(0,\infty)$.

References

1. Pechentsev A.S.: Asymptotic expansions of solutions of linear differential equations containing a parameter. Diff. uravn. **17**, 1611-1620 (1981), Russian.
2. Vagabov A.I.: *Quadratic bundles of ordinary differential operators*. Matematcheskiye zametki, **42**, No3, 381-393.(Russian)
3. Bogomolova E.P., Pechentsev A.S.: *On basis properties of the system of eigen functions of a boundary value problem with multiple root of characteristic polynomial* Vestnik Moskowskogo Universiteta. ser.1, matematica, mekhanika, No4, 17-22 (1989), Russian.
4. Gasymov M.G., Magerramov A.M.: *Investigation of a class of even order operator bundles* DAN SSSR, **265**, No2, 277-280 (1982), Russian.
5. Funtakov V.N.: *On expansion in eigen function of an arbitrary order not self-adjoint differential bundle on a semi-axis* $[0,\infty$ .I-II Izvestia AN Azerb. SSR, ser. fizi. matem.i tekhn. nauk, I: No 6, 3-19 (1960), II: No, 3-21 (1961), Russian.
6. Maksudov F.G., Magerramov A.M., Mamedov M.Z.: *Spectral analysis of a bundle of special type differential operators.* DAN SSSR, **310**, No 1, 24-28 (1990), Russian.
7. Orujov E.G.: *Spectral analysis of differential operators with multiple characteristics on a semi-axis*. Uspekhi matematicheskikh nauk, **54**, No 2(326), 181-182 (1999), Russian.
8. Orujov E.G.: *Resolvent and a spectrum of a class of not self-adjoint differential operators with multiple characteristics* Trudy Inst matem. i mech. AN Azerb. **VI(XII)**, 148-160 (1997), Russian.
9. Mirzoyev S.S., Orujov E.G., Aliev A.R.: *Spectral analysis of a fourth order differential bundle on the axis*. Doklady RAN, **442**, No 3, 312-314 (2012), Russian.
10. Namazova N.M.: *On some boundary value problems for a fourth order differential equations with multiple characteristics* Transactions of NAS of Azerbaijan, **XXXII**, No 4, 79-86 (2012).