Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (1), 3-9 (2016).
Series of Physical-Technical and Mathematical Sciences.

Grounding of the collocation method for a class of second kind integral equation

Fuad A. Abdullayev · Elnur H. Khalilov

Received: 08.01.2016 / Revised: 19.04.2016/ Accepted: 12.05.2016

Abstract. *In the paper, the collocation method for a boundary integral equation of Neumann's external boundary value problem for the Helmholtz equation is grounded.*

Keywords. colloacation method, Helmholts equation, Neumann's external boundary value problem, cubic formula, surface integral

Mathematics Subject Classification (2010): 45E05; 31B10

1 Introduction and problem statement

One of the methods for solving Neumann's external boundary value problem is its reduction to a boundary integral equation (BIE) that is solved in very rare cases. Therefore development of approximate methods for solving BIE with appropriate theoretical ground is of paramount importance.

Let $D \subset R^3$ be a bounded domain with a twice continuously -differentiable boundary S. Recall that Neumann's external boundary value problem for the Helmholtz equation is: to find the function u twice continuously- differentiable on $R^3 \backslash \bar{D}$ and continuous on S, possessing a normal derivative in the sense of uniform convergence, satisfying the Helmholtz equation $\Delta u + k^2 u = 0$ in $R^3 \backslash \bar{D}$, the Zommerfeld radiation at infinity and the boundary condition

$$\frac{\partial u(x)}{\partial \overrightarrow{n}(x)} = g(x) \text{ on } S,$$

where k is a wave number, $\mathrm{Im}k \geq 0$, $\overrightarrow{n}(x)$ is a unit external normal at the point $x \in S$, g is a given continuous function on S. In the paper [5] it is shown that if the function $u(x)$

Fuad A. Abdullayev
Baku State University, Baku AZ 1148, Baku, Azerbaijan
E-mail: fuad.abdullayev.1956 @mail.ru

Elnur H. Khalilov
Azerbaijan State Oil and Industry University
Azadlig str., AZ1010, Baku, Azerbaijan
E-mail: elnurkhalil@mail.ru

has a normal derivative in the sense of uniform convergence, then the Neumann external boundary value problem for the Helmholtz equation is reduced to BIE

$$\varphi - K\varphi = -Fg \tag{1.1}$$

where

$$(K\varphi)(x) = 2\int_S \frac{\partial \Phi_k(x,y)}{\partial \overrightarrow{n}(y)} \varphi(y)\, dS_y,\ \ (F\varphi)(x) = 2\int_S \Phi_k(x,y)\,\varphi(y)\, dS_y,\ \ x \in S,$$

$\Phi_k(x,y) = e^{ik|x-y|}/(4\pi|x-y|)$, $x, y \in R^3$, $x \neq y$ is a fundamental solution of the Helmholtz equation.

Note that a number of papers (see[3-7]) have been devoted to study of approximate solution of BIE of Dirichlet and Neumann's external boundary value problems for the Helmholtz equation. Equation (1.1) has the advantage that its solution is the solution of Neumann's external boundary value problem for the Helmholtz equation on S. And the function

$$u(x) = \int_S \left\{ \varphi(y) \frac{\partial \Phi_k(x,y)}{\partial \overrightarrow{n}(y)} - g(y)\,\Phi_k(x,y) \right\} dS_y,\ \ x \in R^3 \backslash \bar{D}$$

is the solution of Neumann's external boundary value problem for the Helmholtz equation.

This paper is devoted to grounding of the collocation method for integral equation (1.1).

2 Grounding of collocation method

Partition S into elementary domains $S = \bigcup\limits_{l=1}^{N} S_l^N$:

(1) for every $l = \overline{1,N}$ the domains S_l^N is closed and the set of its interior with respect to $\overset{0}{S_l^N}$ points is not emply, $mes\,\overset{0}{S_l^N} = mes S_l^N$ and $\overset{0}{S_l^N} \bigcap \overset{0}{S_j^N} = \varnothing$ for $j \in \{1,2,...,N\}$, $j \neq l$;

(2) for every $l = \overline{1,N}$ the domain S_l^N is a connected segment of the surface S with continuos boundary;

(3) for every $l = \overline{1,N}$ there exists a so called support point $x_l \in S_l^N$ such that:

(3.1) $r_l(N) \sim R_l(N)$ ($r_l(N) \sim R_l(N) \Leftrightarrow C_1 \leq r_l(N)/R_l(N) \leq C_2$, C_1 and C_2- are positive constants independent of N), where $r_l(N) = \min\limits_{x \in \partial S_l^N} |x - x_l|$ and $R_l(N) = \max\limits_{x \in \partial S_l^N} |x - x_l|$;

(3.2) $R_l(N) \leq d/2$, where d is the radius of a standard sphere (see.[10]);

(3.3) for every $j = \overline{1,N}$ $r_j(N) \sim r_l(N)$.

Obviously, $r(N) \sim R(N)$ and $\lim\limits_{N\to\infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,N}} R_l(N)$, $r(N) = \min\limits_{l=\overline{1,N}} r_l(N)$.

Such a partition, as a partition of a unit spere into elementary parts was carlier reduced in [6].

Let $S_d(x)$ and $\Gamma_d(x)$ be the parts of the surface S and tangential plane $\Gamma(x)$ at the point $x \in S$, contained interior to the sphere $B_d(x)$ of radius d cenered at the point x. Furtheremore, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$. Then

$$|x-\tilde{y}| \leq |x-y| \leq c_1(S)\,|x-\tilde{y}| \text{ and } mesS_d(x) \leq c_2(S) mes\Gamma_d(x), \qquad (2.1)$$

where $c_1(S)$ and $c_2(S)$ are positive constants dependent only on S (if S is a sphere, then $c_1(S)=\sqrt{2}$ and $c_2(S)=2$).

The following lemma is valid.

Lemma 2.1 *(see. [6]) There exist independent on N constants $C_0' > 0$ and $C_1' > 0$, for which for $\forall l, j \in \{1,2,...,N\}$, $j \neq l$, and $\forall\ y \in S_j^N$ the following inequality is valid: $C_0'\,|y-x_l| \leq |x_j - x_l| \leq C_1'\,|y-x_l|$.*

For the function $\varphi \in C(S)$ ($C(S)-$ is a space of continuous functions on S with the norm
$\|\varphi\|_\infty = \max\limits_{x\in S} |\varphi(x)|$). Introduce a modulus of continuity of the form

$$\omega(\varphi,\tau) = \underset{\substack{|x-y|\leq\delta \\ x,y\in S}}{max} |\varphi(x)-\varphi(y)|\,,\ \ \delta>0.$$

Furthermore, let

$$a_{lj} = 2\,|sgn\,(l-j)|\,\Phi_k\,(x_l,x_j)\,mesS_j^N \text{ for } l,j=\overline{1,N},$$

$$b_{lj} = 2\,|sgn\,(l-j)|\,\frac{\partial\Phi_k\,(x_l,x_j)}{\partial\overrightarrow{n}\ (x_j)}\,mes\,S_j^N \text{ for } lj=\overline{1,N}.$$

In the paper [4] it was proved that the expressions $\left(F^N g\right)(x_l) = \sum_{j=1}^N a_{lj} g\,(x_j)$ and $\left(K^N\varphi\right)(x_l) = \sum_{j=1}^N b_{lj}\varphi\,(x_j)$ at the points $x_l, l=\overline{1,N}$, are cubic formulas for the integrals $(Fg)\,(x)$ and $(K\varphi)\,(x)$, respectively, and

$$\max_{l=\overline{1,N}} \left|(Fg)\,(x_l) - \left(F^N g\right)(x_l)\right| \leq M^1 \left(\|g\|_\infty R\,(N)\,|\ln R\,(N)| + \omega\,(g,R\,(N))\right), \quad (2.2)$$

$$\max_{l=\overline{1,N}} \left|(K\varphi)\,(x_l) - \left(K^N\varphi\right)(x_l)\right| \leq M \left(\|\varphi\|_\infty R\,(N)\,|\ln R\,(N)| + \omega\,(\varphi,R\,(N)\right). \quad (2.3)$$

For $z^N \in C^N$ (C^N is a space of N dimensional vectors $z^N = \left(z_1^N, z_2^N, ..., z_N^N\right),\ z_l^N \in C,\ l=\overline{1,N}$, with the norm $\left\|z^N\right\| = \max\limits_{l=\overline{1,N}} \left|z_l^N\right|$) we assume

$$K_l^N z^N = \sum_{j=1}^N b_{lj}\,z_j^N,\ l=\overline{1,N},\ \ K^N z^N = \left(K_1^N\,z^N, K_2^N\,z^N, ..., K_N^N\,z^N\right);$$

$$F_l^N g = \left(F^N g\right)(x_l),\ l=\overline{1,N},\ \ F^N g = \left(F_1^N g,\ F_2^N g, ..., F_N^N g\right).$$

Then we substitute BIE (1.1) by the system of algebraic equations with respect to z_l^N-approximate values of $\varphi\,(x_l)\,,\ l=\overline{1,N},$ that write in the form

$$z^N - K^N z^N = -F^N g. \qquad (2.4)$$

[1] Here and in the sequel M denotes positive constants different at various inequalites.

We get the grounding of the collacation method from G.M.Vainicco's theorem on convergence for linear operator equations (see[9]). To formulate it, in denotation of the paper [9] we give necessary definitions and theorem.

Definition 2.1 *([9]) We call the system $Q = \left\{ q^N \right\}$ of operators $q^N : C(S) \to C^N$ a binder for $C(S)$ and C^N if*

$$\left\| q^N \varphi \right\| \to \|\varphi\|_\infty \ as \ N \to \infty, \forall \varphi \in C(S)$$

$$\left\| q^N \left(a\varphi + a'\varphi' \right) - \left(aq^N \varphi + a' q^N \varphi' \right) \right\| \to 0 \ as \ N \to \infty \ \forall \varphi, \varphi' \in C(S), \ a, a' \in C.$$

Definition 2.2 *([9]) The sequence $\{\varphi_N\}$ of elements $\varphi_N \in C^N$ Q converges to $\varphi \in C(S)$, if $\left\| \varphi_N - q^N \varphi \right\| \to 0$ as $N \to \infty$. Therewith we write $\varphi_N \overset{Q}{\to} \varphi$.*

Definition 2.3 *([9]). The sequence $\{ \varphi_N \}$ of elements $\varphi_N \in C^N$ is Q compact if any of its subsequence $\{ \varphi_{N_m} \}$ contains Q convergent subsequence $\left\{ \varphi_{N_{m_k}} \right\}$.*

Proposition 2.1 *([9]) Let $q^N : C(S) \to C^N$ be linear and bounded. Then the following conditions are equally matched; 1) the sequence $\{\varphi_N\}$ i Q compact and the set of its Q limit points is compact in $C(S)$; 2) there exists a relatively compact sequence $\left\{ \varphi^{(N)} \right\} \subset C(S)$ such that $\left\| \varphi_N - q^N \varphi^{(N)} \right\| \to 0$ as $N \to \infty$.*

Definition 2.4 *([9]) The sequence of operators $B^N : C^N \to C^N$ QQ converges to the operator $B : C(S) \to C(S)$ if for any Q- convergent sequence $\{\varphi_N\}$ we have $\varphi_N \overset{Q}{\to} \varphi \Rightarrow B^N \varphi_N \overset{Q}{\to} B\varphi$. Therewith we write $B^N \overset{QQ}{\to} B$.*

Definition 2.5 *([9]) The sequence of operators B^N, boundedly acting in C^N compactly converge to the operator B bounded in $C(S)$ if $B^N \overset{QQ}{\to} B$ and the following compactness condition is fulfilled: $\varphi_N \in C^N, \|\varphi_N\| \le M \Rightarrow \left\{ B^N \varphi_N \right\}$ is $Q-$ compact.*

Theorem 2.1 *([9]) Let the following conditions be fulfilled:*

1) $Ker(I + B) = 0$, where $I-$ is a unit operator in the space $C(S)$;

2)$I^N + B^N$ $(N \ge N_0)$ are Fredholm operators with a zero index where I^N- is a unit operator in the space C^N;

3) $\psi_N \overset{Q}{\to} \psi, \psi_N \in C^N$, $\psi \in C(S)$;

4) $B^N \to B$ is compact.

Then the equation $(I + B)\varphi = \psi$ has a unique solution $\tilde{\varphi} \in C(S)$, the equation

$\left(I^N + B^N \right) \varphi_N = \psi_N$ $(N \ge N_0)$ has a unique solution $\tilde{\varphi}_N \in C^N$, and $\tilde{\varphi}_N \overset{Q}{\to} \tilde{\varphi}$ with the estimation

$$c_1 \left\| \left(I^N + B^N \right) q^N \tilde{\varphi} - \psi_N \right\| \le \left\| \tilde{\varphi}_N - q^N \tilde{\varphi} \right\| \le c_2 \left\| \left(I^N + B^N \right) q^N \tilde{\varphi} - \psi_N \right\|,$$

where

$$c_1 = 1/ \sup_{N \ge N_0} \left\| I^N + B^N \right\| > 0 \ c_2 = \sup_{N \ge N_0} \left\| \left(I^N + B^N \right)^{-1} \right\| < +\infty.$$

Now prove the basic result of this paper.

Theorem 2.2 *Let* $Imk > 0$*, then equations (1.1) and (2.4) have unique solutions* $\varphi_* \in C(S)$ *and* $z_*^N \in C^N$ $(N \geq N_0)$*, respectively, and* $\left\| z_*^N - p^N \varphi_* \right\| \to 0$ *as* $N \to \infty$

$$\left\| z_*^N - p^N \varphi_* \right\| \leq M \left[\|g\|_\infty R(N) \left| \ln R(N) \right| + \omega(g, R(N)) \right],$$

where $p^N \varphi_* = (\varphi_*(x_1), \varphi_*(x_2), ..., \varphi_*(x_N))$.

Proof. In the paper [5], it is shown that if $Imk > 0$, then $Ker(I - K) = 0$. Obviously,the operators $I^N - K^N$ are Fredholm with a zero index and the system of operators $P = \{p^N\}$ is a binder for the spaces $C(S)$ and C^N. Then $F^N g \overset{P}{\to} Fg$ and $I^N - K^N \overset{PP}{\to} I - K$. By definition 2.5 it remains to verity the compactness condition that by preposition 2.1 is equivalent to the condition: $\forall \{z^N\}, z^N \in C^N, \|z^N\| \leq M$ there exists a relatively compact sequence $\{K_N z^N\} \subset C(S)$ such that $\left\| K^N z^N - p^N (K_N z^N) \right\| \to 0$ as $N \to \infty$. As $\{K_N z^N\}$ we choose the sequence

$$\left(K_N z^N\right)(x) = 2 \sum_{j=1}^{N} z_j^N \int_{S_j^N} \frac{\partial \Phi_k(x, y)}{\partial \overrightarrow{n}(y)} dS_y.$$

Take any points $x', x'' \in S$ such that $|x' - x''| = \delta < d/2$. Then

$$\left| \left(K_N z^N\right)(x') - \left(K_N z^N\right)(x'') \right| \leq M \left\| z^N \right\| \int_S \left| \frac{\partial \Phi_k(x', y)}{\partial \overrightarrow{n}(y)} - \frac{\partial \Phi_k(x'', y)}{\partial \overrightarrow{n}(y)} \right| dS_y$$

$$\leq M \left\| z^N \right\| \int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x', y)}{\partial \overrightarrow{n}(y)} \right| dS_y + M \left\| z^N \right\| \int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \overrightarrow{n}(y)} \right| dS_y$$

$$+ M \left\| z^N \right\| \int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \overrightarrow{n}(y)} \right| dS_y + M \left\| z^N \right\| \int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x', y)}{\partial \overrightarrow{n}(y)} \right| dS_y$$

$$+ M \left\| z^N \right\| \int_{S \setminus \left(S_{\delta/2}(x') \bigcup S_{\delta/2}(x'')\right)} \left| \frac{\partial \Phi_k(x', y)}{\partial \overrightarrow{n}(y)} - \frac{\partial \Phi_k(x'', y)}{\partial \overrightarrow{n}(y)} \right| dS_y.$$

Using the inequality,

$$\left| \frac{\partial \Phi_k(x, y)}{\partial \overrightarrow{n}(y)} \right| \leq \frac{M}{|x - y|}, \ \forall x, y \in S, \ x \neq y,$$

and the formula of reduction of a surface integral to double one, we get:

$$\int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x', y)}{\partial \overrightarrow{n}(y)} \right| dS_y \leq M \int_{S_{\delta/2}(x')} \frac{1}{|x' - y|} dS_y \leq M\delta,$$

$$\int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \overrightarrow{n}(y)} \right| dS_y \leq M\delta.$$

Furthermore, taking into account the inequalities $|x'' - y| \geq \delta/2, \forall y \in S_{\delta/2}(x')$ and $|x' - y| \geq \delta/2, \forall y \in S_{\delta/2}(x'')$, we have:

$$\int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \overrightarrow{n}(y)} \right| dS_y \leq M \int_{S_{\delta/2}(x')} \frac{1}{|x'' - y|} dS_y \leq \frac{2M}{\delta} mes\left(S_{\delta/2}(x')\right) \leq M\delta,$$

$$\int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x',y)}{\partial \overrightarrow{n}(y)} \right| dS_y \le M\delta.$$

It is easy to show that

$$\left| \frac{\partial \Phi_k(x',y)}{\partial \overrightarrow{n}(y)} - \frac{\partial \Phi_k(x'',y)}{\partial \overrightarrow{n}(y)} \right| \le \frac{M\delta}{|x'-y|^2}, \forall y \in S \backslash \left(S_{\delta/2}(x') \bigcup S_{\delta/2}(x'') \right).$$

Hence we find

$$\int_{S \backslash \left(S_{\delta/2}(x') \bigcup S_{\delta/2}(x'') \right)} \left| \frac{\partial \Phi_k(x',y)}{\partial \overrightarrow{n}(y)} - \frac{\partial \Phi_k(x'',y)}{\partial \overrightarrow{n}(y)} \right| dS_y \le M\delta \left| \ln \delta \right|.$$

As a result we get

$$\left| \left(K_N z^N \right)(x') - \left(K_N z^N \right)(x'') \right| \le M \left\| z^N \right\| \delta \left| \ln \delta \right|, \tag{2.5}$$

and so, $\left\{ K_N z^N \right\} \subset C(S)$.

Relative compactness of the sequence $\left\{ K_N z^N \right\}$ follows from the Arzela theorem. Indeed, uniform convergence follows immediately from the condition $\left\| z^N \right\| \le M$, while equcontinuity from estimation (2.5).Furthermore, taking into consideration the way of partitoning of the surface into elementary parts, and lemma 2.1, it is easy to prove

$$\left\| K^N z^N - p^N \left(K_N z^N \right) \right\| \to 0 \text{ as } N \to \infty.$$

Then using theorem 2.1, we get that equations (1.1) and (2.4) have unique solutions $\varphi_* \in C(S)$ and $z_*^N \in C^N$ $(N \ge N_0)$, respectively, moreover

$$c_1 \delta_N \le \left\| z_*^N - p^N \varphi_* \right\| \le c_2 \delta_N,$$

where

$$c_1 = 1/ \sup_{N \ge N_0} \left\| I^N - K^N \right\| > 0, \; c_2 = \sup_{N \ge N_0} \left\| \left(I^N - K^N \right)^{-1} \right\| < +\infty,$$

$$\delta_{N(h)} = \max_{l=\overline{1,N}} \left| \varphi_*(x_l) - \left(K^N \varphi_* \right)(x_l) + \left(F^N g \right)(x_l) \right|.$$

Taking into consideration estimations (2.2) and (2.4), we have:

$$\delta_N = \max_{l=\overline{1,N}} \left| (K\varphi_*)(x_l) - (Fg)(x_l) - \left(K^N \varphi_* \right)(x_l) + \left(F^N g \right)(x_l) \right|$$

$$\le M \left[\left\| \varphi_* \right\|_\infty R(N) \left| \ln R(N) \right| + \omega(\varphi_*, R(N)) + \left\| g \right\|_\infty R(N) \left| \ln R(N) \right| + \omega(g, R(N)) \right].$$

As $\varphi_* = -\,(I-K)^{-1} Fg$, then $\left\| \varphi_* \right\|_\infty \le \left\| (I-K)^{-1} \right\| \left\| F \right\| \left\| g \right\|_\infty$. Furthermore, taking into account we find

$$\omega(K\varphi_*, R(N)) \le M \left\| \varphi_* \right\|_\infty R(N) \left| \ln R(N) \right|$$

and

$$\omega(Fg, R(N)) \le M \left\| g \right\|_\infty R(N) \left| \ln R(N) \right|$$

we get:

$$\omega(\varphi_*, R(N)) = \omega(K\varphi_* - Fg, R(N)) \le \omega(Fg, R(N)) + \omega(K\varphi_*, R(N))$$

$$\leq M \left\| g \right\|_{\infty} R\left(N\right) \left|\ln R\left(N\right)\right| .$$

As a result, from the obtained estimations

$$\delta_N \leq M \left[\left\| \right\|_{\infty} R\left(N\right) \left|\ln R\left(N\right)\right| + \omega\ \left(g, R\left(N\right)\right) \right],$$

we find.

The theorem is proved.

References

1. Abdullayev, F.A., Khalilov, E.H.: *Grounding of the collocation method for a class of boundary integral equations*, Differ. Uravn. **40** (1), 82–86 (2004).
2. Kashirin, A.A., Smagin, S.I.: *On numerical solution of Dirichlet problem for Helmholtz equation by the potentials method*, Jornal vichslitelnoy matematiki i matematicheskoy fiziki, **52** (8), 1492–1505 (2012).
3. Khalilov, E.H.: *On approximate solution of external Dirichlet boundary value problem for Laplace equation by collocation method*, Azerbaijan Journal of Mathematics, Baku, **5** (2), 13–20 (2015).
4. Khalilov, E.H.: *Cubic formula for class of weakly singular surface integrals*, Proc. IMM of NAS of Azerb., **(39)** 47, 69–76 (2013)
5. Kussmaul, R.: *Ein numerische Verfahren zur Lsung des Neumannschen Aussenraum-problems fr die Helmholtzsche Schwingungsgleichung*, Computing, **(4)**, 246–273 (1969).
6. Kustov, Yu.A, Musayev, B.I.: *Cubic formula for two-dimensional singular integral and its applications*, M. Dep. in VINITI 4281-81-60.
7. Kolton, D., Kress, R.: Methods of integral equations in scattering theory, *M.Mir*, 311 (1987).
8. Musayev, B.I., Khalilov, E.H.: *On approximate solution of a class of boundary integral equations by collocation method*, Trudy IMM Azerb. Res. **9** (17), 78–84 (1998).
9. Vainicco, G.M. *Regular convergence of operators and approximate solution of equations*, Itogi nauki i tekniki. Mat. analyz **(16)**, 5–53.
10. Vladimirov, V.S.: *Equations of mathematical physis*, M.Nauka, 527p. (1976).