Transactions of National Academy of Sciences of Azerbaijan
Series of Physical-Technical and Mathematical Sciences, vol. XXXV, No 1, 2015.

Integral Limit Theorem For The First Passage Time Of The Level By A Random Walk Described By A Utoregression Process Of Order One $(AR(1))$

Feda H. Rahimov, Tarana E. Hashimova, Aynure D. Farkhadova

Received: 03.10.2014 / Revised: 29.12.2014

Abstract. *In the paper, an integral limit theorem for the first passege time of the level by the random walk, described by the sums of squares of autoregression process of order one is proved.*

1.Introduction. Let on the probability space $(\Omega, \mathcal{F}, P)$ be given a sequence of independent identically distributed random variables $\xi_n = \xi_n(\omega)$ and the random variable $X_0 = X_0(\omega), \omega \in \Omega$, herewith the random variable X_0 is independent of random variables $\xi_n,\ n \geq 1$.

Under the first order autoregression process $(AR(1))$ we understand the solution of the recurrent equation

$$X_n = \beta X_{n-1} + \xi_n, n \geq 1,$$

where β is some non –random constant.

For the case $\beta = 1$ the process X_n is an ordinary random walk, i.e. the sum of independent identically distributed random variables $\xi_n,\ n \geq 1$.

Note that in applied fields of theory of random processes, a lot of mathematical problems are described by the autorepression process. For example, in economic models, the process $(AR(1))$ describes the change of cost, the value of X_n, that at time n depends on its past development (X_{n-1}) and the innovation ξ_n imposed on it is not connected with the past (the random variables ξ_n and X_{n-1} are independent).

Let as consider the sequence of sums

$$S_n = \sum_{k=0}^{n} X_k^2,\ \ n \geq 0$$

and introduce a family of stoping time

$$\tau_\alpha = \inf\{n : S_n > a\},\ a \geq 0. \tag{1}$$

The variable τ_α is the first passage time of the level by the process $S_n, n \geq 0$ and we'll always assume $\inf\{\varnothing\} = \infty$.

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan.Grant N EIF -2013-9(15)-46/13/1.

F.H. Rahimov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ1141, Baku, Azerbaijan
E-mail: ragimovf@rambler.ru

T.E. Hashimova, A.D. Farkhadova
Baku State University,
23, Z.Khalilov str., AZ 1148, Baku, Azerbaijan

Such families of stoping times of type (1) arise in applied fields of theory of random processes and in seqnential analysis [1]-[6].

In particular, if random variables $\xi_1$has a normal distribution with parameters (0,1,) then τ_α is the stopping time of the repeated significance test of $\beta = 0$, wich regects $\beta = 0$ if $\tau_\alpha \leq N_a$ for a suitable chosen constant N_a ([6],[10]).

In these papers, the important boundary problems for random walks described by the autoregression process ($AR(1)$) are studied.

In the present paper we prove an integral limit theorem for the family of stoping time τ_α of the form (1) under which one understands any statement that under some conditions there exist normalizing nonrandom constants $A(a)$ and $B(a) > 0$ (dependent on the parameter a) and a nondegenerate random variable η such that the convergence in distribution:

$$\frac{\tau_\alpha - A(a)}{B(a)} \xrightarrow{d} \eta \;\; as \;\; a \to \infty$$

is fulfilled.

2. Conditions and formulation of the basic result. First of all we note some asymptotic properties of the sum $S_n, n \geq 0$ that were studied in the papers [6] and [7].

In the paper [7] it is shown that the estimation for the parameter β by the least square method that minimizes the sum,

$$\sum_{k=1}^{n} (X_k - \beta X_{k-1})^2$$

has the from

$$\beta_n = \frac{\sum\limits_{k=1}^{n} X_k, X_{k-1}}{\sum\limits_{k=1}^{n} X_{k-1}^2} = \frac{\sum\limits_{k=1}^{n} X_k, X_{k-1}}{\sum\limits_{k=0}^{n-1} X_k^2} = \frac{T_n}{S_{n-1}}, \tag{2}$$

where $T_n = \sum\limits_{k=1}^{n} X_k, X_{k-1}$.

In the paper [7] it is shown that under the condition $|\beta| < 1$ and $EX_0^2 < \infty$, the estimation β_n is asymptotically normal with the parameters $\left(0, 1-\beta^2\right)$, i.e. it is fulfilled the convergence in distribution as $a \to \infty$

$$\sqrt{n}\,(\beta_n - \beta) \xrightarrow{d} N\left(0, 1-\beta^2\right), \tag{3}$$

where $N\left(0,\sigma^2\right)$ denotes a random variable with normal distribution wich parameters $\left(0,\sigma^2\right)$, $(EN\left(0,\sigma^2\right) = 0$, $DN\left(0,\sigma^2\right) = \sigma^2)$.

In the paper [6] it is shown that under the condition $|\beta| < 1$ and $E\,|X_0|^2 < \infty$, the strong law of large numbers for T_n and S_n is fulfilled i.e. the following convergences are almost sure:

$$\frac{T_n}{n} \xrightarrow{a.s} \frac{\beta}{1-\beta^2} \tag{4}$$

and

$$\frac{S_n}{n} \xrightarrow{a.s} \frac{1}{1-\beta^2} \;\; as \;\; n \to \infty. \tag{5}$$

By virtue of these relations, from (2) it follows that $\beta_n \xrightarrow{a.s} \beta$ as $n \to \infty$.

It holds

Theorem. *Let* $E\xi_n = 0, \;\; D\xi_n = 1, 0 < |\beta| < 1$ *and* $E\,|X_0|^2 < \infty$. *Then the convergence in distribution*

$$\frac{\tau_a - \left(1-\beta^2\right)a}{\sqrt{a}} \xrightarrow{d} N\left(0, \sqrt{\frac{(1-\beta^2)}{\beta^2}}\right)$$

as $n \to \infty$ *is fulfilled .*

3. For proving the theorem we need the following facts formulated as lemmas.

Lemma 1. *Let* $\eta_n, n \geq 1$ *be a sequence of arbitrary random variables such that* $\eta_n \xrightarrow{a.s} 1$ *as* $n \to \infty$. *Then for any sequence of random variables* $\dot{Y}_n, n \geq 1$ *it holds*

$$P\left(Y_n \leq x\right) - P\left(Y_n \eta_n \leq x\right) \to 0$$

as $n \to \infty$ *at the continuity point of the distribution function* $F_n(x) = P\left(Y_n \leq x\right)$.

Proof. Assume

$$A = \{\omega : \eta_n \to 1\},\ B = \{\omega : Y_n \eta_n \leq x\}$$

and $G_n(x) = P\left(Y_n \eta_n \leq x\right)$.

It is easy to see that

$$G_n(x) = P(B) = P(AB) = P\left(Y_n \eta_n \leq x, A\right).$$

Let $x \geq 0$. Then from $P(A) = 1$ and for any $\delta > 0$ and sufficiently large n and we have

$$F_n\left(\frac{x}{1+\delta}\right) = P\left(Y_n \leq \frac{x}{1+\delta}, A\right)$$

$$\leq P(AB) \leq P\left(Y_n \leq \frac{x}{1-\delta}, A\right) = F_n\left(\frac{x}{1-\delta}\right).$$

Hence we get

$$F_n(x) - F_n\left(\frac{x}{1-\delta}\right) \leq F_n(x) - G_n(x) \leq F_n(x) - F_n\left(\frac{x}{1+\delta}\right).$$

If x is a continuity point of the distribution function $F_n(x)$, then the values $F_n\left(\frac{x}{1\pm\delta}\right)$ may be made by choosing δ as much as desired close to $F_n(x)$ for sufficiently large n. Hence the required couvlagence follows.

For $x < 0$ it is proved in the same way.

Lemma 2. *Let* $0 < |\beta| < 1$ *and* $EX_0^2 < \infty$. *Then it holds the convergence in distribution*

$$n^{1/2}\left[\frac{S_n}{n} - \frac{1}{1-\beta^2}\right] \xrightarrow{d} N\ \left(0, \frac{1}{\beta^2\left(1-\beta^2\right)}\right)$$

as $n \to \infty$.

Proof. At first consider the case $0 < \beta < 1$. Convergence (2) means that

$$P\left(\sqrt{n}\left(\beta_n - \beta\right) \leq x\right) - \Phi\left(\frac{x}{\sqrt{1-\beta^2}}\right) \to 0 \qquad (6)$$

as $n \to \infty$ uniformly with respect to $x \in R$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-y^{2/2}} dy$.

From (6) it follows that for each $x \in R$

$$P\left(\beta_n \leq x\right) - \Phi\left(\frac{\sqrt{n}(x-\beta)}{\sqrt{1-\beta^2}}\right) \to 0 \qquad (7)$$

as $n \to \infty$.

By virtue of (5) we find

$$\beta_n \frac{1-\beta^2}{\beta} \frac{S_n}{n} \xrightarrow{a.s} 1.$$

Then according to lemma 1 we have

$$P\left(\beta_n \leq x\right) - P\left(\frac{S_n}{n} \leq \frac{x}{\beta^2\left(1-\beta^2\right)}\right) \to 0 \tag{8}$$

as $n \to \infty$ at the points x of continuity of distribution function $P\left(\beta_n \leq x\right)$. From (7) and (8) it follows that

$$P\left(\frac{S_n}{n} \leq \frac{x}{\beta^2\left(1-\beta^2\right)}\right) - \Phi\left(\frac{\sqrt{n}(x-\beta)}{\sqrt{1-\beta^2}}\right) \to 0$$

as $n \to \infty$.

From the last relation we have

$$P\left(\sqrt{n}\left(\frac{S_n}{n} \leq \frac{1}{1-\beta^2}\right) \leq \frac{x}{\beta\left(1-\beta^2\right)}\right) \to \Phi(x),$$

as $n \to \infty$ that is equivalent to the statement of lemma 2 for the case $0<\beta<1$. In the case $1<\beta<0$ it is proved similarly, and the equality $\Phi(x)+\Phi-(x)=1$ is used.

Lemma 3. *If* $|\beta|<1$ and $EX_0^2<\infty$, then *sequence*

$$n^{1/2}\left(\frac{S_n}{n} - \frac{1}{1-\beta^2}\right), \;\; n \geq 1$$

is uniformly continuous in probability.

The statement of this lemma was announced in the paper [6] in which the scheme of the proof of this statement was shown.

Lemma 4. *Let* $|\beta|<1$ *and* $EX_0^2<\infty$.

Then

$$\frac{\tau_a}{a} \xrightarrow{a.s} 1-\beta^2 \;\text{ as }\; a \to \infty.$$

Proof. At first we prove that $\tau_a \xrightarrow{a.s} \infty$ as $n \to \infty$.

Indeed, the process $\tau_a, a>0$ as a function from a increases and therefore there exist the $\lim\limits_{a\to\infty} \tau_a = \tau_\infty \leq \infty$ for all $\omega \in \Omega$.

It is clear that

$$P\left(\tau_\infty > n\right) = \lim_{a\to\infty} P\left(\tau_a > n\right) = \lim_{a\to\infty} P\left(\max_{k\leq n} S_k < a\right) = \lim_{a\to\infty} P\left(S_k < a\right) = 1$$

for all $n \geq 1$.

Therefore $P\left(\tau_\infty = \infty\right) = 1$ or $\tau_a \xrightarrow{a.s} \infty$ as $n \to \infty$.

In what follows, according to the family of the stop moments $\tau_a$we have

$$\frac{S_{\tau_a - 1}}{\tau_a} \leq \frac{a}{\tau_a} < \frac{S_{\tau_a}}{\tau_a}. \tag{9}$$

By theorem 2.1. of the paper [8], from (5) it follows

$$\frac{S_{\tau_a - 1}}{\tau_a} \xrightarrow{a.s} \frac{1}{1-\beta^2} \quad as \quad n \to \infty. \tag{10}$$

Then from (9) and (10) we get the statements of the proved lemma.

The proof of the theorem. Define

$$R_a = S_{\tau_a} - a \;\text{ and } \lambda = \frac{1}{1-\beta^2}.$$

We have

$$\frac{S_{\tau_a} - \lambda\tau_a}{\sqrt{\tau_a}} = \frac{a - \lambda\tau_a}{\sqrt{\tau_a}} + \frac{R_a}{\sqrt{\tau_a}}. \tag{11}$$

By virtue of lemma 2,3 and 4 from the well known theorem of of
Anscombe (see.[8],[9]) it follows that

$$\lim_{a\to\infty} P\left(\frac{S_{\tau_a} - \lambda\tau_a}{\sqrt{\tau_a}} \le x\right) = \Phi\left(x\sqrt{\beta^2\left(1-\beta^2\right)}\right). \tag{12}$$

Prove that

$$\frac{R_a}{\sqrt{\tau_a}} \xrightarrow{a.s} o \ \ as\ a \to \infty. \tag{13}$$

According to definition (1) of the first passage time τ_a we have

$$R_a = S_{\tau_a} - a \le S_{\tau_a} - S_{\tau_a - 1} = X_{\tau_a}^2.$$

Consequently,

$$\frac{R_a}{\sqrt{\tau_a}} \le \frac{X_{\tau_a}^2}{\sqrt{\tau_a}}. \tag{14}$$

Show that

$$\frac{X_n^2}{\sqrt{n}} \xrightarrow{P} o \ \ as\ n \to \infty. \tag{15}$$

From the Chebyshev inequality we have

$$P\left(\frac{X_n^2}{\sqrt{n}} > \varepsilon\right) \le \frac{EX_n^2}{\varepsilon\sqrt{n}}. \tag{16}$$

In the paper [7] it is shown that

$$EX_n^2 \to \frac{1}{1-\beta^2} \ \ as\ \ n \to \infty. \tag{17}$$

Therefore, (16) and (17) follows from (15).
It what follows, it is clear that

$$\frac{X_n^2}{\sqrt{n}} = \frac{S_n - n\lambda}{\sqrt{n}} - \frac{S_{n-1} - (n-1)\lambda}{\sqrt{n}} + \frac{\lambda}{\sqrt{n}}.$$

Hence, by lemma 3 and lemma 1.4 from [9] it follows that sequence $\frac{X_n^2}{\sqrt{n}}$, $n \ge 1$, is uniformly continuous in probability.Therefore, from (15) and lemma (4) it follows that

$$\frac{X_{\tau_a}^2}{\sqrt{\tau_a}} \xrightarrow{P} o \ \ as\ a \to \infty. \tag{18}$$

(13) follows from (18) and (14).
Thus, by virtue of (12) and (13), from equality (11) we find

$$P\left(\frac{a - \lambda\tau_a}{\sqrt{\tau_a}} \le x\right) \to \Phi\left(x\sqrt{\beta^2\left(1-\beta^2\right)}\right) \ \ as\ \ a \to \infty.$$

Hence from lemma 4

$$P\left(\frac{\tau_a - \frac{a}{\lambda}}{\sqrt{\tau_a}} \le x\left(1-\beta^2\right)^{\frac{3}{2}}\right) \to \Phi\left(x\sqrt{\beta^2\left(1-\beta^2\right)}\right)$$

or

$$P\left(\frac{\tau_a - \frac{a}{\lambda}}{\sqrt{\tau_a}} \le x\right) \to \Phi\left(\frac{x\left|\beta\right|}{\sqrt{1-\beta^2}}\right)$$

as $a \to \infty$. The theorem is proved.

References

1. Novikov A.A.: *Some remarks on distribution of the first passage time and optimal stop of $AR(1)$ sequences.* Teoria veroyatni ee prim. **53**, issue 3, 458-471 (2008), Russian.
2. Novikov A.A., Ergashev B.A.: *Limit theorem for the first passage time of the level of autoregression process* Tr. MIAN, **202**, 209-233 (1993).
3. Novikov A.A.: *On the first passage time for the level and one application in "clue" problem.* Teoria veroyat I ee primenenie. **35**, No 2, 282-292 (1990), Russian.
4. Rahimov F.H., Azizov F.J., Khalilov V.S.: *Integral limit theorems for the first passage time for the lever of random Walk, described by a nonlinear function of the Sequence autoregression* . Transaction of NAS of Azerbaijan, **XXXIV**, No 1, 99-104 (2014).
5. Rahimov F.H., Azizov F.J., Khalilov V.S.: *Integral limit theorems for the first passage time for the lever of random Walk, described with sequences.* Transaction of NAS of Azerbaijan, **XXXII**, No 4, 95-100 (2013).
6. Melfi V.F.: *Nonlinear Markov renewal theory with statistical applications*. The Annals of Probability, **20**, No 2, 753-771 (1992).
7. Pollard D.: Convergence of Stochastic Processes. Springer, New-York (1984).
8. Gut A.: Stopped random walks. Limit theorems and applications. Springe, New York (1988).
9. Woodroofe M.: Nonlinear renewal theory in sequential analysis. SIAM. Philadelphia (1982).