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Marcinkiewicz integral and its commutators on local Morrey type spaces

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Abstract. *In this paper, we study the boundedness of the Marcinkiewicz operator μ_Ω and their commutators $[b, \mu_\Omega]$ on local and global Morrey type spaces $LM_{p\theta,w}$ and $GM_{p\theta,w}$, respectively. The problem of boundedness of μ_Ω and their commutators $[b, \mu_\Omega]$ in local Morrey type spaces are reduced to the problem of boundedness of the Hardy operator and general Hardy operator in weighted L_p spaces. This allows obtaining sufficient conditions for boundedness for all admissible values of the parameters.*

Keywords. Local Morrey type spaces; Marcinkiewicz integral, Commutator, BMO, Hardy operator

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1 Introduction and Notation

Morrey spaces and their properties play an important role in the study of local behavior of solutions to elliptic partial differential equations, refer to [21,22]. The authors of [1,9] showed the boundedness in Morrey spaces for some important operators in harmonic analysis such as Hardy-Littlewood operators, Calderon-Zygmund singular integral operators and fractional integral operators. Guliyev in [12] defined local Morrey type spaces and investigated the boundedness of operators above in the new class of spaces.

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma$. Suppose $\Omega \in L_q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ is homogeneous of degree zero and satisfies the cancelation condition

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. Marcinkiewicz operator μ_Ω is defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Let b be a locally integrable function on R^n, the commutator of b and μ_Ω is defined as follows

$$[b, \mu_\Omega]f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

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where

$$F^b_{\Omega,t}(x) = \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)dy.$$

It is well known that Marcinkiewicz operator play an important role in harmonic analysis. Benedek et al. [10] proved that if $\Omega \in C^1(\mathbb{S}^{n-1})$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. The corresponding commutator $[b,\ \mu_\Omega]$ was first considered by Torchinsky and Wang in [24]. In 2002, Ding et al. [11] showed that if $\Omega \in L_q(\mathbb{S}^{n-1})$, $q>1$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$.

Suppose $0<p,\ \theta \le \infty$ and w be a non-negative measurable function on $(0,\infty)$, for any function $f \in L_p^{\rm loc}(\mathbb{R}^n)$, we denote by $LM_{p\theta,w}$, $GM_{p\theta,w}$, the local Morrey-type space, the global Morrey-type space respectively with finite quasinorms

$$\|f\|_{LM_{p\theta,w}} = \|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)},\ \|f\|_{GM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n}\|f(x+\cdot)\|_{LM_{p\theta,w}}.$$

For $w(r)=r^{-\frac{\lambda}{p}},\ 0<\lambda<n$ we get the variant of Morrey type space $GM_{p\theta,r^{-\lambda}}$ introduced by D.R. Adams [1], which were used by G. Lu [19] for studying the embedding theorems for vector fields of Hörmander type. For $\theta=\infty$, $LM_{p,\infty,w} \equiv GM_{p,\infty,w}$ are the generalized Morrey space $M_{p,w}(\mathbb{R}^n)$ introduced by T. Mizuhara [9]. When $\theta=\infty,\ w=r^{-\lambda/p}$, it is the classical Morrey space.

In 1994 the doctoral thesisis [12] by V.S. Guliyev (see, also [13–16]) introduced the local Morrey-type space $LM_{p\theta,w}$. In [12] by V.S. Guliyev intensively studied the classical operators in the local Morrey-type space $LM_{p\theta,w}$, see also the books V.S. Guliyev [13] (1996) and [14] (1999), where these results were presented for the case when the underlying space is the Heisenberg group or a homogeneous group, respectively.

The main purpose of [12] (see also in [13–16]) is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in local Morrey-type space $LM_{p\theta,w}$. In a series of papers by V. Burenkov, H. Guliyev and V. Guliyev (see [3]-[6]) be given some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type space $LM_{p\theta,w}$. Recall that the global Morrey-type space $GM_{p\theta,w}$ were introduced in [3], see also [4].

Therefore, the purpose of this paper is mainly to study the boundedness of Marcinkiewicz operator and its commutators in local Morrey space and global Morrey space for any $0<\theta\le\infty$.

In what follows, we denote by C positive constants which are independent of the main parameters, but it may vary from line to line.

2 Marcinkiewicz integral in local Morrey spaces

In this section, we study the boundedness of integral operators in local Morrey spaces and global Morrey spaces. To state the main results, we first introduce some notations.

Definition 2.1 *Let $0<p,\ \theta\le\infty$, we denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t>0$,*

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega_{p,\theta}$ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t_1,\ t_2>0$,

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty,\ \|w(r)r^{n/p}\|_{L_\theta(0,t_2)} < \infty.$$

In [12], the following result was shown

Lemma 2.1 *Let $0<p,\ \theta\le\infty$ and w be a non-negative measurable function on $(0,\infty)$, then the following is true*

1. *If for all $t>0$, $\|w(r)\|_{L_\theta(t,\infty)}=\infty$, then $LM_{p\theta,w}=GM_{p\theta,w}=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.*

2. *If for all* $t>0,\ \|w(r)r^{n/p}\|_{L_\theta(0,t)}=\infty$, *then any functions* $f\in LM_{p\theta,w}$, *continuous at* $0,\ f(0)=0$, *and for* $0<p<\infty,\ GM_{p\theta,w}=\Theta$. *Consequently, in the sequel, we always assume that either* $w\in\Omega_\theta$ *or* $w\in\Omega_{p,\theta}$.

Let $L_{p,v}(0,\infty)$ be the weighted Lebesgue space of function f on $(0,\infty)$ for which $\|f\|_{L_{p,v}(0,\infty)}=\left(\int_0^\infty |f(x)|^p v(x)dx\right)^{1/p}<\infty$ and let H denote the Hardy operator

$$Hg(r)=\int_0^r g(t)dt,\ 0<r<\infty.$$

Therefore, we have the following theorem

Theorem 2.1 *Let* $\Omega\in L_q(\mathbb{S}^{n-1}),\ 1<q<\infty$, *be a homogeneous of degree zero and satisfy the cancellation condition. If for any* $q'<p<\infty,\ 0<\theta_1,\ \theta_2\le\infty,\ w_1\in\Omega_{\theta_1}$ *an* $dw_2\in\Omega_{\theta_2}$, *suppose that*

$$v(r)=w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-1},\ u(r)=w_2^{\theta_2}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-\theta_2-1}.$$

Assume the operator H *is bounded from* $L_{\theta_1,v}(0,\infty)$ *to* $L_{\theta_2,u}(0,\infty)$ *on the cone of all non-negative non-increasing functions* ϕ *on* $(0,\infty)$ *satisfying the condition* $\lim\limits_{t\to\infty}\phi(t)=0$, *then the Marcinkiewicz operator* μ_Ω *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$ *(in the latter case, it is assume that* $w_1\in\Omega_{p,\theta_1}$ *and* $w_2\in\Omega_{p,\theta_2}$*).*

Proof. For any ball $B=B(x_0,\ r)$, function $f(x)$ can be divided into two parts: $f=f\chi_{4B}+f\chi_{\mathbb{R}^n\setminus 4B}:=f_1+f_2$, thus we have

$$\|\mu_\Omega f\|_{L_p(B)}\le\|\mu_\Omega f_1\|_{L_p(B)}+\|\mu_\Omega f_2\|_{L_p(B)}\equiv I_1+I_2. \tag{2.1}$$

For I_1, by $L_p(\mathbb{R}^n)$ boundedness of μ_Ω in [2], we have

$$I_1\le C\|f\|_{L_p(4B)}\le Cr^{\frac{n}{p}}\int_r^\infty\|f\|_{L_p(B(x,t))}\frac{dt}{t^{\frac{n}{p}+1}}, \tag{2.2}$$

where the constant $C>0$ is independent of f.

For I_2, we first estimate $\mu_\Omega f_2(x)$ for any $x\in B$, since $y\in\mathbb{R}^n\setminus 4B$, it has the following inequality: $|x-y|>|y-x_0|-|x-x_0|>\frac{1}{2}|y-x_0|>3r$, therefore we obtain

$$\begin{aligned}
|\mu_\Omega f_2(x)| &\le \int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_2(y)|\left(\int_{|x-y|<t}\frac{dt}{t^3}\right)^{\frac{1}{2}}dy\\
&= C\int_{\mathbb{R}^n\setminus 4B}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|dy\\
&\le \int_{\mathbb{R}^n\setminus B(0,3r)}\frac{|\Omega(z)|}{|z|^n}f(x-z)dz\\
&= C\int_{\mathbb{R}^n\setminus B(0,3r)}|\Omega(z)f(x-z)|\int_{|z|}^\infty\frac{dt}{t^{n+1}}dz\\
&\le C\int_{3r}^\infty\int_{B(0,t)}|\Omega(z)f(x-z)|dz\frac{dt}{t^{n+1}}\\
&\le C\|\Omega\|_{L_q(\mathbb{S}^{n-1})}\int_{3r}^\infty\left(\int_{B(0,t)}|f(x-z)|^{q'}dz\right)^{\frac{1}{q'}}\frac{dt}{t^{\frac{n}{p}+1}},
\end{aligned}$$

since $q'<p<\infty$, for any $|x-x_0|<r,\ |z|<t$, it has the following inequality: $|x-z-x_0|\le|z|+|x-x_0|<2t$, hence we have

$$\begin{aligned}
|\mu_\Omega f_2(x)| &\le C\|\Omega\|_{L_q(\mathbb{S}^{n-1})}\int_{3r}^\infty(\int_{B(x_0,2t}|f(y)|^p dy)^{\frac{1}{p}}\frac{dt}{t^{\frac{n}{p}+1}}\\
&\le C\|\Omega\|_{L_q(\mathbb{S}^{n-1})}\int_r^\infty\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}.
\end{aligned}$$

Thus for I_2, we have

$$I_2 \le C\|\Omega\|_{L_q(\mathbb{S}^{n-1})} r^{\frac{n}{p}} \int_r^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{2.3}$$

Finally, by the definition of local Morrey space and inequalities of (2.1) − (2.3), we show

$$\begin{aligned} \|\mu_\Omega f\|_{LM_{p\theta_2 w_2}} &= \|w_2(r)\|\mu_\Omega f\|_{L_p(B(0,r))}\|_{L_{\theta_2}(0,\infty)} \\ &\le C\|w_2(r) r^{\frac{n}{p}} \int_r^\infty t^{-n/p-1}\|f\|_{L_p(B(0,t))} dt\|_{L_{\theta_2}(0,\infty)} \\ &= C\|w_2(r^{-\frac{p}{n}})\frac{1}{r}\int_0^r \|f\|_{L_p(B(0,t^{-\frac{p}{n}}))} dt r^{-\frac{p}{n\theta_2}-\frac{1}{\theta_2}}\|_{L_{\theta_2}(0,\infty)}. \end{aligned}$$

Let $g(t) = \|f\|_{L_p(B(0,t^{-\frac{p}{n}}))}$, $u(r) = w_2^{\theta_2}\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}-\theta_2-1}$, then

$$\|\mu_\Omega f\|_{LM_{p\theta_2 w_2}} \le C\|Hg(r)\|_{L_{\theta_2,u}(0,\infty)}. \tag{2.4}$$

Let $v(r) = w_1^{\theta_1}(r^{-\frac{p}{n}}) r^{-\frac{p}{n}-1}$, by the weighted L_p boundedness of Hardy operator H and inequality (2.4), we have

$$\begin{aligned} \|\mu_\Omega f\|_{LM_{p\theta_2 w_2}} &\le C\|g(r)\|_{L_v^{\theta_1}(0,\infty)} \\ &= C\left(\int_0^\infty \|f\|^{\theta_1}_{L_p(B(0,r))} w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}-1} dr\right)^{\frac{1}{\theta_1}} \\ &= C\left(\int_0^\infty \|f\|^{\theta_1}_{L_p(B(0,r))} w_1^{\theta_1}(r) dr\right)^{\frac{1}{\theta_1}} \\ &= C\|\|f\|_{L_p(B(0,r))} w_1(r)\|_{L_{\theta_1}(0,\infty)} = C\|f\|_{LM_{p\theta_1 w_1}}, \end{aligned}$$

where the constant $C > 0$ is independent of f.

On the other hand, by the definition of global Morrey-type spaces, it only need to $g(t) = \|f\|_{L_p(B(x_0,t^{-\frac{p}{n}}))}$, just like local Morrey-type spaces, we also obtain the boundedness in global Morrey spaces.

In order to obtain sufficient conditions of the Marcinkiewicz operator, we shall apply the known necessary and sufficient conditions ensuring boundedness of the Hardy operator H from one weighted Lebesgue space to another one for any non-negative nonincreasing function g (see, for example [7,8]).

Lemma 2.2 *Let g be a non-negative nonincreasing function and u, v weight functions on $(0,\infty)$.*

(a) If $1 < \theta_1 \le \theta_2 < \infty$, then the inequality

$$\left(\int_0^\infty (Hg)^{\theta_2}(t) u(t) dt\right)^{1/\theta_2} \le C\left(\int_0^\infty g^{\theta_1}(t) v(t) dt\right)^{1/\theta_1} \tag{2.5}$$

holds if any only if

$$B_{11} := \sup_{t>0}\left(\int_0^t u(r) r^{\theta_2} dr\right)^{-\frac{1}{\theta_2}} \left(\int_0^t v(r) dr\right)^{\frac{1}{\theta_1}} < \infty,$$

and

$$B_{12} := \sup_{t>0}\left(\int_t^\infty u(r) dr\right)^{\frac{1}{\theta_2}} \left(\int_0^t \frac{v(r) r^{\theta_1'}}{\left(\int_0^r v(\rho) d\rho\right)^{\theta_1'}} dr\right)^{\frac{1}{\theta_1'}} < \infty.$$

(b) If $0 < \theta_1 \le 1, 0 < \theta_1 \le \theta_2 < \infty$, then the inequality (2.5) holds if any only if $B_{11} < \infty$ and

$$B_{22} := \sup_{t>0}\left(\int_t^\infty u(r) dr\right)^{\frac{1}{\theta_2}} \left(\int_0^t v(r) dr\right)^{-\frac{1}{\theta_1'}} < \infty.$$

(c) *If* $1<\theta_1\le\infty,\ 0<\theta_2<\theta_1<\infty,\ \theta_2\ne 1$, *then the inequality* (2.5) *holds if any only if*

$$B_{31}:=\left(\int_0^\infty\left(\frac{\int_0^t u(r)r^{\theta_2}dr}{\int_0^t v(r)dr}\right)^{\frac{\theta_2}{\theta_1-\theta_2}}u(t)t^{\theta_2}dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty,$$

and

$$B_{32}:=\left(\int_0^\infty\left[\left(\int_t^\infty u(r)dr\right)^{\frac{1}{\theta_2}}\left(\int_0^t\frac{v(r)r^{\theta_1'}}{\left(\int_0^r v(\rho)d\rho\right)^{\theta_1'}}dr\right)^{\frac{\theta_2-1}{\theta_2}}\right]^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}\right.$$

$$\left.\times\frac{v(t)t^{\theta_1'}}{\left(\int_0^t v(\rho)d\rho\right)^{\theta_1'}}dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty.$$

(d) *If* $1=\theta_2<\theta_1<\infty$, *then the inequality* (2.5) *holds if any only if*

$$B_{41}:=\left(\int_0^\infty\left(\frac{\int_0^t u(r)rdr}{\int_0^t v(r)dr}\right)^{\frac{1}{\theta_1-1}}u(t)tdt\right)^{\frac{\theta_1-1}{\theta_1}}<\infty,$$

and $B_{42}:=\sup_{t>0}\left[\left(\frac{\int_0^t u(r)rdr+t\int_t^\infty u(r)dr}{\int_0^t v(r)dr}\right)^{\theta_1'-1}\times\left(\int_t^\infty u(r)dr\right)dt\right]^{\theta_1'}<\infty.$

(e) *If* $0<\theta_2<\theta_1=1$, *then the inequality* (2.5) *holds if any only if*

$$B_{51}:=\left(\int_0^\infty\left(\frac{\int_0^t u(r)r^{\theta_2}dr}{\int_0^t v(r)dr}\right)^{\frac{\theta_2}{1-\theta_2}}u(t)t^{\theta_2}dt\right)^{\frac{1-\theta_2}{\theta_2}}<\infty,$$

and

$$B_{52}:=\left(\int_0^\infty\left(\int_t^\infty u(r)dr\right)^{\frac{\theta_2}{1-\theta_1}}\left(\inf_{0<s<t}\frac{1}{s}\int_0^s v(\rho)d\rho\right)^{\frac{\theta_2}{\theta_2-1}}\times u(t)dt\right)^{\frac{1-\theta_2}{\theta_2}}<\infty.$$

(f) *If* $0<\theta_2<\theta_1<1$, *then the inequality* (2.5) *holds if any only if* $B_{31}<\infty$ *and*

$$B_{62}:=\left(\int_0^\infty\sup_{0<s\le t}\frac{s^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}}{\left(\int_0^s v(\rho)d\rho\right)^{\frac{\theta_2}{\theta_1-\theta_2}}}\left(\int_t^\infty u(r)dr\right)^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}\times u(t)dt\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}}<\infty.$$

(g) *If* $0<\theta_1\le 1,\ \theta_2=\infty$, *then the inequality* (2.5) *holds if any only if*

$$B_7:=\operatorname*{ess\,sup}_{0<s\le t}\frac{su(t)}{\left(\int_0^s v(r)dr\right)^{\frac{1}{\theta_1}}}<\infty.$$

(h) *If* $1<\theta_1<\infty,\ \theta_2=\infty$, *then the inequality* (2.5) *holds if any only if*

$$B_8:=\operatorname*{ess\,sup}_{t>0}u(t)\left(\int_0^t\frac{r^{\theta_1'-1}}{\int_0^r v(s)}dr\right)^{\frac{1}{\theta_1{}'}}<\infty.$$

(i) If $\theta_1 = \infty,\ 0 < \theta_2 < \infty$, then the inequality (2.5) *holds if any only if*

$$B_9 := \left(\int_0^\infty \left(\int_0^t \frac{dr}{\operatorname*{ess\,sup}\limits_{0<y<r} v(y)} \right)^{\theta_2} u(t)dt \right)^{\frac{1}{\theta_2}} < \infty.$$

(j) If $\theta_1 = \theta_2 = \infty$, then the inequality (2.5) *holds if any only if*

$$B_{10} := \operatorname*{ess\,sup}_{t>0} u(t) \int_0^t \frac{dr}{\operatorname*{ess\,sup}\limits_{0<y<r} v(y)} < \infty.$$

From Theorem 2.1 and Lemma 2.2, we obtain the following result.

Corollary 2.1 *Let $\Omega \in L_q(\mathbb{S}^{n-1})$, for any $q' < p < \infty,\ 0 < \theta_1,\ \theta_2 \le \infty,\ w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$, suppose that any of condition $(a) - (j)$ is satisfied, then the Marcinkiewicz operator μ_Ω is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$ (in the latter case, it assumes that $w_1 \in \Omega_{p,\theta_1}$ and $w_2 \in \Omega_{p,\theta_2}$).*

Note that if $\theta_1 = \theta_2 = \infty$, that is, condition (j) is satisfied, then the operator μ_Ω is bounded from generalized Morrey space M_{p,ω_1} to generalized Morrey space M_{p,ω_2}, which extend to the result of Guliyev et al. in [17].

3 Commutators of Marcinkiewicz integral in Local Morrey spaces

In this section, we consider the commutators generalized by the singular integral operator, Marcinkiewicz operator and BMO function. A local integrable function $f \in L^{\mathrm{loc}}(\mathbb{R}^n)$, if it satisfies

$$\|b\|_* \equiv \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $B(x,\ r)$ is ball centered at x and radius of r and $b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)dy$, then b belongs to BMO, and $\|\cdot\|_*$ is the norm in BMO. Meantime, it has the following equivalent condition

$$\sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}} < \infty$$

for any $1 < p < \infty$. Besides this equivalent property, the following estimate is very convenient in applications.

Lemma 3.1 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose $1 \le p < \infty,\ x \in \mathbb{R}^n$, and $R > 2r > 0$, there exist constant $C > 0$, such that*

$$|b_{B(x,R)} - b_{B(x,r)}| \le C \ln \frac{R}{r} \|f\|_{\mathrm{BMO}}.$$

These lemmas are obvious, we omit here, reader can consult [15].

In the discussion of boundedness of Marcinkiewicz operator in local Morrey-type space, we use the $L_{p,w}$ boundedness of the Hardy operator. However, when we consider its commutator, it is not enough to the weighted L_p boundedness of the Hardy operator. In the following, we introduce a general Hardy operator.

Definition 3.1 *We will say that K is a general Hardy-type operator if it has the form*

$$Kg(x) := \int_0^x k(x,\ t)g(t)dt,$$

where the kernel $k(x,y)$ satisfies

(i) $k(x,t) \ge 0,\ \ 0 < t < x$;

(ii) $k(x,t)$ *is increasing in* x *and decreasing in* t*;*
(iii) $k(x,t) \approx k(x,z) + k(z,t), \;\; 0 < t < z < x.$

Such kernels are called Oinarov kernels.

Remark 3.1 $k(x,\; t) \equiv 1$, then K is the classical Hardy operator; $k(x,t) = \Phi\left(\frac{x}{t}\right)$, where Φ satisfies $\Phi(ab) \approx \Phi(a) + \Phi(b), \;\; 0 < a < b < \infty$, meets the demands.

Therefore, we get the following theorem

Theorem 3.1 *Let* $\Omega \in L_q(\mathbb{S}^{n-1})$*, for any* $q' < p < \infty, \; 0 < \theta_1, \; \theta_2 \leq \infty, \; w_1 \in \Omega_{\theta_1}, w_2 \in \Omega_{\theta_2}$ *and* $b \in \mathrm{BMO}$*, and*

$$v(r) = w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}-1}, \;\; u(r) = w_2^{\theta_2}\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}-\theta_2-1}.$$

If the Marcinkiewicz operator μ_Ω *is bounded from* $L_{\theta_1,v}(0,\infty)$ *to* $L_{\theta_2,u}(0,\infty)$*, then the commutator* $[b,\; \mu_\Omega]$ *is bounded from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$ *(in the latter case, it is assume that* $w_1 \in \Omega_{p,\theta_1}$ *and* $w_2 \in \Omega_{p,\theta_2}$*).*

Proof. For any ball $B = B(x_0,\; r)$, function $f(x)$ can be divided into two parts: $f = f\chi_{4B} + f\chi_{\mathbb{R}^n\setminus 4B} := f_1 + f_2$, thus, we have

$$\|[b,\; \mu_\Omega]f\|_{L_p(B)} \leq \|[b,\; \mu_\Omega]f_1\|_{L_p(B)} + \|[b,\; \mu_\Omega]f_2\|_{L_p(B)} \equiv J_1 + J_2. \tag{3.1}$$

For J_1, by $L_p(\mathbb{R}^n)$ boundedness of $[b,\; \mu_\Omega]$ in [3], we have

$$J_1 \leq C\|f\|_{L_p(4B)} \leq Cr^{\frac{n}{p}} \int_{3r}^{\infty} \|f\|_{L_p(B(x,2t))} \frac{dt}{t^{\frac{n}{p}+1}}, \tag{3.2}$$

where the constant $C > 0$ is independent of f.

For J_2, observe that for any $x \in B$, since $y \in \mathbb{R}^n \setminus 4B$, it has the following inequality: $|x-y| > |y-x_0| - |x-x_0| > \frac{1}{2}|y-x_0| > 3r$, therefore we obtain

$$\begin{aligned}
|[b,\mu_\Omega]f_2(x)| &\leq \int_{\mathbb{R}^n\setminus 4B} \frac{\Omega(x-y)}{|x-y|^n}|b(x)-b(y)|\,|f(y)|dy \\
&\leq \int_{\mathbb{R}^n\setminus B(0,3r)} \frac{|\Omega(z)|}{|z|^n}|b(x)-b(x-z)|\,|f(x-z)|dz \\
&= C\int_{\mathbb{R}^n\setminus B(0,3r)} |\Omega(z)|\,|b(x)-b(x-z)|\,|f(x-z)| \int_{|z|}^{\infty} \frac{dt}{t^{n+1}}dz \\
&\leq C\int_{3r}^{\infty}\int_{B(0,t)} |\Omega(z)|\,|b(x)-b_B|\,|f(x-z)|dz\frac{dt}{t^{n+1}} \\
&+ C\int_{3r}^{\infty}\int_{B(0,t)} |\Omega(z)|\,|b_{B(x_0,2t)}-b_B|\,|f(x-z)|dz\frac{dt}{t^{n+1}} \\
&+ C\int_{3r}^{\infty}\int_{B(0,t)} |\Omega(z)|\,|b(x-z)-b_{B(x_0,2t)}|\,|f(x-z)|dz\frac{dt}{t^{n+1}} \\
&:= K_1 + K_2 + K_3,
\end{aligned} \tag{3.3}$$

since $q' < p < \infty$, for any $|x - x_0| < r,\ |z| < t$, it has the following inequality: $|x - z - x_0| \leq |z| + |x - x_0| < 2t$, hence we have

$$\begin{aligned} K_1 &\leq C|b(x) - b_B| \int_{3r}^{\infty} \int_{B(0,t)} |\Omega(z) f(x-z)| dz \frac{dt}{t^{n+1}} \\ &\leq C|b(x) - b_B| \int_{3r}^{\infty} \left(\int_{B(0,t)} |\Omega(z)|^q dz \right)^{\frac{1}{q}} \left(\int_{B(0,t)} |f(x-z)|^{q'} dz \right)^{\frac{1}{q'}} \frac{dt}{t^{n+1}} \\ &\leq C|b(x) - b_B| \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_{3r}^{\infty} \left(\int_{B(x_0,2t)} |f(y)|^{q'} dy \right)^{\frac{1}{q'}} \frac{dt}{t^{\frac{n}{q'}+1}} \\ &\leq C|b(x) - b_B| \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_{3r}^{\infty} \|f\|_{L_p(B(x_0,2t))} \frac{dt}{t^{\frac{n}{p}+1}}, \end{aligned}$$

thus, we obtain

$$\begin{aligned} \|K_1\|_{L_p(B)} &\leq C \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \left(\int_B |b(x) - b_B|^p dx \right)^{\frac{1}{p}} \int_r^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q'}+1}} \\ &\leq C \|b\|_* \|\Omega\|_{L_q(\mathbb{S}^{n-1})} r^{\frac{n}{p}} \int_{3r}^{\infty} \|f\|_{L_p(B(x_0,2t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{3.4}$$

Next, we consider the third part of K_3, for any $q' < p < \infty$ and some $1 < s < \frac{pq}{p+q}$, we have

$$\begin{aligned} K_3 &\leq C \int_{3r}^{\infty} \left(\int_{B(0,t)} |b(x-z) - b_{B(x_0,2t)}|^{s'} dz \right)^{\frac{1}{s'}} \left(\int_{B(0,t)} |\Omega(z) f(x-z)|^s dz \right)^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\ &\leq C \int_{3r}^{\infty} \left(\int_{B(x_0,2t)} |b(y) - b_{B(x_0,2t)}|^{s'} dy \right)^{\frac{1}{s'}} \\ &\times \left(\int_{B(0,t)} |\Omega(z)|^q dz \right)^{\frac{1}{q}} \left(\int_{B(0,t)} |f(x-z)|^{\frac{sq}{q-s}} dz \right)^{\frac{q-s}{qs}} \frac{dt}{t^{n+1}} \\ &\leq C \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \|b\|_* \int_{3r}^{\infty} \|f\|_{L_p(B(x_0,2t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{3.5}$$

Finally, for the second part of K_2, by the lemma 3.1, we obtain

$$\begin{aligned} K_2 &\leq C \|b\|_* \int_{3r}^{\infty} \int_{B(0,t)} |\Omega(z) f(x-z)| dz \ln\left(\frac{2t}{r}\right) \frac{dt}{t^{n+1}} \\ &\leq C \|b\|_* \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_{3r}^{\infty} \left(\int_{B(0,t)} |f(x-z)|^{q'} dz \right)^{\frac{1}{q'}} \ln\left(\frac{t}{r}\right) \frac{dt}{t^{\frac{n}{q'+1}}} \\ &\leq C \|b\|_* \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_{3r}^{\infty} \left(\int_{B(x_0,2t)} |f(y)|^{q'} dy \right)^{\frac{1}{q'}} \ln\left(\frac{t}{r}\right) \frac{dt}{t^{\frac{n}{q'}+1}} \\ &= C \|b\|_* \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_{3r}^{\infty} \|f\|_{L_p(B(x_0,2t))} \ln\left(\frac{t}{r}\right) \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{3.6}$$

Therefore, by the inequalities (3.1)-(3.6), we show

$$\begin{aligned} \|[b,\ \mu_\Omega] f\|_{L_p(B)} &\leq C \|b\|_* \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_{3r}^{\infty} \|f\|_{L_p(B(x_0,2t))} \ln\left(\frac{t}{r}\right) \frac{dt}{t^{\frac{n}{p}+1}} \\ &\leq C \|b\|_* \|\Omega\|_{L_q(\mathbb{S}^{n-1})} \int_r^{\infty} \|f\|_{L_p(B(x_0,t))} \ln\left(\frac{t}{r}\right) \frac{dt}{t^{\frac{n}{p}+1}}, \end{aligned}$$

where the constant $C > 0$ is independent of f.

Thus, by the definition of local Morrey space, we have

$$\begin{aligned}\|[b,\ \mu_\Omega]f\|_{LM_{p\theta_2 w_2}} &= \|w_2(r)\|[b,\ \mu_\Omega]f\|_{L_p(B(0,r))}\|_{L_{\theta_2}(0,\infty)} \\ &\le C\|w_2(r)r^{\frac{n}{p}}\int_r^\infty t^{-n/p-1}\|f\|_{L_p(B(0,t))}\ \ln\left(\frac{t}{r}\right)\ dt\|_{L_{\theta_2}(0,\infty)} \\ &= C\|w_2\left(r^{-\frac{p}{n}}\right)\frac{1}{r}\int_0^r \|f\|_{L_p(B(x_0,t^{-\frac{p}{n}}))}\ \ln\left(\frac{t}{r}\right)\ dtr^{\frac{-p}{n\theta_2}-\frac{1}{\theta_2}}.\end{aligned}$$

Let $g(t) = \|f\|_{L_p(B(0,t^{-\frac{p}{n}}))}$, $u(r) = w_2^{\theta_2}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-\theta_2-1}$ and $k(r,t) = \ln\frac{r}{t}$, for any $0 < t < r$, then

$$\|[b,\mu_\Omega]f\|_{LM_{p\theta_2,w_2}} \le C\|Kg(r)\|_{L_{\theta_2,u}(0,\infty)}. \tag{3.7}$$

Let $v(r) = w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}}$, by the weighted L_p boundedness of general Hardy operator K and inequality (3.7), we have

$$\begin{aligned}\|[b,\mu_\Omega]f\|_{LM_{p\theta_2 w_2}} &\le C\|g(r)\|_{L_{\theta_1,v}(0,\infty)} \\ &= C\left(\int_0^\infty \|f\|_{L_p(B(0,r^{-\frac{p}{n}}))}w_1^{\theta_1}\left(r^{-\frac{p}{n}}\right)r^{-\frac{p}{n}-1}dr\right)^{\frac{1}{\theta_1}} \\ &= C\left(\int_0^\infty \|f\|^{\theta_1}_{L_p(B(0,r))}w_1^{\theta_1}(r)dr\right)^{\frac{1}{\theta_1}} \\ &= C\|\|f\|_{L_p(B(0,r))}w_1(r)\|_{L_{\theta_1}(0,\infty)} \\ &= C\|f\|_{LM_{p\theta_1,w_1}},\end{aligned}$$

where the constant $C > 0$ is independent of f.

On the other hand, by the definition of global Morrey-type spaces, it only need to $g(t) = \|f\|_{L_p(B(x_0,t^{-\frac{p}{n}}))}$, just like local Morrey-type spaces, we also obtain the boundedness in global Morrey spaces.

Note that, in the proof of Theorem 3.1, we assume $k(r,t) = \ln\left(\frac{r}{t}\right),\ \ 0 < t < r$. According to [16], it has known the necessary and sufficient conditions on the weight functions u and v which ensured that

$$\left(\int_0^\infty |(Kf)(x)|^{\theta_2}u(x)dx\right)^{\theta_2} \le C\left(\int_0^\infty |f(x)|^{\theta_1}v(x)dx\right)^{\theta_1} \tag{3.8}$$

holds. Next, it supposes the kernel $k(r,\ t) = \ln\left(\frac{r}{t}\right),\ \ 0 < t < r$, we have the following lemma

Lemma 3.2 *Let g be a non-negative function and u, v weight functions on $(0,\infty)$.*

(i) If $(\theta_1,\ \theta_2) \in D_1 \equiv \{(\theta_1,\ \theta_2) : 1 < \theta_1 \le \theta_2 < \infty\}$, then the inequality (3.8) *holds if and only if*

$$B_{11} = \sup_{t>0}\left(\int_t^\infty \left(\ln\frac{s}{t}\right)^{\theta_2}u(s)ds\right)^{1/\theta_2}\left(\int_0^t v^{1-\theta_1'}(s)ds\right)^{1/\theta_1} < \infty,$$

and

$$B_{12} = \sup_{t>0}\left(\int_t^\infty u(s)ds\right)^{1/\theta_2}\left(\int_0^t \left(\ln\frac{t}{s}\right)^{\theta_1'}v^{1-\theta'}(s)ds\right)^{1/\theta_1} < \infty,$$

(ii) If $(\theta_1,\ \theta_2) \in D_2 \equiv \{(\theta_1,\ \theta_2) : 1 < \theta_2 < \theta_1 < \infty\}$, $\frac{1}{r} = \frac{1}{\theta_2} - \frac{1}{\theta_1}$, then the inequality (3.8) *holds if and only if*

$$B_1^2 = \left\{\int_0^\infty \left(\int_t^\infty \left(\ln\frac{s}{t}\right)^{\theta_2}u(s)ds\right)^{r/\theta_2}\left(\int_0^t v^{1-\theta_1'}(s)ds\right)^{r/\theta_2'}v^{1-\theta_1'}(t)dt\right\}^{1/r} < \infty,$$

and

$$B_{22} = \left\{\int_0^\infty \left(\int_t^\infty u(s)ds\right)^{r/\theta_1}\left(\int_0^t \left(\ln\frac{t}{s}\right)^{\theta_1'}v^{1-\theta_1'}(s)ds\right)^{r/\theta_1'}u(t)dt\right\}^{1/r} < \infty.$$

(iii) *Let* $(\theta_1,\ \theta_2) \in D_3 \equiv \{(\theta_1,\ \theta_2) : 0 < \theta_2 < 1 < \theta_1 < \infty\}$, $\frac{1}{r} = \frac{1}{\theta_2} - \frac{1}{\theta_1}$, *if* $B_1^2 < \infty$, *then the inequality* (3.8) *holds. Conversely if* (3.8) *holds, then*

$$B_{32} = \left(\int_0^\infty \left(\int_t^\infty \left(\ln \frac{s}{t} \right)^{\theta_2} u(s) ds \right)^{\theta_1'/\theta_2} v^{1-\theta_1'}(t) dt \right)^{1/\theta_1'} < \infty.$$

Moreover, if g is a non-negative nonincreasing function, for parameter: $0 < \theta_1 \le 1,\ \theta_1 \le \theta_2 < \infty$, we have the following lemma

Lemma 3.3 *Let* g *is a non-negative nonincreasing function, and* $u,\ v$ *weight functions on* $(0, \infty)$, *for* $(\theta_1,\ \theta_2) \in D_4 \equiv \{(\theta_1,\ \theta_2) : 0 < \theta_2 \le 1,\ \theta_1 \le \theta_2 < \infty\}$, *the inequality* (3.8) *holds if and only if*

$$\sup_{r>0} \left(\int_r^\infty t^{\theta_2} u(t) dt \right)^{1/\theta_2} \left(\int_0^r v(t) dt \right)^{-1/\theta_1} < \infty.$$

Note that Lemma 3.2 and Lemma 3.3 were proved in [18] (see theorem 2.10, 2.15, 2.17 and corollary 6.15). Now, from Theorem 2 and Lemmas 4 and 5, we have the following result.

Corollary 3.1 *Let* $\Omega \in L_q(\mathbb{S}^{n-1}),\ 1 < q < \infty$, *for any* $q' < p < \infty$, $(\theta_1,\ \theta_2) \in D_1 \cup D_2 \cup D_3 \cup D_4,\ w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *suppose that any of condition of Lemma 3.2 or Lemma 3.3 is satisfied. Then for any* $b \in \mathrm{BMO}$, *the commutator* $[b,\ \mu_\Omega]$ *is bounded from* $LM_{p\theta_1, w_1}$ *to* $LM_{p\theta_2, w_2}$ *and from* $GM_{p\theta_1, w_1}$ *to* $GM_{p\theta_2, w_2}$ *(in the latter case, it is assume that* $w_1 \in \Omega_{p,\theta_1}$ *and* $w_2 \in \Omega_{p,\theta_2}$*).*

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References

1. Adams, D.R.: *A note on Riesz potentials*, Duke Math. J., 42, 765-778 (1975).
2. Alvarez, J., Bagby, R.J., Kurtz, D.S., Perez C.: *Weighted estimates for commutators of linear operators*. Studia Math. 104, 195-209 (1993).
3. Burenkov, V.I., Guliyev, H.V.: *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Studia Math., 163 (2), 157-176 (2004).
4. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces*, J. of Comt. Appl. Math. 208 (1), 280-301 (2007).
5. Burenkov, V.I., Guliyev, V.S.: *Necessary and sufficient conditions for the boundedness of the Riesz operator in local Morrey-type spaces*, Potential Anal. 30 (3), 211-249 (2009).
6. Burenkov, V.I., Guliyev, V.S., Serbetci, A., Tararykova, T.V.: *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces*, Eurasian Math. J. **1** (1), 32-53 (2010).
7. Carro, M., Pick, L., Soria, J., Stepanov, V.D.: *On embeddings between classical Lorentz spaces*. Math. Inequal. Appl. **4** (3), 397-428 (2001).
8. Carro, M., Gogatishvili, A., Martin, J., Pick, L.: *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces*. J. Operator Theory 59(2), 309-332 (2008).
9. Chiarenza, F., Frasca, M.: *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. **7**, 273-279 (1987).
10. Benedek, A., Calderon, A.P., Panzone, R.: *Convolution operators on Banach space valued functions*, Proc. Natl. Acad. Sci. USA 48, 356-365 (1965).
11. Ding, Y., Lu, S., Yabuta, K.: *On commutators of Marcinkiewicz integrals with rough kernel*, J. Math. Anal. Appl. **275**, 60-68 (2002).
12. Guliyev, V.S.: *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctor's degree dissertation, Moscow, Mat. Inst. Steklov, 1-329 (1994) (Russian).
13. Guliyev, V.S.: Integral operators, function spaces and questions of approximation on Heisenberg groups. Baku. 1-200 (1996) (Russian).
14. Guliyev, V.S.: Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications. Baku. 1-332 (1999) (Russian).
15. Guliyev, V.S., Mustafayev, R.Ch.: *Integral operators of potential type in spaces of homogeneous type,* Dokl. Akad. Nauk, Matematika, **354** (6), 730-732 (1997) (Russian).
16. Guliyev, V.S., Mustafayev, R.Ch.: *Fractional integrals in spaces of functions defined on spaces of homogeneous type,* J. Anal. Math., **24** (3), 181-200 (1998) (Russian).
17. Guliyev, V.S., Aliyev, S.S., Karaman, T.: *Boundedness for a class of sublinear operators and their commutators on generalized Morrey spaces*, Abstr. Appl. Anal. Article ID 356041, 18 p. (2011).
18. Kufner, A., Persson, L.E.: Weighted inequalities of Hardy type. World Sci. Pub. Co. Ltd. (2003).

19. Lu, G.: Embedding theorems on Campanato-Morrey spaces for vector fields and applications. C.R.Acad. Sci. Paris. **3** (320), 429-434 (1995).
20. Mizuhara, T.: Boundedness of some classical operators on generalized Morrey spaces. Harmonic Analysis, ICM 90 Statellite Proceedings, Springer, Tokyo. 183-189 (1991).
21. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
22. Peetre, J.: *On the theory of* $M_{p,\lambda}$, J. Funct. Anal. **4**, 71-87 (1969).
23. Softova, L.: *Singular integrals and commutators in generalized Morrey spaces*. Acta Math. Appl. Sin. Engl. Ser., **22** (3), 757-766 (2006).
24. Torchinsky, A, Wang, S.: *A note on the Marcinkiewicz integral*. Colloq. Math. **60** (61), 235-243 (1990).