Transactions of NAS of Azerbaijan, Issue Mathematics, **35** (4), 102-112 (2015).
Series of Physical-Technical and Mathematical Sciences.

Zigmund estimations for the direct value of the derivative of a simple layer acoustic potential

Elnur H. Khalilov

Received: 09.02.2015 / Accepted: 12.11.2015

Abstract. In the paper, the Zigmund estimation for the direct value of a simple layer acoustic potential is proved.

Keywords. derivative of simple layer acoustic potential, Zigmund estimation, Maxwell equation, Helmholts equation, surface singular integral.

Mathematics Subject Classification (2010): 45E05, 31B10

1 Introduction

It is known that boundary value problems for the Maxwell equations and boundary value problems for the Helmholts vector equations are reduced to the system of singular integral equations dependent on the direct value of the derivative of the simple layer acoustic potential (see[1])

$$V_{k,\rho}(x) = \int\limits_S grad_x \Phi_k\left(x,y\right)\rho\left(y\right)dS_y, \quad x \in S, \tag{1.1}$$

where $S \subset \mathbf{R}^3$ is the Lyapunov surface with the exponent $0 < \alpha \leq 1$, $\overrightarrow{n}\ (y)$ is the external unit normal at the point $y \in S$, $\Phi_k\left(x,y\right) = e^{ik|x-y|}/\left(4\pi\left|x-y\right|\right)$ is the fundamental solution of the Helmholts equation, k is a wave number, moreover $Imk \geq 0$, and $\rho(y)$ is a continuous function on the surface S.

Note that the counterexamples constructed by Gunter (see[2]) show that for simple and double layer potential with continuous density, the derivatives, generally speaking, do not exist. However, in the paper [3], a formula for calculating the derivative of an acoustic double layer potential is given and some basic properties of the operator generated by the acoustic double layer potential are studied, while in the paper [4], it was proved that if $\int\limits_0^{diam\ S} \frac{\omega(\rho,t)}{t}dt < +\infty$, then integral (1.1) exists in the sense of the Cauchy principal value, and

$$\sup_{x\in S}\left|V_{k,\rho}(x)\right| \leq M^1 \left(\left\|\rho\right\|_\infty + \int\limits_0^{diam\ S}\frac{\omega(\rho,t)}{t}dt\right),$$

where $\left\|\rho\right\|_\infty = \max\limits_{x\in S}\left|\rho(x)\right|, w(\rho,t)$ is the continuity of the functions ρ. Furthermore, in [1] the boundedness of the operator $\left(A\rho\right)(x) = V_{k,\rho}\left(x\right),\ x \in S$ in Holder's class is proved. Some properties of the

E.H. Khalilov
Azerbaijan State Oil Academy
20, Azadlig av. AZ 1601, Baku, Azerbaijan E-mail: elnurkhalil@mail.ru

[1] here and in the sequel, by M we denote positive constants different at various inequalities.

operator A in generalized Holder spaces have not been studied yet. For that as is known, at first it is necessary to prove the Zigmund estimation for the direct value of the derivative of a simple layer acoustic potential, and this paper is devoted to this issue.

2 Main results

Assume that $V_{k,\rho}(x) = \left(V_{k,\rho}^{(1)}(x), V_{k,\rho}^{(2)}(x), V_{k,\rho}^{(3)}(x)\right)$,where

$$V_{k,\rho}^{(m)}(x) = \int\limits_S \frac{\partial \Phi_k(x,y)}{\partial x_m} \rho\,(y)\,dS_y,\ x = (x_1, x_2, x_3) \in S\ \ (m = 1, 2, 3)\,.$$

Lemma 2.1 *Let S be a Lyapunov surface with the exponent, $0 < \alpha \le 1$, and $\int\limits_0^{diam\ S} \frac{\omega(\rho,t)}{t} dt < +\infty$. Then for any $m = \overline{1,3}$ and for any points $x', x'' \in S$ the following estimation is valid:*

$$\left| V_{k,\rho}^{(m)}(x') - V_{k,\rho}^{(m)}(x'') \right|$$

$$\le M_\rho \left(h^\alpha + \omega\,(\rho, h) + \int\limits_0^h \frac{\omega(\rho,t)}{t} dt + h \int\limits_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt \right) \quad \textit{for} \quad 0 < \alpha < 1,$$

$$\left| V_{k,\rho}^{(m)}(x') - V_{k,\rho}^{(m)}(x'') \right|$$

$$\le M_\rho \left(h\,|\ln h| + \omega\,(\rho, h) + \int\limits_0^h \frac{\omega(\rho,t)}{t} dt + h \int\limits_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt \right) \quad \textit{for} \quad \alpha = 1,$$

where $h = \left|x' - x''\right|$, and M_ρ is a positive constant dependent only on S, k and ρ.

Proof. Let $0 < \alpha < 1$ and $m = 1$. It is easy to calculate

$$V_{k,\rho}^{(1)}(x) = \int\limits_S \frac{(ik\,|x-y| \exp(ik\,|x-y|) - (\exp ik\,|x-y| - 1))\,(x_1 - y_1)}{4\pi\,|x-y|^3}$$

$$\times \rho(y) dS_y + \int\limits_S \frac{y_1 - x_1}{4\pi\,|x-y|^3} \left(\rho(y) - \rho(x)\right) dS_y$$

$$+ \rho(x) \int\limits_S \frac{y_1 - x_1}{4\pi\,|x-y|^3} dS_y,\ x \in S,$$

where the last integral exists in the sense of the Cauchy principal value . Take any points $x', x'' \in S$ such that the quantity h be rather small. Then

$$V_{k,\rho}^{(1)}(x') - V_{k,\rho}^{(1)}(x'')$$

$$= \frac{1}{4\pi} \int\limits_S \left(\frac{\left(y_1 - x_1'\right)\left(\rho(y) - \rho(x')\right)}{|x' - y|^3} - \frac{\left(y_1 - x_1''\right)\left(\rho(y) - \rho(x'')\right)}{|x'' - y|^3} \right) dS_y$$

$$+ \int\limits_S \left[\frac{\left(ik\left|x' - y\right| \exp(ik\left|x' - y\right|) - \left(\exp(ik\left|x' - y\right|) - 1\right)\right)(x_1' - y_1)}{4\pi\,|x' - y|^3} \right.$$

$$\left. - \frac{\left(ik\left|x'' - y\right| \exp(ik\left|x'' - y\right|) - \left(\exp(ik\left|x'' - y\right|) - 1\right)\right)(x_1' - y_1)}{4\pi\,|x' - y|^3} \right]$$

$$\times\rho(y)dS_y+\left(\frac{\rho(x')}{4\pi}\int\limits_S\frac{y_1-x_1'}{|x'-y|^3}dS_y-\frac{\rho(x'')}{4\pi}\int\limits_S\frac{y_1-x_1''}{|x''-y|^3}dS_y\right). \tag{2.1}$$

Denote the summands in the right side of equality (2.1) by $F\left(x',x''\right)$, $L\left(x',x''\right)$ and $G\left(x',x''\right)$, respectively.

Let d be a radius of standard sphere for S (see [5]). Then it is obvious that

$$F\left(x',x''\right)=\int\limits_{S\backslash S_d(x')}\left(\frac{\left(y_1-x_1'\right)\left(\rho(y)-\rho(x')\right)}{4\pi\left|x'-y\right|^3}-\frac{\left(y_1-x_1''\right)\left(\rho(y)-\rho(x'')\right)}{4\pi\left|x''-y\right|^3}\right)dS_y$$

$$+\int\limits_{S_{h/2}(x')}\frac{\left(y_1-x_1'\right)\left(\rho(y)-\rho(x')\right)}{4\pi\left|x'-y\right|^3}dS_y-\int\limits_{S_{h/2}(x'')}\frac{\left(y_1-x_1''\right)\left(\rho(y)-\rho(x'')\right)}{4\pi\left|x''-y\right|^3}dS_y$$

$$-\int\limits_{S_{h/2}(x')}\frac{\left(y_1-x_1''\right)\left(\rho(y)-\rho(x'')\right)}{4\pi\left|x''-y\right|^3}dS_y+\int\limits_{S_{h/2}(x'')}\frac{\left(y_1-x_1'\right)\left(\rho(y)-\rho(x')\right)}{4\pi\left|x'-y\right|^3}dS_y$$

$$+\int\limits_{S_d(x')\backslash(S_{h/2}(x')\cup S_{h/2}(x''))}\left(y_1-x_1'\right)\left(\rho(y)-\rho(x')\right)$$

$$\times\left(\frac{1}{4\pi\left|x'-y\right|^3}-\frac{1}{4\pi\left|x''-y\right|^3}\right)dS_y$$

$$+\int\limits_{S_d(x')\backslash(S_{h/2}(x')\cup S_{h/2}(x''))}\frac{\left(x_1''-x_1'\right)\left(\rho(y)-\rho(x')\right)}{4\pi\left|x''-y\right|^3}dS_y$$

$$+\int\limits_{S_d(x')\backslash(S_{h/2}(x')\cup S_{h/2}(x''))}\frac{\left(y_1-x_1''\right)\left(\rho(x'')-\rho(x')\right)}{4\pi\left|x''-y\right|^3}dS_y$$

$$=F_1(x',x'')+F_2(x',x'')+F_3(x',x'')+F_4(x',x'')$$

$$+F_5(x',x'')+F_6(x',x'')+F_7(x',x'')+F_8(x',x'').$$

It is obvious that $\left|F_1(x',x'')\right|\le M\left\|\rho\right\|_\infty h$.

Applying the formula of reduction of a surface integral to double one we have

$$\left|F_2(x',x'')\right|\le M\int\limits_0^h\frac{\omega(\rho,t)}{t}dt,\quad \left|F_3(x',x'')\right|\le M\int\limits_0^h\frac{\omega(\rho,t)}{t}dt.$$

Furthermore, taking into account the inequality

$$h/2\le\left|y-x''\right|\le 3h/2,\ y\in S_{h/2}(x'), \tag{2.2}$$

we get

$$F_4(x',x'')\le M\frac{\omega(\rho,3h/2)}{(h/2)^2}mesS_{h/2}(x')\le M\omega\left(\rho,h\right).$$

In the similar way, taking into account the inequality

$$h/2\le\left|y-x'\right|\le 3h/2,\ y\in S_{h/2}(x''), \tag{2.3}$$

we get $F_4(x',x'')\le M\omega\left(\rho,h\right)$.

As for any $y\in S_d(x')\backslash(S_{h/2}(x')\cup S_{h/2}(x''))$

$$\left|x'-y\right|\le\left|x'-x''\right|+\left|x''-y\right|\le 3\left|x''-y\right|, \tag{2.4}$$

and also

$$\left|x'' - y\right| \le 3\left|x' - y\right|, \tag{2.5}$$

passing to double integral we find:

$$\left|F_6(x', x'')\right| \le Mh\int_h^d \frac{\omega(\rho, t)}{t^2}dt, \quad \left|F_7(x', x'')\right| \le Mh\int_h^d \frac{\omega(\rho, t)}{t^2}dt.$$

Estimate the expression $\left|F_8(x', x'')\right|$. As for the point $x' \in S$ the vicinity $S_d(x') = \left\{y \in S \left\|y - x'\right| < d\right\}$ intersects the straighline parallel to the normal $\overrightarrow{n}(x')$ at a unique point thus, the set $S_d(x')$ is uniquely projected onto the set $\Omega_d(x')$ lying in the circle of radius d centered at the point x' in the tangential plane $\Gamma(x')$ to S at the point x'. On the segment $S_d(x')$ we select a local rectangular system of coordinates (u, v, w) with the origin at the point x', where the axis is directed along the normal $\overrightarrow{n}(x')$, and the axes u and v will lie on the tangential plane $\Gamma(x')$. Then in these coordinates, the vicinity $S_d(x')$ may be given by the equation $w = f(u, v)$, $(u.v) \in \Omega_d(x')$, moreover

$$f \in C^{1,\alpha}\left(\Omega_d(x')\right) \text{ and } f(0,0) = 0, \frac{\partial f(0,0)}{\partial u} = 0, \frac{\partial f(0,0)}{\partial v} = 0. \tag{2.6}$$

Furthermore, if $\widetilde{y} \in \Gamma(x')$ is the projection of the point $y \in S$, then $\left|x' - \widetilde{y}\right| \le \left|x' - y\right| \le C_1\left|x' - \widetilde{y}\right|$, where C_1 is a positive constant dependent only on S (see [6]). And the coordinates of the points x' will be $(0,0,0)$, while the coodinates of the points x'' will be denoted by $\left(u'', v'', f(u'', v'')\right)$. Let $d_0 = d/C_1, h_0 = \sqrt{(u'')^2 + (v'')^2}$, $O_p(x') = \left\{(u, v, 0) \left|u^2 + v^2 < p^2\right.\right\}$ and

$$O_p(x'') = \left\{(u, v, 0) \left|(u - u'') + (v - v'') < p^2\right.\right\},$$

denote by $\Omega_{h/2}(x', x'')$ the projection of the set $S_{h/2}(x') \cup S_{h/2}(x'')$ on the tangential plane $\Gamma(x')$.

By the formula of reduction of a surface integral to repeated one, we get

$$\begin{aligned}
F_8(x', x'') &= \int\limits_{\Omega_d(x')\backslash(\Omega_{h/2}(x', x'')} \frac{\left(\rho\left(x''\right) - \rho\left(x'\right)\right)u}{4\pi\left(\sqrt{u^2 + v^2 + f^2(u, v)}\right)^3} \\
&\quad \times\sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}dudv \\
&= \int\limits_{\Omega_d(x')\backslash(\Omega_{h/2}(x', x'')} \frac{\left(\rho\left(x''\right) - \rho\left(x'\right)\right)u}{4\pi\left(\sqrt{u^2 + v^2 + f^2(u, v)}\right)^3} \\
&\quad \times\left(\sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} - 1\right)dudv \\
&+ \int\limits_{\Omega_d(x')\backslash(\Omega_{h/2}(x', x'')} \left(\rho\left(x''\right) - \rho\left(x'\right)\right)\left(\frac{1}{\left(\sqrt{u^2 + v^2 + f^2(u, v)}\right)^3}\right. \\
&\quad \left. - \frac{1}{\left(\sqrt{u^2 + v^2)}\right)^3}\right)ududv \\
&+ \int\limits_{\Omega_d(x')\backslash(\Omega_{h/2}(x', x'')} \frac{\left(\rho\left(x''\right) - \rho\left(x'\right)\right)u}{4\pi\left(\sqrt{u^2 + v^2}\right)^3}dudv \\
&= F_8^{(1)}(x', x'') + F_8^{(2)}(x', x'') + F_8^{(3)}(x', x'').
\end{aligned}$$

Taking into account the inequality

$$|f(u,v)| \leq M\left(\sqrt{u^2+v^2}\right)^{1+\alpha}, \ \forall\,(u,v)\in\Omega_d(x'). \tag{2.7}$$

it is easy to show

$$\left|\sqrt{1+f_u^2+f_v^2}-1\right| \leq M\left(\sqrt{u^2+v^2}\right)^{2\alpha}, \ \forall\,(u,v)\in\Omega_d(x'), \tag{2.8}$$

$$\left|\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2)}\right)^3}\right|$$
$$\leq M\frac{1}{\left(\sqrt{u^2+v^2)}\right)^{3-2\alpha}}, \ \ \forall\,(u,v)\in\Omega_d(x')\backslash O_p(x'),$$

where $0<p<d$. Then

$$\left|F_8^{(1)}(x',x'')\right| \leq M\omega(\rho,h), \ \ \left|F_8^{(2)}(x',x'')\right| \leq M\omega(\rho,h).$$

As

$$\int\limits_{O_{d_0}(x')\backslash O_{2h}(x')} \frac{\left(\rho\left(x''\right)-\rho\left(x'\right)\right)u}{\left(\sqrt{u^2+v^2}\right)^3}dudv$$
$$=\left(\rho\left(x''\right)-\rho\left(x'\right)\right)\int\limits_0^{2\pi}\int\limits_{2h}^{d_0}\frac{\cos\varphi}{r}drd\varphi=0,$$

we have

$$F_8^{(3)}(x',x'')=\int\limits_{\Omega_{d_0}(x')\backslash O_{d_0}(x')}\frac{\left(\rho\left(x''\right)-\rho\left(x'\right)\right)u}{\left(\sqrt{u^2+v^2}\right)^3}dudv$$
$$+\int\limits_{O_{2h}(x')\backslash \Omega_{h/2}(x',x'')}\frac{\left(\rho\left(x''\right)-\rho\left(x'\right)\right)u}{\left(\sqrt{u^2+v^2}\right)^3}dudv.$$

Hence we get

$$\left|F_8^{(3)}(x',x'')\right| \leq M\left(\omega(\rho,h)+\omega(\rho,h)\int\limits_{h/C_1}^{2h}\frac{1}{dt}\right) \leq M\omega(\rho,h),$$

and so $\left|F_8(x',x'')\right| \leq M\omega(\rho,h)$. As a result, summing up the obtained estimations for the expressions $F_j(x',x''), j=\overline{1,8}$ we find:

$$F(x',x'') \leq M\left(\|\rho\|_\infty h+\omega(\rho,h)+\int\limits_0^d\frac{\omega(\rho,t)}{t}dt+h\int\limits_h^{diamS}\frac{\omega(\rho,t)}{t^2}dt\right).$$

Similarly, taking into account inequalities (2.2),(2.4),(2.5) and

$$|\exp\left(ik\,|x-y|\right)-1| \leq M\,|x-y|\,, \forall x,y\in S,$$

it is easy to prove that $\left|L(x',x'')\right| \leq M\,\|\rho\|_\infty\,h\,|\ln h|\,.$

Now estimate the expression $G\left(x',x''\right)$. Obviously,

$$G\left(x',x''\right)=\frac{\rho\left(x''\right)-\rho\left(x'\right)}{4\pi}\int\limits_S\frac{y_1-x_1'}{|x'-y|}dS_y+\frac{\rho\left(x''\right)}{4\pi}$$

$$\times\left(\int\limits_{S\backslash S_d(x')}\frac{y_1-x_1'}{|x'-y|^3}dS_y-\int\limits_{S\backslash S_d(x')}\frac{y_1-x_1''}{|x''-y|^3}dS_y\right)$$

$$+\frac{\rho\left(x''\right)}{4\pi}\left(\int\limits_{S_d(x')}\frac{y_1-x_1'}{|x'-y|^3}dS_y-\int\limits_{S_d(x')}\frac{y_1-x_1''}{|x''-y|^3}dS_y\right)$$

$$=G_1(x',x'')+G_2(x',x'')+G_3(x',x'').$$

As the integral $\int\limits_S\frac{y_1-x_1'}{|x'-y|^3}dS_y$ converges in the sense of the Cauchy principal value, we have $\left|G_1(x',x'')\right|\le M\omega\left(\rho,h\right).$

Obviously, $\left|G_2(x',x'')\right|\le M\left\|\rho\right\|_\infty h.$

Estimate the expression $G_3(x',x'')$. On the segment $S_d(x')$ chose a local rectangular system of coordinates (u,v,w) with the origin at the point x', where the axis w is directed along the normal $\overrightarrow{n}\left(x'\right)$ and the axis u and v lie on the tangential plane $\Gamma(x')$. Passing to double integral, it is easy to show that the expression $G_3(x',x'')$ may be represented in the form

$$G_3(x',x'')=M\rho(x'')\left[\int\limits_{\Omega_{d_0}(x')\backslash O_{d_0}(x')}\left(\frac{u}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}\right.\right.$$

$$\left.-\frac{u-u''}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f\left(u'',v''\right))^2}\right)^3}\right)$$

$$\times\sqrt{1+f_u^2+f_v^2}dudv$$

$$+\int\limits_{O_{d_0}(x')}\left(\frac{u}{\left(\sqrt{u^2+v^2}\right)^3}-\frac{u-u''}{\left(\sqrt{(u-u'')^2+(v-v'')^2}\right)^3}\right)dudv$$

$$+\int\limits_{O_{d_0}(x')}\left(\frac{u}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}\right.$$

$$\left.-\frac{u-u''}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f\left(u'',v''\right))^2}\right)^3}\right)$$

$$\times\left(\sqrt{1+f_u^2+f_v^2}-1\right)dudv$$

$$\int\limits_{O_{d_0}(x')}\left\{u\left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2}\right)^3}\right)\right.$$

$$-\left(u-u''\right)\left(\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f\left(u'',v''\right))^2}\right)^3}\right.$$

$$- \left.\left.\left. \frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2}\right)^3} \right)\right\} dudv \right]$$

$$= M\rho(x'') \left\lfloor G_{3,1}(x',x'') + G_{3,2}(x',x'') + G_{3,3}(x',x'') + G_{3,4}(x',x'') \right\rfloor .$$

Obviously, the subintegrand expression of the integral $G_{3,1}(x',x'')$ is bounded in the set $\Omega_{d_0}(x') \backslash O_{d_0}(x')$. Therefore, $\left|G_{3,1}(x',x'')\right| \leq Mh_0$. It is known that in the sense of the Cauchy principal value

$$\int\limits_{O_{d_0}(x')} \frac{u}{\left(\sqrt{(u^2+v^2)}\right)^3} dudv = 0,$$

and also

$$\int\limits_{O_{d_0-h_0}(x'')} \frac{u}{\left(\sqrt{(u+u'')+(v-v'')^2}\right)^3} dudv = 0.$$

Then for the integral $G_{3,2}(x',x'')$ we have :

$$\left|G_{3,2}(x',x'')\right| = \left| \int\limits_{O_{d_0}(x') \backslash O_{d_0-h_0}(x'')} \frac{u-u''}{\left(\sqrt{(u-u'')+(v-v'')^2}\right)^3} dudv \right|$$

$$\leq M \; mes \; \left(O_{d_0}(x') \backslash O_{d_0-h_0}(x'')\right) \leq Mh_0.$$

Represent the integral $G_{3,3}(x',x'')$ in the form

$$G_{3,3}(x',x'') = \int\limits_{O_{d_0}(x') \backslash O_{h_0/2}(x') \cup O_{h_0/2}(x''))} \frac{u''}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}$$

$$\times \left(\sqrt{1+f_u^2+f_v^2}-1\right) dudv + \int\limits_{O_{d_0}(x') \backslash O_{h_0/2}(x') \cup O_{h_0/2}(x''))} \left(u-u''\right)$$

$$\times \frac{\left(\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f(u'',v''))^2}\right)^3 - \left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3\right)}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3 \left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f(u'',v''))^2}\right)^3}$$

$$\times \left(\sqrt{1+f_u^2+f_v^2}-1\right) dudv$$

$$+ \int\limits_{O_{h_0/2}(x')} \frac{u\left(\sqrt{1+f_u^2+f_v^2}-1\right)}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3} dudv + \int\limits_{O_{h_0/2}(x'')} \frac{u\left(\sqrt{1+f_u^2+f_v^2}-1\right)}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3} dudv$$

$$- \int\limits_{O_{h_0/2}(x')} \frac{u-u''}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f(u'',v''))^2}\right)^3}$$

$$\times \left(\sqrt{1+f_u^2+f_v^2}-1\right) dudv$$

$$- \int\limits_{O_{h_0/2}(x'')} \frac{u-u''}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f(u'',v''))^2}\right)^3}$$

$$\times\left(\sqrt{1+f_u^2+f_v^2}-1\right)dudv$$

$$=G_{3,3}^{(1)}(x',x'')+G_{3,3}^{(2)}(x',x'')+G_{3,3}^{(3)}(x',x'')+G_{3,3}^{(4)}(x',x'')+G_{3,3}^{(5)}(x',x'')+G_{3,3}^{(6)}(x',x'').$$

Taking into account inequality (2.8), we get

$$\left|G_{3,3}^{(1)}(x',x'')\right|\le Mh_0\int\limits_{O_{d_0}(x')\backslash O_{h_0\backslash 2}(x')}\frac{dudv}{\left(\sqrt{u^2+v^2}\right)^{3-2\alpha}}\le Mh_0.$$

It is clear that for any $(u,v)\in O_{d_0}(x')\backslash\left(O_{h_0/2}(x')\cup O_{h_0/2}(x'')\right)$

$$\sqrt{(u-u'')^2+(v-v'')^2}\le\sqrt{u^2+v^2}+\sqrt{(u'')^2+(v'')^2}$$

$$\le\sqrt{u^2+v^2}+h_0^2\le 3\sqrt{u^2+v^2},$$

and

$$\sqrt{u^2+v^2}\le 3\sqrt{(u-u'')^2+(v-v'')^2}.$$

Then applying elementary transformations and taking into account inequalities (2.7) and (2.8), we can show that

$$\Bigg|\left(u-u''\right)$$

$$\times\frac{\left(\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f(u'',v''))^2}\right)^3-\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3\right)}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3\left(\sqrt{(u-u'')^2+(v-v'')^2+(f(u,v)-f(u'',v''))^2}\right)^3}$$

$$\times\left(\sqrt{1+f_u^2+f_v^2}-1\right)\Bigg|\le\frac{h_0}{\left(\sqrt{u^2+v^2}\right)^{3-2\alpha}},$$

$$\forall\,(u,v)\in O_{d_0}(x')\backslash\left(O_{h_0/2}(x')\cup O_{h_0/2}(x'')\right).$$

And so, $\left|G_{3,3}^{(2)}(x',x'')\right|\le Mh_0^{\alpha}$.

Taking into account (2.8), we find:

$$\left|G_{3,3}^{(3)}(x',x'')\right|\le M\int\limits_0^{h_0/2}\frac{dr}{r^{1-2\alpha}}\le Mh_0^{2\alpha}.$$

Obviously, $h_0\backslash 2<\sqrt{u^2+v^2}<3h_0\backslash 2,\ \forall\,(u,v)\in O_{h_0\backslash 2}(x'')$. Then

$$\left|G_{3,3}^{(4)}(x',x'')\right|\le M\int\limits_{O_{h_0/2}(x'')}\frac{dudv}{\left(\sqrt{u^2+v^2}\right)^{2-2\alpha}}\le Mh_0^{2\alpha-2}mesO_{h_0\backslash 2}(x'')\le Mh_0^{2\alpha}.$$

Now, using inequality (2.8), we have: $\left|G_{3,3}^{(5)}(x',x'')\right|\le Mh_0^{2\alpha-2}mesO_{h_0\backslash 2}(x')\le Mh_0^{2\alpha}$.

It is clear that these exists a point $(u_*,v_*)=\left(u''+a\left(u-u''\right),v''+b(v-v'')\right)$, such that

$$f\left(u,v\right)-f\left(u'',v''\right)=f_u\left(u_*,v_*\right)\left(u-u''\right)+f_v\left(u_*,v_*\right)\left(v-v^*\right),$$

here $a,b\in(0,1)$.Then representing the integral $G_{3,3}^{(6)}(x',x'')$ in the form

$$G_{3,3}^{(6)}(x',x'')=-\int\limits_{O_{h_0/2}(x'')}\left(u-u''\right)$$

$$\times\frac{\left(\sqrt{1+f_u^2+f_v^2}-\sqrt{1+f_u^2\left(u'',v''\right)+f_u^2\left(u'',v''\right)}\right)}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f\left(u,v\right)-f\left(u'',v''\right)\right)^2}\right)^3}dudv-\int\limits_{O_{h_0/2}(x'')}(u-u'')$$

$$\times\frac{\left(\sqrt{1+f_u^2\left(u'',v''\right)+f_v^2\left(u'',v''\right)}-1\right)}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f_u\left(u'',v''\right)(u-u'')+f\left(u'',v''\right)(v-v'')\right)^2}\right)^3}dudv$$

$$-\int\limits_{O_{h_0/2}(x'')}(u-u'')\left(\sqrt{1+f_u^2\left(u'',v''\right)+f_v^2\left(u'',v''\right)}-1\right)$$

$$\times\left(\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f_u\left(u_*,v_*\right)(u-u'')+f_v\left(u_*,v_*\right)(v-v'')\right)^2}\right)^3}\right.$$

$$\left.-\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f_u\left(u'',v''\right)(u-u'')+f_v\left(u'',v''\right)(v-v'')\right)^2}\right)^3}\right)dudv,$$

it is easy to prove that $\left|G_{3,3}^{(6)}(x',x'')\right|\leq Mh_0^{\alpha}$.

As a result we find: $\left|G_{3,3}(x',x'')\right|\leq Mh_0^{\alpha}$.

As

$$\int\limits_{O_{d_0-h_0}(x'')}\frac{u-u''}{\left(\sqrt{(u-u'')^2+(v-v'')^2}\right)^3}dudv=0,$$

$$\int\limits_{O_{d_0-h_0}(x'')}(u-u'')$$

$$\times\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f_u\left(u'',v''\right)(u-u'')+f_v\left(u'',v''\right)(v-v'')\right)^2}\right)^3}$$

$$\times dudv=0.$$

Then the integral $G_{3,4}(x',x'')$ may be represented in the form:

$$G_{3,4}(x',x'')=\int\limits_{O_{d_0}(x')\backslash O_{d_0-h_0}(x'')}\left(u\left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2}\right)^2}\right)-(u-u'')\right.$$

$$\left.\times\left(\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f_u\left(u,v\right)-f\left(u'',v''\right)\right)^2}\right)^3}-\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2}\right)^3}\right)\right)dudv$$

$$+\int\limits_{O_{d_0-h_0}(x'')\backslash O_{h_0/2}(x')\cup O_{h_0/2}(x''))}u''\left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2}\right)^3}\right)dudv$$

$$+\int\limits_{O_{d_0-h_0}(x'')\backslash O_{h_0/2}(x')\cup O_{h_0/2}(x''))}(u-u'')\left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2}\right)^3}\right.$$

$$+\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+\left(f_u\left(u'',v''\right)(u-u'')+f_v\left(u'',v''\right)(v-v'')\right)^2}\right)^3}$$

$$- \frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f\,(u,v)-f\,(u'',v''))^2}\right)^3}\Bigg) dudv$$

$$+ \int\limits_{O_{h_0/2}(x')} u \left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3} - \frac{1}{\left(\sqrt{u^2+v^2}\right)^3} \right) dudv$$

$$+ \int\limits_{O_{h_0/2}(x')} u \left(\frac{1}{\left(\sqrt{u^2+v^2+f^2(u,v)}\right)^3} - \frac{1}{\left(\sqrt{u^2+v^2}\right)^3} \right) dudv$$

$$+ \int\limits_{O_{d_0-h_0}(x'')} \left(u-u''\right)$$

$$\times \Bigg(\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f_u\,(u'',v'')\,(u-u'')+f_v\,(u'',v'')\,(v-v''))^2}\right)^3}$$

$$- \frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f\,(u,v)-f\,(u'',v''))^2}\right)^3}\Bigg) dudv$$

$$+ \int\limits_{O_{h_0/2}(x'')} \left(u-u''\right)$$

$$\times \Bigg(\frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f_u\,(u'',v'')\,(u-u'')+f_v\,(u'',v'')\,(v-v''))^2}\right)^3}$$

$$- \frac{1}{\left(\sqrt{(u-u'')^2+(v-v'')^2+(f\,(u,v)-f\,(u'',v''))^2}\right)^3}\Bigg) dudv.$$

Behaving as in the proof of the estimation for the expression $G_{3,3}(x',x'')$ we can show that $\left|G_{3,3}(x',x'')\right| \le M h_0^{\alpha}$.

Summing up the obtained estimations for the expressions$G_{3,j}(x',x''),\ j=\overline{1,4}$ and taking into account $h_0 \le h$ we have:$\left|G_3(x',x'')\right| \le M h^{\alpha}$. And so,$\left|G(x',x'')\right| \le M\left(\omega(\rho,h)+\|\rho\|_{\infty}\,h^{\alpha}\right)$.

As a result, taking into account the obtained estimations for the expressions $F\left(x',x''\right), L\left(x',x''\right)$ and $G\left(x',x''\right)$ we get that if $0<\alpha<1$, then

$$\left|V_{k,\rho}^{(1)}(x') - V_{k,\rho}^{(1)}(x'')\right|$$

$$\le M_{\rho}\left(\left(h^{\alpha}+\omega(\rho,h)\right)+\int\limits_0^h \frac{\omega(\rho,t)}{t}dt + h\int\limits_0^{diam\ S}\frac{\omega(\rho,t)}{t^2}dt\right).$$

Behaving in the same way, it is easy to show that

$$\left|V_{k,\rho}^{(m)}(x') - V_{k,\rho}^{(m}(x'')\right|$$

$$\le M_{\rho}\left(\left(h^{\alpha}+\omega(\rho,h)\right)+\int\limits_0^h \frac{\omega(\rho,t)}{t}dt + h\int\limits_0^{diam\ S}\frac{\omega(\rho,t)}{t^2}dt\right), m=2;3.$$

From the proof of the theorem it is clear that if $\alpha = 1$, then

$$\left| V_{k,\rho}^{(m)}(x') - V_{k,\rho}^{(m)}(x'') \right|$$

$$\leq M_\rho \left(h \left| \ln h \right| + \omega\left(\rho, h\right) + \int_0^h \frac{\omega(\rho,t)}{t} dt + h \int_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt \right), m = \overrightarrow{1,3}.$$

The lemma 2.1 is proved.

Theorem 2.1 *Let S be a Lyapunov surface with the exponent $0 < \alpha \leq 1$, and*

$$\int_0^{diam\ S} \frac{\omega(\rho,t)}{t} dt < +\infty.$$

Then the following estimation is valid

$$\omega\left(V_{k,\rho}, h\right) \leq M_\rho \left(h^\alpha + \omega\left(\rho, h\right) + \int_0^h \frac{\omega(\rho,t)}{t} dt + h \int_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt \right) \quad for\ 0 < \alpha < 1,$$

$$\omega\left(V_{k,\rho}, h\right) \leq M_\rho \left(h \left| \ln h \right| + \omega\left(\rho, h\right) + \int_0^h \frac{\omega(\rho,t)}{t} dt + h \int_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt \right) \ for\ \alpha = 1,$$

where M_ρ is a positive constant dependent only on S, k and ρ.

Proof. Consider the function

$$\psi(h) = \begin{cases} h^\alpha + \omega\left(\rho, h\right) + \int_0^h \frac{\omega(\rho,t)}{t} dt + h \int_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt, & if\ \ 0 < \alpha < 1, \\ h \left| \ln h \right| + \omega\left(\rho, h\right) + \int_0^h \frac{\omega(\rho,t)}{t} dt + h \int_0^{diam\ S} \frac{\omega(\rho,t)}{t^2} dt, & if\ \ \alpha = 1. \end{cases}$$

It is easy to show that $\lim_{h \to 0} \psi(h) = 0$, the function $\psi(h)$ doesn't decrease, the function $\psi(h)/h$ doesn't increase. Then applying the lemma 2.1, we get the proof of the theorem.

References

1. Kolton, D., Cress, R.: Method of integral equations in scattering theory. *M. Mir*, 311 p. (1987) (Russian).
2. Gunter, N.M.: Potential theory and its application to basic problems of mathematical physics. *M. Gostechizdat*, 415 p. (1957) (Russian).
3. Khalilov, E.H.: *Some properties of an operator generated by the derivative of a double layer acoustic potential.* Siberian Math. J., **55** (3), 690-700 (2014).
4. Khalilov, E.H.: *On direct value of the derivative of an acoustic single layer potential.* Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **34** (4), 143-148 (2014).
5. Vladimirov, V.S.: Equations of mathematical physics. *M., Nauka*, 527 p. (1976) (Russian).
6. Kustov, Yu.A., Musayev, B.I.: *Cubic formula for two-dimensions singular integral and its applications*. Dep.in VINITI, No 281-81-60 p. (Russian)