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(L_p, L_q) boundedness of the fractional integral operator on the dual of Laguerre hypergroup

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Abstract. *Let $\mathbb{K} = [0,\infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper, we are interested in the dual of the Laguerre hypergroup $\hat{\mathbb{K}}$ which can be topologically identified with the so-called Heisenberg fan, the subset of $\mathbb{R}^2$:*

$$\Big(\bigcup_{m\in\mathbb{N}} \{(\lambda,\mu) \in \mathbb{R}^2 : \mu = |\lambda|(2m+\alpha+1), \lambda \neq 0\}\Big) \bigcup \{(0,\mu) \in \mathbb{R}^2 : \mu \geq 0\}.$$

We obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional integral operator on the dual of Laguerre hypergroup $\hat{\mathbb{K}}$ from the spaces $L_p(\hat{\mathbb{K}})$ to the spaces $L_q(\hat{\mathbb{K}})$ for $1 < p < q < \infty$ and from the spaces $L_p(\hat{\mathbb{K}})$ to the weak spaces $WL_q(\hat{\mathbb{K}})$ for $1 \leq p < q < \infty$.

Keywords. Dual of Laguerre hypergroup · generalized translation operator · Fourier-Laguerre transform · fractional integral.

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1 Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. The maximal function was firstly introduced by Hardy and Littlewood in 1930 (see [16]) for functions defined on the circle. It was extended to the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted measure spaces (see [8], [9], [20], [23], [25]). In the setting of hypergroups versions of Hardy–Littlewood maximal functions were given in [6] for the Jacobi hypergroups of compact type, in [7] for the Jacobi-type hypergroups, in [4] for the one-dimensional Chebli-Trimeche hypergroups, in [21] for the

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one-dimensional Bessel-Kingman hypergroups, in [10] (see also [11–13]) for the n-dimensional Bessel-Kingman hypergroups ($n \geq 1$), and in [15] for the dual of Laguerre hypergroups.

In the present work, we study fractional integral on the dual of Laguerre hypergroup [5,17], so we fix $\alpha \geq 0$ and $\hat{\mathbb{K}} \cong \mathbb{R} \times \mathbb{N}$ and we define fractional integral using the harmonic analysis on the Laguerre hypergroup and its dual which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2,19,22]).

The functional analysis and Fourier analysis on $\mathbb{K}$ and its dual have been extensively studied in [3] and [19], and hence, it is well known that the Fourier-Laguerre transform defined on $\mathbb{K}$ is a topological isomorphism from the Schwartz space $\mathcal{S}(\mathbb{K})$ onto $\mathcal{S}(\hat{\mathbb{K}})$: the Schwartz space on $\hat{\mathbb{K}}$ (see [[19], Proposition II.1]). Its inverse is given by

$$g^{\vee}(\xi) = \int_{\hat{\mathbb{K}}} \varphi(\xi) g d\gamma_{\alpha}, \tag{1.1}$$

where $d\gamma$ is the Plancherel measure on $\hat{\mathbb{K}}$ given by $d\gamma(\lambda, m) = |\lambda|^{\alpha+1} d\lambda \otimes L_m^{\alpha}(0)\delta_m$. The topology on $\mathbb{K}$ is given by the norm $N(x,t) = (x^4 + 4t^2)^{1/4}$, while we assign to $\hat{\mathbb{K}}$ the topology generated by the quasi-semi-norm $\mathcal{N}(\lambda, m) = |\lambda|(m + \frac{\alpha+1}{2})$.

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. In the present work, we study the fractional maximal function and fractional integral on the dual of Laguerre hypergroup. We define the fractional maximal function and the fractional integral using harmonic analysis on dual of Laguerre hypergroups which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2,18,19, 22]). We obtain the necessary and sufficient conditions for the boundedness of the fractional maximal operator and the fractional integral operator on the dual of Laguerre hypergroup from the spaces $L_p(\hat{\mathbb{K}})$ to the spaces $L_q(\hat{\mathbb{K}})$ and from the spaces $L_1(\hat{\mathbb{K}})$ to the weak spaces $WL_q(\hat{\mathbb{K}})$.

The paper organized as follows. In Section 2, we give the our main result on the boundness of the fractional integral on the dual of Laguerre hypergroup. In Section 3, we present some definitions and auxiliary results. In section 4, we give polar coordinates in dual of Laguerre hypergroup and some lemmas needed to facilitate the proofs of our theorems. The main result of the paper is the inequality of Hardy-Littlewood-Sobolev type for the fractional integral, established in Section 5. We prove the boundedness of the fractional maximal operator and fractional integral operator from the spaces $L_p(\hat{\mathbb{K}})$ to $L_q(\hat{\mathbb{K}})$ and from the spaces $L_1(\hat{\mathbb{K}})$ to the weak Lebesgue spaces $WL_q(\hat{\mathbb{K}})$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

2 Main result

Let $\alpha \geq 0$ be a fixed number, $\hat{\mathbb{K}} = \mathbb{R} \times \mathbb{N}$ and where $d\gamma$ is the Plancherel measure on $\hat{\mathbb{K}}$ given by $d\gamma(\lambda, m) = |\lambda|^{\alpha+1} d\lambda \otimes L_m^{\alpha}(0)\delta_m$.

For every $1 \leq p \leq \infty$, we denote by $L_p(\hat{\mathbb{K}}) = L_p(\hat{\mathbb{K}}; d\gamma_\alpha)$ the spaces of complex-valued functions f, measurable on $\hat{\mathbb{K}}$ such that

$$\|f\|_{L_p(\hat{\mathbb{K}})} = \left(\int_{\hat{\mathbb{K}}} |f(\lambda, m)|^p \, d\gamma_\alpha(\lambda, m) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1, \infty),$$

and

$$\|f\|_{L_\infty(\hat{\mathbb{K}})} = \underset{(\lambda,m)\in\hat{\mathbb{K}}}{ess\sup} |f(\lambda, m)| \quad \text{if} \quad p = \infty.$$

We recall here that $d\gamma_\alpha$ is the positive measure defined on $\hat{\mathbb{K}}$ by

$$\int_{\hat{\mathbb{K}}} f(\lambda, m) d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} f(\lambda, m) |\lambda|^{\alpha+1} d\lambda.$$

For $1 \leq p < \infty$ we denote by $WL_p(\hat{\mathbb{K}})$, the weak $L_p(\hat{\mathbb{K}})$ spaces defined as the set of locally integrable functions $f(\lambda, m)$, $(\lambda, m) \in \hat{\mathbb{K}}$ with the finite norm

$$\|f\|_{WL_p(\hat{\mathbb{K}})} = \sup_{r>0} r \left(\gamma_\alpha\{(\lambda, m) \in \hat{\mathbb{K}} : |f(\lambda, m)| > r\} \right)^{1/p}.$$

For $m \in \mathbb{N}$ we denote by $\hat{\mathbb{K}}_m = \{(0,0)\} \cup \{R \setminus \{0\} \times \{0,1,2,\ldots,2m\}\}$ and $\mathbf{1}_{\hat{\mathbb{K}}_m}$ the characteristic function of $\hat{\mathbb{K}}_m$. In this section we introduce the ball in $\hat{\mathbb{K}}$ with center (λ, m) and radius $r > 0$ (for shortness B_r) to be the set

$$B_r(\lambda, m) = \{(\mu, \eta) \in \hat{\mathbb{K}}_m, \, \mathcal{N}(\lambda - \mu, \max(n-m), 0) < r\}.$$

We denote by

$$f_r(\lambda, m) = r^{-(\alpha+2)} f\left(\delta_{\frac{1}{r}}(\lambda, m)\right)$$

the dilated of the function f defined on $\hat{\mathbb{K}}$ preserving the mean of f with respect to the measure dm_α, in the sense that

$$\int_{\hat{\mathbb{K}}} f_r(\lambda, m) d\gamma_\alpha(\lambda, m) = \int_{\hat{\mathbb{K}}} f(\lambda, m) d\gamma_\alpha(\lambda, m), \;\; \forall r > 0 \text{ and } \; f \in L_1(\hat{\mathbb{K}}).$$

The generalized translation operators $T^{(\alpha)}_{(\lambda,m)}$ on $\hat{\mathbb{K}}$ are given for a suitable function f by

$$[T^{(\alpha)}_{(\lambda,m)} f](\mu, \eta) = \sum_{j \in \mathbb{N}_{m,n}} f(\lambda + \mu, j) C^\alpha_j((\lambda, m)(\mu, \eta)),$$

where

$$C^\alpha_j((\lambda, m)(\mu, \eta)) = \frac{L^\alpha_j(0)}{\Gamma(\alpha+1)} \int_0^\infty \mathcal{L}^\alpha_m\Big(\Big|\frac{\lambda}{\lambda+\mu}\Big| x\Big) \mathcal{L}^\alpha_n\Big(\Big|\frac{\mu}{\lambda+\mu}\Big| x\Big) \mathcal{L}^\alpha_j(x) x^\alpha dx,$$

and

$$\mathbb{N}_{m,n} = \begin{cases} \{0, 1 \ldots, m+n\}, & \text{if} \quad \lambda\mu > 0, \\ \mathbb{N}, \quad \text{if} \quad \lambda\mu \leq 0 \end{cases}$$

with the assumption $C^\alpha_j((\lambda, m)(\mu, \eta)) = 0$ if $j \geq m + n + 1$ and $\lambda\mu > 0$.

The generalized translation operator above satisfies the following contraction property,

$$\|T^{(\alpha)}_{(\lambda,m)} f\|_{p,\gamma_\alpha} \leq \|f\|_{p,\gamma_\alpha} \quad \forall f \in L_p(\hat{\mathbb{K}}).$$

The generalized convolution product on $\hat{\mathbb{K}}$ is defined for a suitable pair of functions f and g by

$$f \sharp g(\lambda, m) = \int_{\hat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)} f(\mu, \eta) g(-\mu, \eta) d\gamma_\alpha(\mu, \eta),$$

and satisfies for f in $L_p(\hat{\mathbb{K}})$ and g in $L_q(\hat{\mathbb{K}})$, $1 \leq p, q \leq \infty$, $f\sharp g$ belongs to $L_r(\hat{\mathbb{K}})$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and

$$\|f \sharp g\|_{L_r(\hat{\mathbb{K}})} \leq \|f\|_{L_p(\hat{\mathbb{K}})} \|g\|_{L_q(\hat{\mathbb{K}})}.$$

For the maximal operator defined on $\hat{\mathbb{K}}$ by

$$Mf(\lambda, m) = \sup_{r>0} \frac{1}{\gamma_\alpha(B_r)} \int_{B_r} T^{(\alpha)}_{(\lambda,m)} |f(\mu, \eta)| d\gamma_\alpha(\mu, \eta)$$

and the fractional integral by

$$I_\beta f(\lambda, m) = \int_{\hat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)} |(\mu, \eta)|^{\beta-\alpha-2}_{\hat{\mathbb{K}}} f(\mu, \eta) d\gamma_\alpha(\mu, \eta), \quad 0 < \beta < \alpha + 2.$$

We have the following result

Theorem 2.1 *(see [1]) 1. If $f \in L_1(\hat{\mathbb{K}})$, then for every $\beta > 0$*

$$\gamma_\alpha\{(\lambda, m) \in \hat{\mathbb{K}} : Mf(\lambda, m) > \beta\} \le \frac{C}{\beta}\int_{\hat{\mathbb{K}}} |f(\lambda, m)| d\gamma_\alpha(\lambda, m),$$

where $C > 0$ is independent of f.

2. If $f \in L_p(\hat{\mathbb{K}})\lambda$, $1 < p \le \infty$, then $Mf \in L_p(\hat{\mathbb{K}})$ and

$$\|Mf\|_{L_p(\hat{\mathbb{K}})} = C_p \|f\|_{L_p(\hat{\mathbb{K}})},$$

where $C_p > 0$ is independent of f.

Corollary 2.1 *If $f \in L_{loc}(\hat{\mathbb{K}})$, then*

$$\lim_{r\to 0} \frac{1}{\gamma_\alpha B_r} \int_{B_r} \left| T^{(\alpha)}_{(\lambda,m)} f(y, s) - f(\lambda, m)\right| dm_\alpha(y, s) = 0$$

for a. e. $(\lambda, m) \in \hat{\mathbb{K}}$.

As an application, we give a result about approximations of the identity. The maximal function can be used to study almost everywhere convergence of $f * \varphi_\varepsilon$ as they can be controlled by the Hardy-Littlewood maximal function Mf under some conditions on φ.

Theorem 2.2 *[15] Let ψ a nonnegative and decreasing function on $[0, \infty)$, $|\varphi(\lambda, m)| \le \psi(|(\lambda, m)|_{\hat{\mathbb{K}}})$ and $\psi(|(\lambda, m)|_{\hat{\mathbb{K}}}) \in L_1(\hat{\mathbb{K}})$. Then there exists a constant $C > 0$ such that*

$$M_\varphi f(\lambda, m) \equiv \sup_{r>0} |(f * \varphi_r)(\lambda, m)| \le CMf(\lambda, m).$$

Corollary 2.2 *Let $\varphi \in L_1(\hat{\mathbb{K}})$ and assume $\int_{\hat{\mathbb{K}}} \varphi(\lambda, m)\, dm_\alpha(\lambda, m) = 1$. Then for $f \in L_p(\hat{\mathbb{K}})$, $1 \le p < \infty$*

$$\lim_{r\to 0} \|f * \varphi_r - f\|_{L_p(\hat{\mathbb{K}})} = 0.$$

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the fractional integral operator I_β to be bounded from the spaces $L_p(\hat{\mathbb{K}})$ to $L_q(\hat{\mathbb{K}})$, $1 < p < q < \infty$ and from the spaces $L_1(\hat{\mathbb{K}})$ to the weak spaces $WL_q(\hat{\mathbb{K}})$, $1 < q < \infty$.

Theorem 2.3 *Let $0 < \beta < \alpha + 2$ and $1 \le p < \frac{\alpha+2}{\beta}$.*

1) If $1 < p < \frac{\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\alpha+2}$ is necessary and sufficient for the boundedness of I_β from $L_p(\hat{\mathbb{K}})$ to $L_q(\hat{\mathbb{K}})$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{\alpha+2}$ is necessary and sufficient for the boundedness of I_β from $L_1(\hat{\mathbb{K}})$ to $WL_q(\hat{\mathbb{K}})$.

Recall that, for $0 < \beta < \alpha + 2$, the following inequality hold

$$M_\beta f(\lambda, m) \le \Omega_2^{\frac{\beta}{\alpha+2}-1} I_\beta(|f|)(\lambda, m),$$

where Ω_2 is the volume of the unit ball in $\hat{\mathbb{K}}$. Hence the boundedness of the fractional integral operator I_β implies the boundedness of the fractional maximal operator M_β.

Corollary 2.3 *Let $0 < \beta < \alpha + 2$ and $1 \le p < \frac{\alpha+2}{\beta}$.*

1) If $1 < p < \frac{\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\alpha+2}$ is necessary and sufficient for the boundedness of M_β from $L_p(\hat{\mathbb{K}})$ to $L_q(\hat{\mathbb{K}})$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{\alpha+2}$ is necessary and sufficient for the boundedness of M_β from $L_1(\hat{\mathbb{K}})$ to $WL_q(\hat{\mathbb{K}})$.

For $1 \le p, q \le \infty$ and $0 < s < 2$, the Besov space on the dual of Laguerre hypergroup $B^s_{p,q}(\hat{\mathbb{K}})$ consists of all functions f in $L_p(\hat{\mathbb{K}})$ so that

$$\|f\|_{B^s_{p,q}(\hat{\mathbb{K}})} = \|f\|_{L_p(\hat{\mathbb{K}})} + \left(\int_{\hat{\mathbb{K}}} \frac{\|T^{(\alpha)}_{(\lambda,m)} f(\cdot) - f(\cdot)\|^q_{L_p(\hat{\mathbb{K}})}}{|(\lambda, m)|^{\alpha+2+sq}_{\hat{\mathbb{K}}}} d\gamma_\alpha(\lambda, m) \right)^{1/q} < \infty. \tag{2.1}$$

Besov spaces in the setting of the Laguerre hypergroups studied by Assal and Ben Abdallah ([2]). In the following theorem we prove the boundedness of the maximal operator in Besov spaces on the dual of Laguerre hypergroups.

Theorem 2.4 *For $1 < p < \infty$, $1 \le q \le \infty$ and $0 < s < 2$ the Hardy-Littlewood maximal function operator is bounded on $B^s_{pq}(\hat{\mathbb{K}})$. More precisely, there is a constant $C > 0$ such that*

$$\|Mf\|_{B^s_{pq}(\hat{\mathbb{K}})} \le C \|f\|_{B^s_{pq}(\hat{\mathbb{K}})}$$

hold for all $f \in B^s_{pq}(\hat{\mathbb{K}})$.

3 Preliminaries

The harmonic analysis on the Laguerre hypergroup $\mathbb{K}$ (see [21]) is generated by the singular operator $\mathcal{L}_\alpha = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}$ and the norm $N(x,t) = (x^4 + t^2)^{1/4}$, $(x,t) \in \mathbb{K}$, while its dual $\hat{\mathbb{K}}$ is generated by the differential difference operator $\Lambda = \Lambda_1^2 - (2\Lambda_2 + 2\frac{\partial}{\partial \lambda})^2$, where $\Lambda_1 = \frac{1}{|\lambda|}\Big(m\triangle_+\triangle_- + (\alpha+1)\triangle_+\Big)$ and $\Lambda_2 = \frac{-1}{2\lambda}\Big((\alpha + j + 1)\triangle_+ + m\triangle_-\Big)$ and the quasinorm $\mathcal{N}(\lambda, m) = |\lambda|(m + \frac{\alpha+1}{2})$, $(\lambda, m) \in \hat{\mathbb{K}}$, where the difference operators $\triangle_\pm$ are given for a suitable function Φ by: $\triangle_+\Phi(\lambda, m) = \Phi(\lambda, m+1) - \Phi(\lambda, m)$, $\triangle_-\Phi(\lambda, m) = \Phi(\lambda, m) - \Phi(\lambda, m-1)$, if $m \ge 1$ and $\triangle_-\Phi(\lambda, 0) = \Phi(\lambda, 0)$.

These operators satisfy some basic properties which can be found in [2], [3] and [19], namely one has

$$\mathcal{L}_\alpha \varphi_{(\lambda,m)} = -\mathcal{N}(\lambda, m)\varphi(\lambda, m) \text{ and } \Lambda\varphi_{(\lambda,m)}(x,t) = N^4(x,t)\varphi_{(\lambda,m)}. \tag{3.1}$$

For $f \in L_1(\hat{\mathbb{K}})$ the Fourier-Laguerre transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f)(\lambda, m) = \int_{\hat{\mathbb{K}}} \varphi_{-\lambda,m}(\lambda, m) f(\lambda, m) d\gamma_\alpha(\lambda, m)$$

such that

$$\|\mathcal{F}(f)\|_{L_\infty(\hat{\mathbb{K}})} \le \|f\|_{L_1(\hat{\mathbb{K}})}.$$

The generalized translation operators $T^{(\alpha)}_{(\lambda,m)}$ on the dual of Laguerre hypergroup satisfies the following properties

$$\begin{gathered} T^{(\alpha)}_{(\lambda,m)} f(\mu, \eta) = T^{(\alpha)}_{(\mu,\eta)} f(\lambda, m), \quad T^{(\alpha)}_{(0,0)} f(\mu, \eta) = f(\mu, \eta), \\ \|T^{(\alpha)}_{(\lambda,m)} f\|_{L_p(\hat{\mathbb{K}})} \le \|f\|_{L_p(\hat{\mathbb{K}})} \quad \text{for all} \quad f \in L_p(\hat{\mathbb{K}}),\ 1 \le p \le \infty, \\ \mathcal{F}\big(T^{(\alpha)}_{(\lambda,m)} f\big)(\lambda, m) = \mathcal{F}(f)(\lambda, m)\, \varphi_{\lambda,m}(\lambda, m). \end{gathered} \tag{3.2}$$

The translation operator $T^{(\alpha)}_{(\lambda,m)}$ is defined by

$$T^{(\alpha)}_{(\lambda,m)} f(\mu, \eta) = \int_{\hat{\mathbb{K}}} f(z, v) W_\alpha((\lambda, m), (\mu, \eta), (z, v)) z^{2\alpha+1} dz dv,$$

where $dzdv$ is the Lebesgue measure on $\hat{\mathbb{K}}$, and W_α is an appropriate kernel satisfying

$$\int_{\hat{\mathbb{K}}} W_\alpha((\lambda, m), (\mu, \eta), (z, v)) z^{2\alpha+1} dz dv = 1$$

(see [18]). For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the function $\varphi_{\lambda,m}(\lambda, m)$ satisfies the following product formula

$$\varphi_{\lambda,m}(\lambda, m)\,\varphi_{\lambda,m}(\mu, \eta) = T^{(\alpha)}_{(\lambda,m)}\varphi_{\lambda,m}(\mu, \eta).$$

By using the generalized translation operators $T^{(\alpha)}_{(\lambda,m)}$, $(\lambda, m) \in \hat{\mathbb{K}}$, we define a generalized convolution product $*$ on $\hat{\mathbb{K}}$ by

$$\left(\delta_{(\lambda,m)} * \delta_{(\mu,\eta)}\right)(f) = T^{(\alpha)}_{(\lambda,m)} f(\mu, \eta),$$

where $\delta_{(\lambda,m)}$ is the Dirac measure at (λ, m).

We define the convolution product on the space $M_b(\hat{\mathbb{K}})$ of bounded Radon measures on $\hat{\mathbb{K}}$ by

$$(\mu * \nu)(f) = \int_{\hat{\mathbb{K}}\times\hat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)} f(\mu, \eta)\, d\mu(\lambda, m)\, d\nu(\mu, \eta).$$

If $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, then we have

$$\mu * \nu = (h * \check{g}) \cdot m_\alpha, \quad \text{with} \;\; \check{g}(\mu, \eta) = g(y, -s),$$

where, h and g belong to the space $L_1(\hat{\mathbb{K}})$ of the integrable functions on $\hat{\mathbb{K}}$ with respect to the measure $d\gamma_\alpha(\lambda, m)$, and $h * g$ is the convolution product defined by

$$(h * g)(\lambda, m) = \int_{\hat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)} h(\mu, \eta)\, g(y, -s)\, dm_\alpha(\mu, \eta), \quad \text{for all} \quad (\lambda, m) \in \hat{\mathbb{K}}.$$

Note that, for the convolution operators the Young inequality is valid: If $1 \le p,\ r \le q \le \infty$, $1/p' + 1/q = 1/r$, $f \in L_p(\hat{\mathbb{K}})$, and $g \in L_r(\hat{\mathbb{K}})$, then $f * g \in L_q(\hat{\mathbb{K}})$ and

$$\|f * g\|_{L_q(\hat{\mathbb{K}})} \le \|f\|_{L_p(\hat{\mathbb{K}})}\, \|g\|_{L_r(\hat{\mathbb{K}})}, \tag{3.3}$$

where $p' = p/(p-1)$.

4 Polar coordinates in dual of Laguerre hypergroup and some lemmas

Let $\Sigma = \Sigma_2$ be the unit sphere in $\hat{\mathbb{K}}$. We denote by ω_2 the surface area of Σ and by Ω_2 its volume (see [14,15]). For $\xi = (\lambda, m) \in \hat{\mathbb{K}}$, consider the transformation given by

$$x = r(\cos\varphi)^{1/2}, \; t = r^2 \sin\varphi,$$

where $-\pi/2 \le \varphi \le \pi/2$, $r = |\xi|_{\hat{\mathbb{K}}}$ and $\xi' = ((\cos\varphi)^{1/2}, \sin\varphi) \in \Sigma$.

The Jacobian of the above transformation is $r^{2\alpha+3}(\cos\varphi)^\alpha$. If f is integrable in $\hat{\mathbb{K}}$, then

$$\int_{\hat{\mathbb{K}}} f(\lambda, m)\, dm_\alpha(\lambda, m)$$
$$= \frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2}\int_0^\infty f\left(r(\cos\varphi)^{1/2}, r^2\sin\varphi\right) r^{2\alpha+3}(\cos\varphi)^\alpha dr d\varphi.$$

Since

$$\frac{1}{2\pi\Gamma(\alpha+1)} \int_{-\pi/2}^{\pi/2} (\cos\varphi)^\alpha d\varphi = \int_\Sigma d\xi',$$

we get

$$\int_{\hat{\mathbb{K}}} f(\lambda, m)\, d\gamma_\alpha(\lambda, m) = \int_\Sigma \int_0^\infty r^{\alpha+1} f(\delta_r \xi')\, dr d\xi'. \tag{4.1}$$

Here $d\xi'$ is the surface area element on Σ.

Lemma 4.1 *[14, 15] The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}, \ \ \Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

Note that for any $x \in \hat{\mathbb{K}}$ and $r > 0$, the area of the sphere $S_r(\lambda, m)$ is $r^{2\alpha+3}\omega_2$ and its volume is $r^{\alpha+2}\Omega_2 = r^{\alpha+2}\frac{\omega_2}{\alpha+2}$.

Lemma 4.2 *[14, 15] The function $f(\lambda, m) = |(\lambda, m)|_{\hat{\mathbb{K}}}^{\lambda}$ is integrable in any neighborhood of the origin if and only if $\lambda > -\alpha - 2$, and f is integrable in the complement of any neighborhood of the origin if and only if $\lambda < -\alpha - 2$.*

5 Hardy-Littlewood-Sobolev theorem for the fractional integral on the dual of Laguerre hypergroup

The examples considered below show that if $p \geq \frac{\alpha+2}{\beta}$, then I_β is not defined for all functions $f \in L_p(\hat{\mathbb{K}})$.

Example 1. Let $(\lambda, m) \in \hat{\mathbb{K}}, \quad 0 < \beta < \alpha + 2$, $f(\lambda, m) = \frac{1}{|(\lambda,m)|_{\hat{\mathbb{K}}}^{\beta} \ln |(\lambda,m)|_{\hat{\mathbb{K}}}}\chi_{\complement B_2}(\lambda, m)$, where $\complement B_r = \hat{\mathbb{K}} \setminus B_r$, $r > 0$. For $p = \frac{\alpha+2}{\beta}$, we have $f \in L_p(\hat{\mathbb{K}})$ and $I_\beta f(\lambda, m) = +\infty$.

Example 2. Let $(\lambda, m) \in \hat{\mathbb{K}}, \quad 0 < \beta < \alpha + 2$, $f(\lambda, m) = |(\lambda, m)|_{\hat{\mathbb{K}}}^{-\beta}\chi_{\complement B_2}(\lambda, m)$. For $p > \frac{\alpha+2}{\beta}$, we have $f \in L_p(\hat{\mathbb{K}})$ and $I_\beta f(\lambda, m) = +\infty$.

For the fractional integral on the Laguerre group the following analogue of Hardy-Littlewood-Sobolev theorem is valid.

Theorem 5.1 *Let $0 < \beta < \alpha + 2$ and $1 \leq p < \frac{\alpha+2}{\beta}$.*

1) If $1 < p < \frac{\alpha+2}{\beta}$, $f \in L_p(\hat{\mathbb{K}})$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\alpha+2}$, then $I_\beta f \in L_q(\hat{\mathbb{K}})$ and

$$\left\|I_\beta f\right\|_{L_q(\hat{\mathbb{K}})} \leq C_{pq}\|f\|_{L_p(\hat{\mathbb{K}})}, \tag{5.1}$$

where $C_{pq} = 2(C_3)^{1-p/q}(C_2C_p)^{p/q}$, $C_2 = \Omega_2 2^{\alpha+2}/(2^\beta - 1)$, $C_3 = \left(\Omega_2 q/p'\right)^{1/p'}$.

2) If $f \in L_1(\hat{\mathbb{K}})$ and $1 - \frac{1}{q} = \frac{\beta}{\alpha+2}$, then $I_\beta f \in WL_q(\hat{\mathbb{K}})$ and

$$\left\|I_\beta f\right\|_{WL_q(\hat{\mathbb{K}})} \leq C_{1q}\|f\|_{L_1(\hat{\mathbb{K}})}, \tag{5.2}$$

where $C_{1q} = 2(C_1C_2)^{1/q}$.

Proof. 1) Let $f \in L_p(\hat{\mathbb{K}})$, $1 < p < \frac{\alpha+2}{\beta}$. Then we write

$$I_\beta f(\lambda, m) = \left(\int_{B_r} + \int_{\complement B_r}\right) T_{(\lambda,m)}^{(\alpha)} f(\mu, \eta)\ |(\mu, \eta)|_{\hat{\mathbb{K}}}^{\beta-\alpha-2} d\gamma_\alpha(\mu, \eta) = A(\lambda, m) + B(\lambda, m).$$

By taking sum with respect to all integer $k > 0$, we get

$$\begin{aligned}|A(\lambda,m)| &\le \int_{B_r} T^{(\alpha)}_{(\lambda,m)} |f(\mu,\eta)|\, |(\mu,\eta)|^{\beta-\alpha-2}_{\hat{\mathbb{K}}} dm_\alpha(\mu,\eta) \\ &= \sum_{k=1}^{\infty} \int_{B_{2^{-k+1}r}\setminus B_{2^{-k}r}} T^{(\alpha)}_{(\lambda,m)} |f(\mu,\eta)|\, |(\mu,\eta)|^{\beta-\alpha-2}_{\hat{\mathbb{K}}} dm_\alpha(\mu,\eta) \\ &\le \sum_{k=1}^{\infty} \left(2^{-k} r\right)^{\beta-\alpha-2} \int_{B_{2^{-k+1}r}\setminus B_{2^{-k}r}} T^{(\alpha)}_{(\lambda,m)} |f(\mu,\eta)|\, dm_\alpha(\mu,\eta) \\ &\le \Omega_2 r^\beta (Mf)(\lambda,m) \sum_{k=1}^{\infty} \left(2^{-k}\right)^{\beta-\alpha-2} \left(2^{-k+1}\right)^{\alpha+2} \\ &= \Omega_2 2^{\alpha+2} r^\beta (Mf)(\lambda,m) \sum_{k=1}^{\infty} 2^{-k\beta} \le \frac{\Omega_2 2^{\alpha+2}}{2^\beta - 1} r^\beta (Mf)(\lambda,m).\end{aligned}$$

Therefore it follows that

$$|A(\lambda,m)| \le C_2 r^\beta Mf(\lambda,m). \tag{5.3}$$

By Hölder's inequality and the inequality (3.2) we have

$$\begin{aligned}|C(\lambda,m)| &\le \left(\int_{\complement_{B_r}} \left(T^{(\alpha)}_{(\lambda,m)} |f(\mu,\eta)|\right)^p d\gamma_\alpha(\mu,\eta)\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\complement_{B_r}} |(\mu,\eta)|^{(\beta-\alpha-2)p'}_{\hat{\mathbb{K}}} d\gamma_\alpha(\mu,\eta)\right)^{\frac{1}{p'}} \\ &\le \left\| T^{(\alpha)}_{(\lambda,m)} |f| \right\|_{L_p(\hat{\mathbb{K}})} \left(\int_{\complement_{B_r}} |(\mu,\eta)|^{(\beta-\alpha-2)p'}_{\hat{\mathbb{K}}} d\gamma_\alpha(\mu,\eta)\right)^{\frac{1}{p'}} \\ &\le \|f\|_{L_p(\hat{\mathbb{K}})} \left(\int_{\complement_{B_r}} |(\mu,\eta)|^{(\beta-\alpha-2)p'}_{\hat{\mathbb{K}}} d\gamma_\alpha(\mu,\eta)\right)^{\frac{1}{p'}} = C_3 r^{-(\alpha+2)/q} \|f\|_{L_p(\hat{\mathbb{K}})}.\end{aligned}$$

Consequently, we get

$$|B(\lambda,m)| \le C_3 r^{-(\alpha+2)/q} \|f\|_{L_p(\hat{\mathbb{K}})}. \tag{5.4}$$

Thus, from the inequalities (5.3) and (5.4), we have

$$\left| I_\beta f(\lambda,m) \right| \le C_2 r^\beta Mf(\lambda,m) + C_3 r^{-(\alpha+2)/q} \|f\|_{L_p(\hat{\mathbb{K}})}.$$

The minimum value of the right-hand side is attained at

$$r = \left[\left(C_2 Mf(\lambda,m)\right)^{-1} C_3 \|f\|_{L_p(\hat{\mathbb{K}})} \right]^{p/(\alpha+2)},$$

and hence

$$\left| I_\beta f(\lambda,m) \right| \le 2 \left(C_2 Mf(\lambda,m)\right)^{p/q} \left(C_3 \|f\|_{L_p(\hat{\mathbb{K}})} \right)^{1-p/q}.$$

By Theorem 2.1, we have

$$\begin{aligned}\int_{\hat{\mathbb{K}}} \left| I_\beta f(\lambda,m) \right|^q d\gamma_\alpha(\lambda,m) &\le 2^q \left(C_3 \|f\|_{L_p(\hat{\mathbb{K}})} \right)^{q-p} \int_{\hat{\mathbb{K}}} \left(C_2 Mf(\lambda,m)\right)^p d\gamma_\alpha(\lambda,m) \\ &\le 2^q (C_3)^{q-p} (C_2 C_p)^p \|f\|^q_{L_p(\hat{\mathbb{K}})}.\end{aligned}$$

Then we get

$$\left\| I_\beta f \right\|_{L_q(\hat{\mathbb{K}})} \le 2 (C_3)^{1-p/q} (C_2 C_p)^{p/q} \|f\|_{L_p(\hat{\mathbb{K}})}.$$

2) Let $f \in L_1(\hat{\mathbb{K}})$. We have

$$\gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} : |I_\beta f(\lambda, m)| > 2\tau\Big\}$$
$$\leq \gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} : |A(\lambda, m)| > \tau\Big\} + \gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} : |B(\lambda, m)| > \tau\Big\}.$$

Taking into account the inequality (5.3) and applying Theorem 1 we have

$$\tau\, \gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} : |A(\lambda, m)| > \tau\Big\} \leq \tau \int_{\big\{(\lambda,m)\in\hat{\mathbb{K}}:\ C_2 r^\beta Mf(\lambda,m)>\tau\big\}} d\gamma_\alpha(\lambda, m)$$
$$= \tau\, \gamma_\alpha \left\{(\lambda, m) \in \hat{\mathbb{K}} :\ Mf(\lambda, m) > \frac{\tau}{C_2 r^\beta}\right\} \leq C_1\, r^\beta \int_{\hat{\mathbb{K}}} |f(\lambda, m)| d\gamma_\alpha(\lambda, m)$$
$$= C_1 C_2\, r^\beta\, \|f\|_{L_1(\hat{\mathbb{K}})},$$

and

$$|C(x, t)| \leq \int_{\complement_{B_r}} T^{(\alpha)}_{(\lambda,m)}\, |f(\mu, \eta)|\, |(\mu, \eta)|^{\beta-\alpha-2}_{\hat{\mathbb{K}}} d\gamma_\alpha(\mu, \eta)$$
$$\leq r^{\beta-\alpha-2} \int_{\complement_{B_r}} T^{(\alpha)}_{(\lambda,m)}\, |f(\mu, \eta)|\, d\gamma_\alpha(\mu, \eta)$$
$$\leq r^{-\frac{\alpha+2}{q}} \int_{\hat{\mathbb{K}}} |f(\mu, \eta)| d\gamma_\alpha(\mu, \eta) = r^{-\frac{\alpha+2}{q}}\, \|f\|_{L_1(\hat{\mathbb{K}})}.$$

If $r^{-\frac{\alpha+2}{q}}\, \|f\|_{L_1(\hat{\mathbb{K}})} = \tau$, then $|C(\lambda, m)| \leq \tau$, and hence

$$\gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} :\ |C(\lambda, m)| > \tau\Big\} = 0.$$

Then we get

$$\gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} :\ |I_\beta f(\lambda, m)| > 2\tau\Big\}$$
$$\leq \gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} : |A(\lambda, m)| > \tau\Big\} + \gamma_\alpha \Big\{(\lambda, m) \in \hat{\mathbb{K}} : |B(\lambda, m)| > \tau\Big\}$$
$$\leq \frac{C_1 C_2}{\tau} r^\beta\, \|f\|_{L_1(\hat{\mathbb{K}})} = C_1 C_2 r^{\beta+\frac{\alpha+2}{q}}$$
$$= C_1 C_2 r^{\alpha+2} = C_1 C_2 \tau^{-q}\, \|f\|^q_{L_1(\hat{\mathbb{K}})} = \frac{C_1 C_2}{\tau^q} \|f\|^q_{L_1(\hat{\mathbb{K}})}$$

and hence

$$\big\|I_\beta f\big\|_{WL_q(\hat{\mathbb{K}})} \leq 2(C_1 C_2)^{1/q} \|f\|_{L_1(\hat{\mathbb{K}})}.$$

Therefore the proof of the theorem is completed.

Proof of Theorem 2.3. Sufficiency part of the proof follows from Theorem 5.1.
Necessity. 1) Let $1 < p < \frac{\alpha+2}{\beta}$, $f \in L_p(\hat{\mathbb{K}})$ and assume that the inequality

$$\big\|I_\beta f\big\|_{L_q(\hat{\mathbb{K}})} \leq C\, \|f\|_{L_p(\hat{\mathbb{K}})} \tag{5.5}$$

holds, where C depends only on p, q and α.

Define $f_r(\lambda, m) := f(rx, r^2 t)$, then

$$\|f_r\|_{L_p(\hat{\mathbb{K}})} = r^{-\frac{\alpha+2}{p}}\, \|f\|_{L_p(\hat{\mathbb{K}})}$$

and

$$\big\|I_\beta f_r\big\|_{L_q(\hat{\mathbb{K}})} = r^{-\beta-\frac{2\alpha+4}{q}}\, \big\|I_\beta f\big\|_{L_q(\hat{\mathbb{K}})}.$$

By the inequality (5.5)

$$\left\|I_\beta f\right\|_{L_q(\hat{\mathbb{K}})} \le C r^{\beta+\frac{\alpha+2}{q}-\frac{\alpha+2}{p}} \|f\|_{L_p(\hat{\mathbb{K}})}.$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_p(\hat{\mathbb{K}})$ we have $\left\|I_\beta f\right\|_{L_q(\hat{\mathbb{K}})} = 0$ as $r \to 0$, which is impossible. Similarly, if $\frac{1}{p} < \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_p(\hat{\mathbb{K}})$ we obtain $\left\|I_\beta f\right\|_{L_q(\hat{\mathbb{K}})} = 0$ as $r \to \infty$, which is also impossible.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{\alpha+2}$.

Necessity. Let I_β bounded from $L_1(\hat{\mathbb{K}})$ to $WL_q(\hat{\mathbb{K}})$. We have

$$\left\|I_\beta f_r\right\|_{WL_q(\hat{\mathbb{K}})} = r^{-\beta-\frac{\alpha+2}{q}} \left\|I_\beta f\right\|_{WL_q(\hat{\mathbb{K}})}.$$

By the boundedness I_β from $L_1(\hat{\mathbb{K}})$ to $WL_q(\hat{\mathbb{K}})$

$$\left\|I_\beta f\right\|_{WL_q(\hat{\mathbb{K}})} = r^{\beta+\frac{\alpha+2}{q}} \left\|I_\beta f_r\right\|_{WL_q(\hat{\mathbb{K}})}$$

$$\le C r^{\beta+\frac{\alpha+2}{q}} \|f_r\|_{L_1(\hat{\mathbb{K}})} = C r^{\beta+\frac{\alpha+2}{q}-(\alpha+2)} \|f\|_{L_1(\hat{\mathbb{K}})},$$

where C depends only on q and α.

If $1 < \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_1(\hat{\mathbb{K}})$ we have $\left\|I_\beta f\right\|_{WL_q(\hat{\mathbb{K}})} = 0$ as $r \to 0$. Similarly, if $1 > \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_1(\hat{\mathbb{K}})$ we obtain $\left\|I_\beta f\right\|_{WL_q(\hat{\mathbb{K}})} = 0$ as $r \to \infty$.

Hence we get $1 = \frac{1}{q} + \frac{\beta}{\alpha+2}$. Thus the proof of Theorem 2.3 is completed.

Proof of Corollary 2.3. Sufficiency part of the proof follows from Theorem 5.1 and the inequality

$$M_\beta f(\lambda, m) \le \Omega_2^{\frac{\beta}{\alpha+2}-1} I_\beta(|f|)(\lambda, m), \quad 0 < \beta < \alpha + 2.$$

Necessity. 1) Let M_β be bounded from $L_p(\hat{\mathbb{K}})$ to $L_q(\hat{\mathbb{K}})$ for $1 < p < \frac{\beta}{\alpha+2}$, $1 < p < q < \infty$. Then we have

$$M_\beta f_r(\lambda, m) = r^{-\beta} M_\beta f(r\lambda, r^2 m),$$

and

$$\left\|M_\beta f_r\right\|_{L_q(\hat{\mathbb{K}})} = r^{-\beta-\frac{\alpha+2}{q}} \left\|M_\beta f\right\|_{L_q(\hat{\mathbb{K}})}.$$

By the same argument in Theorem 2.3 we obtain $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{\alpha+2}$.

2) Let M_β be bounded from $L_1(\hat{\mathbb{K}})$ to $WL_q(\hat{\mathbb{K}})$. Then we have

$$\left\|M_\beta f_r\right\|_{WL_q(\hat{\mathbb{K}})} = r^{-\beta-\frac{\alpha+2}{q}} \left\|M_\beta f\right\|_{WL_q(\hat{\mathbb{K}})}.$$

Hence it is not hard to verify that $1 = \frac{1}{q} + \frac{\beta}{\alpha+2}$. Thus the proof of Corollary 2.3 is completed.

Proof of Theorem 2.4. For $(\lambda, m) \in \hat{\mathbb{K}}$, let $T^{(\alpha)}_{(\lambda,m)}$ be the generalized translation by (λ, m). By definition of the Besov spaces it suffices to show that

$$\|T^{(\alpha)}_{(\lambda,m)} Mf - Mf\|_{L_p(\hat{\mathbb{K}})} \le C \|T^{(\alpha)}_{(\lambda,m)} f - f\|_{L_p(\hat{\mathbb{K}})}.$$

It is easy to see that $T^{(\alpha)}_{(\lambda,m)}$ commutes with M, i.e. $T^{(\alpha)}_{(\lambda,m)} Mf = M(T^{(\alpha)}_{(\lambda,m)} f)$. Hence we have

$$|T^{(\alpha)}_{(\lambda,m)} Mf - Mf| = |M(T^{(\alpha)}_{(\lambda,m)} f) - Mf| \le M(|T^{(\alpha)}_{(\lambda,m)} f - f|).$$

Taking $L_p(\hat{\mathbb{K}})$ norm on both ends of the above inequality, by the boundedness of M on $L_p(\hat{\mathbb{K}})$ (see [1]), we obtain the desired result. Theorem 2.4 is proved.

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