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On An Optimal Control Of The Coefficients Of An Elliptic Equation With A Quality Criterion On The Boundary Of Domain

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Abstract. *We consider an optimal control problem for a linear elliptic equation with controls in the coefficients and with a quality criterion on the boundary of domain. The well-posedness of the problem statement is studied and a necessary optimality condition is established.*

Keywords. optimal control · elliptic equation · necessary optimality condition

The optimal control problems for elliptic equations with controls in the coefficients arise for example in optimization problems of continuum mechanics, designing of constructions, theory of elasticity, convection-diffusion reactions, ecological prediction [1-3]. The problems of optimal control of coefficients of elliptic equations with quality criterion on the domain under consideration were studied in the papers [4-10] and so on. Such problems have been studied not enough in the cases when the quality criterion on the boundary is considered [4, p.92]

In the present paper we consider an optimal control problem for a linear elliptic equation with controls in the coefficients of the state equation and in the coefficient of the boundary condition. The quality criterion is expressed by the integral on the part of the boundary of the domain under consideration. The well-posedness of the problem in the weak topology is studied and a necessary optimality condition is established.

1. Problem statement. Let $\Omega = \{x = (x_1, x_2) : 0 < x_i < l,\ i = 1, 2\}$ be a rectangle with the boundary $\Gamma, \Gamma_{-1} = \{x = (x_1, x_2) : x_1 = 0,\ 0 < x_2 < l_2\}$ be the left side of the rectangle Ω. Let the controlled system be described in Ω by the following mixed boundary value problem for an elliptic type linear equation:

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial u}{\partial x_i}\right) + q(x)u = f(x),\ x \in \Omega, \tag{1}$$

$$-k_1(x)\frac{\partial u}{\partial x_1} + p(x)u = g(x),\ x \in \Gamma_{-1}, \tag{2}$$

$$u(x) = 0, x \in \Gamma \backslash \Gamma_{-1}. \tag{3}$$

Here $f(x) \in L_2(\Omega),\ g(x) \in W_2^{1/2}(\Gamma_{-1})$ are the given functions, $\nu(x) = (k_1(x)\,, k_2(x), q(x, p(x))$ is a control, $u = u(x) = u(x; \nu(\cdot))-$ is the state function the solution of the problem (1) (2) corresponding to the control $\nu = \nu(x)$.

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Introduce the set of admissible controls

$$V = \prod_{k=1}^{4} V_k \subset H = W_2^1(\Omega) \times W_2^1(\Omega) \times L_2(\Omega) \times W_2^1(\Gamma_{-1})$$

for

$$V_i = \Big\{ k_i(x) \in W_2^1(\Omega) : 0 < \nu_i \le k_i(x) \le \mu_i,$$

$$\left| \frac{\partial k_i(x)}{\partial x_j} \right| \le d_j^{(i)} \ \ (j = 1, 2) \text{ a.e.in } \Omega \Big\} \quad (i = 1, 2),$$

$$V_3 = \{ q(x) \in L_2(\Omega) : 0 \le q_0 \le q(x) \le q_1 \text{ a.e.in } \Omega \}, \tag{4}$$

$$V_4 = \left\{ p(x) \in W_2^1(\Gamma_{-1}) : 0 < p_0 \le p(x) \le p_1, \left| \frac{\partial p(x)}{\partial x_2} \right| \le p_2 \text{ a.e.in } \Gamma_{-1} \right\},$$

where $\nu_i, \mu_i, d_j^{(i)} (i, j = 1, 2), q_1, p_0, p_1, p_2 > 0, q_0 \ge 0$ are the given numbers.

State the following problem: on the solutions $u = u(x, \nu)$ of problem (1)-(3), corresponding to all admissible controls $\nu \in V$, to minimize the functional

$$J(\nu) = \int_{\Gamma_{-1}} |u(x; \nu(\cdot)) - z(x)|^2 \, dx, \tag{5}$$

where $z(x) \in W_2^1(\Gamma_{-1})$ is the given function. The denotation of functional spaces and their norms used in the paper correspond to [11, pp.27-30].

Under the solution of boundary value problem (1)-(3) at each given control $\nu \in V$ we understand the generalized solution from the class $W_{2.0}^1(\Omega)$ i.e. the function $u = u(x; \nu(\cdot))$ from $W_{2,0}^1(\Omega)$ satisfying for all $\eta = \eta(x) \in W_{2,0}^1(\Omega)$ the integral identity

$$\iint_{\Omega} \left(\sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_i} + qu\eta \right) dx + \int_{\Gamma_{-1}} pu\eta ds = \iint_{\Omega} f\eta dx + \int_{\Gamma_{-1}} g\eta ds. \tag{6}$$

Here $W_{2,0}^1(\Omega)$ is the subspace of the space $W_2^1(\Omega)$ the dense set of which is the set of all functions from $C^1(\overline{\Omega})$ equal to zeno near $\Gamma \backslash \Gamma_{-1}$.

Under the made assumptions, the problem (1)-(3) has a unique generalized solution from $W_{2,0}^1(\Omega)$ for each fixed control $\nu \in V$, this solution also belongs to the space $W_{2,0}^2(\Omega) = W_2^2(\Omega) \cap W_{2,0}^1(\Omega)$ and the following a prior estimation [11, pp.200-226], [12,pp.112-134] is valid:

$$\|u\|_{2,\Omega}^{(2)} \le M_1 \left(\|f\|_{2,\Omega} + \|g\|_{2,\Gamma_{-1}}^{1/2} \right). \tag{7}$$

Here and in the sequel, by M_i $(i = 1, 2...)$ we will denote positive constants independent of the estimated values and of the control $\nu \in V$.

2.Well posedness of the problem statement. Show that problem (1)-(5) is well posed in the weak topology of the space H

Theorem 1. *Let the conditions accepted at the statement of problem (1)-(5) be fulfilled. Then*

$$V_* = \{ v_* \in V : J(v_*) = J_* \equiv \inf \{ J(v) : \nu \in V \} \} \neq \varnothing,$$

V_* is weakly compact in H and any minimizing sequence $\{\nu_n\} \subset V$ of the functional $J(v)$ weakly in H converges to the set V_*.

Proof. The set $V \subset H$ is bounded, closed and convex in the Hilbert space H,and therefore is weakly compact in H [13, p 51].

Show that the functional (5) is continuous on the set V in the weak topology of the space H. Let $v(x) = (k_1(x), k_2(x), q(x), p(x)) \in V$ be some control,

$$\left\{v^{(n)}(x) = (k_1^{(n)}(x), k_2^{(n)}(x), q^{(n)}(x), p^{(n)}(x))\right\} \subset V$$

be on arbitrary sequence of controls, weakly in $H = W_2^1(\Omega) \times W_2^1(\Omega) \times L_2(\Omega) \times W_2^1(\Gamma_{-1})$ convergent to $v(x)$ i.e.

$$k_i^{(n)}(x) \to k_i(x)\ (i = 1, 2) \text{ weakly in } W_2^1(\Omega), \tag{8}$$

$$q^{(n)}(x) \to q(x) \text{ weakly in } L_2(\Omega), \tag{9}$$

$$p^{(n)}(x) \to p(x) \text{ weakly in } W_2^1(\Gamma_{-1}). \tag{10}$$

By a univalent solvability of problem (1)-(3) to each control $v^{(n)}(x)$ there corresponds a unique solution $u^{(n)} = u(x; v^{(n)}(\cdot))$ of problem (1)-(3) for $v(x) = v^{(n)}(x)$, furthermore,

$$\left\|u^{(n)}\right\|_{2,\Omega}^{(2)} \le M_2,\ \forall n = 1, 2..., \tag{11}$$

i.e. the sequence $\left\{u^{(n)}(x)\right\}$ is uniformly bounded in the norm of the space $W_{2,0}^2(\Omega)$. From relations (8-11) and compactness of imbeddings $W_2^1(\Omega) \to L_2(\Omega)$, $W_{2,0}^2(\Omega) \to W_2^1(\Omega)$, $W_2^2(\Omega) \to L_2(\Gamma_{-1})$, $W_2^1(\Gamma_{-1}) \to L_2(\Gamma_{-1})$ [11,pp.77-79] it follows that from the sequence $\left\{\nu^{(n)}, u^{(n)}\right\}$ we can extract the subsequence $\left\{v^{(n_k)}, u^{(n_k)}\right\}$ such that

$$k_i^{(n_k)}(x) \to k_i(x)\ (i = 1, 2) \text{ strongly in } L_2(\Omega), \tag{12}$$

$$q^{(n_k)}(x) \to q(x) \text{ weakly in } L_2(\Omega), \tag{13}$$

$$p^{(n_k)}(x) \to p(x) \text{ strongly in } L_2(\Gamma_{-1}). \tag{14}$$

$$u^{(n_k)} \to u(x) \text{ weakly in } W_{2,0}^2(\Omega), \text{ strongly in } W_{2,0}^1(\Omega) \text{ andstrongly in } L_2(\Gamma_{-1}). \tag{15}$$

where $u = u(x)$ is some element from $W_{2,0}^2(\Omega)$.

Show that $u = u(x)$ is the solution of problem (1)-(3) corresponding to the control $v = v(x)$ i.e. $u = u(x; v(\cdot))$ It is clear that the following identities are valid

$$\iint\limits_{Q} \left(\sum_{i=1}^{2} k_i^{(n_k)} \frac{\partial u^{(n_k)}}{\partial x_i} \frac{\partial \eta}{\partial x_i} + q^{(n_k)} u^{(n_k)} \eta\right) dx + \int\limits_{\Gamma_{-1}} q^{(n_k)} u^{(n_k)} \eta ds$$

$$= \iint\limits_{\Omega} f \eta dx + \int\limits_{\Gamma_{-1}} g \eta ds.\ k = 1, 2, ..., \forall \eta = \eta(x) \in W_{2,0}^1(\Omega) \tag{16}$$

Passing to limit in (16) as $k \to \infty$ and taking into account relations (12)-(15), we can show that $u(x)$ satisfies the identity (6), i.e. it is a generalized solution from $W_{2,0}^1(\Omega)$ of problem (1)-(3) corresponding to the control $\nu(x) \in V$ .Hence and from the inclusion $u(x) \in W_{2,0}^2(\Omega)$ it follows that $u(x) = u(x; v(\cdot))$.

Thus, it is established that subject to relations (8)-(10) from the sequence $\left\{u^{(n)}(x) = u(x; v^{(n)}(\cdot))\right\}$ we can isolate the subsequence $\left\{u^{(n_k)}(x) = u(x; v^{(n_k)}(\cdot))\right\}$. For the which the relation (15) is valid . Using the uniqueness of the solution of problem (1)-(3) corresponding to the control $\nu(x) \in V$, we can show that the relation (15) is valid not only for the subsequence $\left\{u^{(n_k)}(x)\right\}$ but also for all the sequence $\left\{u^{(n_k)}(x)\right\}$, i.e.

$$u^{(n)}(x) = u(x; v^{(n)}(\cdot)) \to u(x) = u(x; v(\cdot)) \text{ weakly in } W_{2,0}^2(\Omega)$$

$$\text{strongly in } W_{2,0}^{1}(\Omega) \text{ and strongly in } L_2(\Gamma_{-1}). \tag{17}$$

Prove now that $J(v^{(n)}) \to J(v)$ as $n \to \infty$.Using (5), it is easy to be convinced that the following estimation is valid

$$\left|J(v^{(n)}) \to J(v)\right| \le \left(\left\|u^{(n)}\right\|_{2,\Gamma_{-1}} + \|u\|_{2,\Gamma_{-1}} + 2\,\|z\|_{2,\Gamma_{-1}}\right)\left\|u^{(n)} - u\right\|_{2,\Gamma_{-1}}$$

Then, by using estimations (7),(11) and relation (17), we get that $J(v^{(n)}) \to J(v)$ as $n \to \infty$, i.e. the functional (5) weakly in H is continuous on V . Furthermore, as it was above noted, the set V is weakly compact in H. Then by using the result from [13, p.49] we get that the problem (1)-(5) is well-posed in the weak topology of the space H, i.e. all the statements of theorem 1 are valid. Theorem 1 is proved.

Remark. The existence of the solution of problem (1)-(5) follows from theorem 1. However as the following example shows, the solution of this problem, generally speaking, may not be unique.

Example 1. Let in the problem (1)-(5)

$$\nu_1 = \nu_2 = 1,\ \mu_1 = \mu_2 = 4,\ d_j^{(i)} = 4,\ (i.j = 1,2)\ q_0 = 1,\ q_1 = 2\pi^2$$

$$p_0 = 1,\ p_1 = 2,\ p_2 = 2,\ z(x)\sin\pi x_2,$$

$$f(x) = -4\pi^2 \sin\frac{\pi}{2}(x_1 - 1)\sin\pi x_2,\ g(x) = \sin\pi x_2.$$

Then it is easy to be convinced that the minimum value of the functional $J(v)$ is achieved on two controls

$$v_{1*} = \Big(k_{1*}(x) = 4, k_{2*}(x) = 1, q_*(x) = 2\pi^2, p_*(x) = 1\Big),$$

$$v_{2*} = \Big(k_{1*}(x) = 4, k_{2*}(x) = 2, q_*(x) = \pi^2, p_*(x) = 1\Big)$$

and

$$J(v_{1*}) = J(v_{2*}) = J_* = 0,\ u(x, v_{1*}) = u(x, v_{2*})$$
$$= -\sin\frac{\pi}{2}(x_1 - 1)\sin\pi x_2,\ x = (x_1, x_2) \in \Omega,$$

i.e. the solution of problem (1)-(5) is not unique.

Remark 2. It follows from theorem 1 that the solution of problem (1)-(5) is stable in weak topology of the space H. However, as the following example shows, the solution of this problem, generally speaking, is unstable in the norm of the space H.

Example 2. Let's consider the optimal control problem from example 1. Then $\nu_* = \big(k_{1*}(x) = 4,$ $k_{2*}(x) = 2,\ q_*(x) = \pi^2,\ p_*(x) = 1\big)$ is the optimal control, and $u(x;\nu_*) = -\sin\frac{\pi}{2}(x_1 - 1)\sin\pi x_2$, $x = (x_1, x_2) \in \Omega$, $J_* = J(v_*) = 0$. Take the sequence of the controls

$$v^{(n)} = v^{(n)}(x) = (k_1^{(n)}(x) = 4, k_2^{(n)}(x) = 2,$$

$$q^{(n)}(x) = \pi^2 + \sin\pi n x_1, p^{(n)}(x) = 1) \in V,\ n = 1, 2, ...$$

Then $v^{(n)}(x) \to v_*(x)$ weakly in H and therefore from relation (17) and weak continuity of the functional $J(v)$ it follows that $J(v^{(n)}) \to J(v_*) = 0$ i.e. the sequence $\left\{v^{(n)}\right\}$ is minimizing for the functional $J(v)$. However, this sequence has no limit in the norm of the space H since $\{\sin\pi x_1\}$ strongly doesn't converge in $L_2(\Omega)$.

3. Necessary optimality condition. For the problem (1)-(5) introduce the conjugated state $\psi = \psi(x) = \psi(x; v(\cdot))$ as the solution of the boundary value problem

$$-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial\psi}{\partial x_i}\right) + q(x)\psi = 0,\ x \in \Omega, \tag{18}$$

$$-k_1(x)\frac{\partial\psi}{\partial x_1}+p(x)\psi=-2(u(x,v)-z(x)),\ x\in\Gamma_{-1}, \tag{19}$$

$$\psi(x,v)=0,\ \ x\in\Gamma\backslash\Gamma_{-1}. \tag{20}$$

Under the solution of the conjugated boundary value problem (18)-(20), at the fixed control $\nu(x)\in V$ we understand the function $\psi(x)=\psi(x,v(\cdot))\in W^1_{2,0}(\Omega)$ satisfying for all $\eta=\eta(x)\in W^1_{2,0}(\Omega)$ the identity

$$\iint\limits_{\Omega}\left(\sum_{i=1}^{2}k_i\frac{\partial\psi}{\partial x_i}\frac{\partial\eta}{\partial x_i}+q\psi\eta\right)dx+\int\limits_{\Gamma_{-1}}p\psi\eta ds=2\int\limits_{\Gamma_{-1}}(u-z)\eta ds. \tag{21}$$

Under the made suppositions, the problem (18)-(20) under each fixed control $\nu\in V$ has a unique generalized solution from $W^1_{2,0}(\Omega)$ and this solution belongs also to the space $W^2_{2,0}(\Omega)$ and the following a priori estimation [11. pp.200-226], [12, pp.112-134] is true :

$$\|\psi\|^{(2)}_{2,\Omega}\le M_2\,\|n-z\|^{(1/2)}_{2,\Gamma_{-1}}\,. \tag{22}$$

Taking into account estimations (7) and boundedness of the imbedding $W^2_{2,0}(\Omega)\to W_2^{1/2}(\Gamma_{-1})$ [14, pp.26,27], we get the estimation

$$\|\psi\|^{(2)}_{2,\Omega}\le M_3\left(\|f\|_{2,\Omega}+\|g\|^{(1/2)}_{2,\Gamma_{-1}}+\|z\|^{(1/2)}_{2,\Gamma_{-1}}\right). \tag{23}$$

Theorem 2. *Let the conditions accepted at the statement of problem (1)-(5) be fulfilled,* $v_*=v_*(x)=(k_{1*}(x),k_{2*}(x),q_*(x),p_*(x))\in V$ *be an optimal control in the problem (1)-(5) and* $u_*(x)=u(x;v_*(\cdot))$, $\psi_*(x)=\psi(x;v_*(\cdot))$ *be the solution of the problems (1)-(3) and (18)-(20) corresponding to the control* v_*. *Then for any* $k_i(x)\in V_i$ $(i=1,2)$, $q\in[q_0,q_1]$, $p(x)\in V_4$ *and for almost all* $\xi\in\Omega$ *the following inequality is valid*

$$\iint\limits_{\Omega}\sum_{i=1}^{2}\frac{\partial u_*}{\partial x_i}\frac{\partial\psi}{\partial x_i}(k_i-k_{i*})dx+(q-q_*(\xi))u_*(\xi)\psi_*(\xi)+$$

$$+\int\limits_{\Gamma_{-1}}u_*(s)\psi_*(s)(p(s)-p_*(s))ds\ge 0. \tag{24}$$

Proof. For the control functions $k_{i*}(x)$, $(i=1,2)$, $p_*(s)$ we determine the classic variations

$$k_{i\varepsilon}(x)=k_{i*}(x)+\varepsilon^2(k_i(x)-k_{i*}(x)),(i=1,2),x\in\Omega, \tag{25}$$

$$p_\varepsilon(s)=p_*(s)+\varepsilon^2(p(s)-p_*(s)),\ \ s\in\Gamma_{-1} \tag{26}$$

where $\varepsilon\in(0,1)$ is an arbitrary number, $k_i(x)\in V_i(i=1,2)$, $p(s)\in V_4$ are arbitrary elements. Obviously, $k_{i\varepsilon}(x)\in V_i(i=1,2)$, $p_\varepsilon(s)\in V_4$ for all $\varepsilon\in(0,1)$ and

$$k_{i\varepsilon}(x)\to k_{i*}(x)(i=1,2)\ \ \text{strongly in } W^1_\infty(\Omega), \tag{27}$$

$$p_\varepsilon(s)\to p_*(s)\ \ \text{strongly in } W^1_\infty(\Gamma_{-1}), \tag{28}$$

as $\varepsilon\to 0$.

For the functions $q_*(x)$ we construct the impulse variation

$$q_\varepsilon(x)=\begin{cases}q, & \text{if } x\in\Pi_\varepsilon,\\ q_*(x) & \text{if } x\in\Omega\backslash\Pi_\varepsilon,\end{cases} \tag{29}$$

where $q\in[q_0,q_1]$ is an arbitrary constant. $\Pi_\varepsilon=\left\{x\in R^2\ :\xi_i-\varepsilon/2<x_i<\xi_i+\varepsilon/2,\,(i=1,2)\right\}$ is a rectangle in R^2, $\xi=(\xi_1,\xi_2)\in\Omega$ is the Lebesgue aritrary point of all the functions contained in the conditions of problems (1)-(5) and (18)-(20). Obviously, $\Pi_\varepsilon\subset Q$, $q_\varepsilon(x)\in V_3$ for all sufficiently small $\varepsilon\in(0,1)$, and

$$q_\varepsilon(x)\to q_*(x)\ \ \text{strongly in } \ L_r(\Omega) \tag{30}$$

as $\varepsilon \to 0$, where $r \in [2, \infty)$ is an arbitrary finite number.

Let $u_\varepsilon(x) = u(x; v_\varepsilon(\cdot))$ be the solution of boundary value problem (1)-(3) corresponding to the control $\nu_\varepsilon(x) = (k_{1\varepsilon}(x), \ k_{2\varepsilon}(x), q_\varepsilon(x), p_\varepsilon(x))$. Denote

$$\Delta u_\varepsilon = u_\varepsilon(x) - u_*(x), \Delta_\varepsilon k_i = k_{i\varepsilon}(x) - k_{i*}(x), (i = 1, 2),$$

$$\Delta_\varepsilon q = q(x) - q_*(x),\ x \in \Omega,\ ,\Delta_\varepsilon p = p_\varepsilon(s) - p(s),\ s \in \Gamma_{-1}.$$

From (6) it follows that the function $\Delta u_\varepsilon(x) \in W_{2,0}^1(\Omega)$ satisfies for all $\eta = \eta(x) \in W_{2,0}^1(\Omega)$ the integral identity

$$\iint\limits_{\Omega} \left(\sum_{i=1}^{2} k_{i\varepsilon} \frac{\partial \Delta u_\varepsilon}{\partial x_i} \frac{\partial \eta}{\partial x_i} + q_\varepsilon \Delta u_\varepsilon \eta \right) dx + \int\limits_{\Gamma_{-1}} p_\varepsilon \Delta u_\varepsilon \eta ds$$

$$= -\iint\limits_{Q} \left(\sum_{i=1}^{2} \Delta_\varepsilon k_i \frac{\partial u_*}{\partial x_i} \frac{\partial \eta}{\partial x_i} + \Delta_\varepsilon q u_* \eta \right) dx - \int\limits_{\Gamma_{-1}} (\Delta_\varepsilon p u_*) \eta ds, \tag{31}$$

and for the function Δu_ε the following estimation is valid [12,pp.112-116], [14, p.42]

$$\|\Delta u_\varepsilon\|_{2,\Omega}^{(1)} \le M_4 \left[\sum_{i=1}^{2} \left\| \Delta_\varepsilon k_i \frac{\partial u_*}{\partial x_i} \right\|_{2,\Omega} + \|\Delta_\varepsilon q u_*\|_{2,\Omega} + \|\Delta_\varepsilon p u_*\|_{2,\Gamma_{-1}} \right]. \tag{32}$$

Then, taking into account relations (27),(28),(30) and estimation (7), we get that the right side of inequality (32) tends to zero as $\varepsilon \to 0$. Thus, we get the convergence

$$\|\Delta u_\varepsilon\|_{2,\Omega}^{(1)} = \|u_\varepsilon - u_*\|_{2,\Omega}^{(1)} \to 0 \tag{33}$$

as $\varepsilon \to 0$.

Let $\psi_\varepsilon(x) = \psi(x; \nu_\varepsilon(\cdot))$ be the generalized solution from the class $W_{2,0}^1(\Omega)$ of the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(k_{i\varepsilon} \frac{\partial \psi_\varepsilon}{\partial x_i} \right) + q_\varepsilon(x) \psi_\varepsilon = 0,\ x \in \Omega, \tag{34}$$

$$-k_{1\varepsilon} \frac{\partial \psi_\varepsilon}{\partial x_i} + p_\varepsilon(x) \psi_\varepsilon = -2 \left(u_* + \frac{1}{2} \Delta u_\varepsilon - z(x) \right), x \in \Gamma_{-1}, \tag{35}$$

$$\psi_\varepsilon(x) = 0,\ x \in \Gamma \backslash \Gamma_{-1}. \tag{36}$$

It is clear that the function $\psi_\varepsilon(x)$ satisfies for all $\eta = \eta(x) \in W_{2,0}^1(\Omega)$ the identity

$$\iint\limits_{\Omega} \left(\sum_{i=1}^{2} k_{i\varepsilon} \frac{\partial \psi_\varepsilon}{\partial x_i} \frac{\partial \eta}{\partial x_i} + q_\varepsilon \psi_\varepsilon \eta \right) dx + \int\limits_{\Gamma_{-1}} p_\varepsilon \psi_\varepsilon \eta ds = 2 \int\limits_{\Gamma_{-1}} \left(u_* + \frac{1}{2} \Delta u_\varepsilon - z \right) \eta ds \ , x \in \Gamma_{-1}, \tag{37}$$

Denote $\Delta \psi_\varepsilon(x) = \psi(x) - \psi_*(x),\ x \in \Omega$. The function $\Delta \psi_\varepsilon(x)$ will satisfy the identity similar to (31). Using this identity, estimation (23), relation (33)and arguing as obtaining (33), we have

$$\|\Delta \psi_\varepsilon\|_{2,\Omega}^{(1)} = \|\psi_\varepsilon - \psi_*\|_{2,\Omega}^{(1)} \to 0 \tag{38}$$

as $\varepsilon \to 0$.

The increment of the functional (5) at the point $\nu_* \in V$ has the from

$$\Delta_\varepsilon J(\nu_*) = J(\nu_\varepsilon) - J(\nu_*) = 2 \int\limits_{\Gamma_{-1}} \left(u^* + \frac{1}{2} \Delta u_\varepsilon - z \right) \Delta u_\varepsilon dx. \tag{39}$$

Using identities (31)and (37), transform the equality (39) . It in (31) we set $\eta = \psi_\varepsilon(x)$in (37), we set $\eta = \Delta u_\varepsilon(x)$ and subtract the obtained relations, we have

$$2\int\limits_{\Gamma_{-1}} \left(u^* + \frac{1}{2}\Delta u_\varepsilon - z\right) \Delta u_\varepsilon dx = \iint\limits_{\Omega} \sum_{i=1}^{2} \Delta_\varepsilon k_i \frac{\partial u_*}{\partial x_i} \frac{\partial \psi_\varepsilon}{\partial x_i} dx + \iint\limits_{\Pi_\varepsilon} \Delta_\varepsilon q u_* \psi_\varepsilon dx + \int\limits_{\Gamma_{-1}} \Delta_\varepsilon p u_* \psi_\varepsilon ds$$

Taking into account this equality in (39), we have

$$\Delta_\varepsilon J(\nu_*) = \iint\limits_{\Omega} \sum_{i=1}^{2} \Delta_\varepsilon k_i \frac{\partial u_*}{\partial x_i} \frac{\partial \psi_\varepsilon}{\partial x_i} dx + \iint\limits_{\Pi_\varepsilon} \Delta_\varepsilon q u_* \psi_\varepsilon dx + \int\limits_{\Gamma_{-1}} \Delta_\varepsilon p u_* \psi_\varepsilon ds.$$

Using this equality and arguing similar to the paper [9], for the first variation of the functional $J(\nu_*)$ we get the formula

$$\delta J(\nu_*) = \lim_{\varepsilon \to 0} \frac{\Delta_\varepsilon J(\nu_*)}{\varepsilon^2} = \iint\limits_{\Omega} \sum_{i=1}^{2} \frac{\partial u_*}{\partial x_i} \frac{\partial \psi_*}{\partial x_i} (k_i + k_{i*}) dx$$

$$+ (q - q_*(\xi)) u_*(\xi) \psi_*(\xi) + \int\limits_{\Gamma_{-1}} u_* \psi_* (p - p_*) ds \qquad (40)$$

almost for all $\xi \in \Omega$.

From the optimality of the control $\nu_*(x) \in V$ it follows that $\delta J(\nu_*) \geq 0$. Hence and from equality (40) and everywhere density in Ω of Lebesgue points the validity of inequality (24) follows. Theorem 2 is proved.

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