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Approximation of analytic functions by sequences of linear operators in the closed domain

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Abstract. *We consider the space of analytic functions in the closed domain, where convergence is a uniform convergence in closed domain that contains the original domain strictly inside itself and prove the theorems on the approximation and statistical approximation of functions in this space by the sequences of linear operators.*

Keywords. space of analytical functions; Faber polynomials; conformal mapping; Korovkin type theorem; linear k - positive operators; statistical convergence.

Mathematics Subject Classification (2010): Primary 41A35, 47A58.

1 Introduction

It is well known that the sequences of linear positive operators play an important role in the theory of approximation of functions (see [14], [1]). Some authors explored the problem of approximation of analytic functions by the sequences of linear operators with the use of k-positivity properties of linear operators. The concept of k-positivity was first introduced in [4] to obtain Korovkin-type approximation theorems in the space of analytic functions. Using this definition of k-positivity, some results on approximation of analytic functions by means of k-positive linear operators in the unit disk were obtained in [4]. Various problems of approximation of analytic functions by k-positive linear operators have later been studied extensively in [2, 5-9, 12, 13, 15]. The approximation of analytic functions by linear operators without k-positivity has been considered in [10], [11].

This paper is devoted to the approximation of analytic functions in the closed domain by sequences of linear operators, where convergence is a uniform convergence in another closed domain that contains the original domain strictly inside itself. The basis for this space is formed by the system of Faber polynomials [16]. The paper is structured as follows. In Section 2, we prove theorems on the approximation by linear operators. In Section 3, we

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present similar results for statistical approximation of analytic functions by sequences of linear operators.

2 Approximation of analytic functions by sequences of linear operators

Let D be a simply connected domain with simply connected complement, and let $A\left(\bar{D}\right)$ be the space of analytic functions in a closed domain $\bar{D}$. The convergence in $A\left(\bar{D}\right)$ is a uniform convergence in some closed domain D_1 such that $\bar{D}\subset D_1$.

Let $\varphi\left(z\right)$ be any function mapping the exterior of D conformally and one-to-one into the exterior of the unit circle. We set

$$\varphi_n\left(z\right)=\frac{1}{2\pi i}\int_C\left[\varphi\left(t\right)\right]^n\frac{dt}{t-z}, n\in Z_+,$$

where the contour C contains D inside itself. The system $\left\{\varphi_n\left(z\right)\right\}$ is called the system of Faber polynomials and, as mentioned above, forms a basis for $A\left(\bar{D}\right)$. This means that each function $f\in A\left(\bar{D}\right)$ has the expansion

$$f\left(z\right)=\sum_{k=0}^{\infty}f_k\varphi_k\left(z\right), \tag{2.1}$$

where corresponding Fourier coefficients $f_k,\ k\in Z$ are defined by the formula

$$f_k=\frac{1}{2\pi i}\int_C\frac{f\left(t\right)\varphi'\left(t\right)}{\left[\varphi\left(t\right)\right]^{k+1}}dt.$$

We need the following theorem proved in [9] on the convergence to zero in $A\left(\bar{D}\right)$.

Theorem A [9]. *The sequence $f_n\left(z\right)$ convergence to zero in $A\left(\bar{D}\right)$ if and only if the coefficients of expansion $f_n\left(z\right)=\sum\limits_{k=0}^{\infty}f_k^{(n)}\varphi_k\left(z\right)$ satisfy the conditions*

$$\left|f_k^{(n)}\right|<\varepsilon_n\left(1+\delta\right)^{-k} \tag{2.2}$$

for any $n\in N$, $k\in Z_+$, where

$$\lim_{n\to\infty}\varepsilon_n=0, \delta>0. \tag{2.3}$$

Now we consider the linear operators in $A\left(\bar{D}\right)$. It follows from (2.1) that for any linear operator $T:A\left(\bar{D}\right)\to A\left(\bar{D}\right)$ the expansion

$$\left(Tf\right)\left(z\right)=\sum_{k=0}^{\infty}\left(\sum_{p=0}^{\infty}T_{k,p}f_p\right)\varphi_k\left(z\right)$$

is valid, where $f\left(z\right)=\sum\limits_{k=0}^{\infty}f_k\varphi_k\left(z\right)$ and $T\left(\varphi_p\left(z\right)\right)=\sum\limits_{k=0}^{\infty}T_{k,p}\varphi_k\left(z\right)$.

Let the sequence $g=\left\{g_k\right\}_{k=0}^{\infty}$ of positive numbers satisfy the conditions

$$\begin{gathered}\forall\, k\in Z_+:\varDelta_k\left(g\right)=\inf_{p\in Z_+,p\neq k}\left|\sqrt{g_k}-\sqrt{g_p}\right|>0,\\ \lim_{k\to\infty}\left(\varDelta_k\left(g\right)\right)^{\frac{1}{k}}=1,\ \lim_{k\to\infty}\left(g_k\right)^{\frac{1}{k}}=1.\end{gathered} \tag{2.4}$$

Definition 2.1 *By $A_g^{(\delta_0)}\left(\bar{D}\right)$ we denote the set of analytic functions*

$$f(z)=\sum_{k=0}^{\infty} f_k\varphi_k(z)\in A\left(\bar{D}\right)$$

whose coefficients satisfy the following conditions:

$$|f_k|\leq \frac{M_f^{(\delta_0)}g_k}{(1+\delta_0)^k}, \tag{2.5}$$

where $\delta_0>0$ and $M_f^{(\delta_0)}$ is a constant independent of k.

Theorem 2.1 *Let $T_n : A\left(\bar{D}\right)\to A\left(\bar{D}\right)$ be a sequence of linear operators*

$$(T_nf)(z)=\sum_{k=0}^{\infty}\left(\sum_{p=0}^{\infty}T_{k,p}^{(n)}f_p\right)\varphi_k(z), \tag{2.6}$$

where $f(z)=\sum\limits_{k=0}^{\infty}f_k\varphi_k(z)\in A\left(\bar{D}\right)$. If there exist sequence ε_n satisfying (2.3) *and positive number $\delta_1>0$ such that the inequalities*

$$\left|\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}-(1+\delta_0)^{-k}\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.7}$$

$$\left|\sum_{p=0}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}-(1+\delta_0)^{-k}\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.8}$$

$$\left|\sum_{p=0}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}\sqrt{g_p}-(1+\delta_0)^{-k}\sqrt{g_k}\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.9}$$

$$\left|\sum_{p=-\infty}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}g_p-(1+\delta_0)^{-k}g_k\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.10}$$

hold, then for any function $f\in A_g^{(\delta_0)}\left(\bar{D}\right)$ the sequence $T_nf(z)$ tends to $f(z)$ in $A\left(\bar{D}\right)$.

Proof. From (2.8) - (2.10) we have

$$\sum_{p=0}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}\left(\sqrt{g_p}-\sqrt{g_k}\right)^2\leq 4\varepsilon_n(1+\delta_1)^{-k}(1+\sqrt{g_k})^2. \tag{2.11}$$

Consequently,

$$\sum_{\substack{p=0\\p\neq k}}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}\leq\frac{4\varepsilon_n(1+\delta_1)^{-k}\left(1+\sqrt{g_k}\right)^2}{\Delta_k^2(g)}. \tag{2.12}$$

For every $f(z)=\sum\limits_{k=0}^{\infty} f_k\varphi_k(z)\in A_g^{(\delta_0)}\left(\bar{D}\right)$ we have

$$T_n f(z)-f(z)=\sum_{k=0}^{\infty}\left\{\sum_{p=0}^{\infty}T_{k,p}^{(n)}f_p-f_k\right\}\varphi_k(z)$$

$$=\sum_{k=0}^{\infty}\left\{\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}-(1+\delta_0)^{-k}\right\}f_k(1+\delta_0)^k\varphi_k(z)$$

$$+\sum_{k=0}^{\infty}\left\{\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}\left(f_p(1+\delta_0)^p-f_k(1+\delta_0)^k\right)\right\}\varphi_k(z)$$

$$=J_n^{(1)}(z)+J_n^{(2)}(z). \tag{2.13}$$

Then from (2.5), (2.7), (2.11) and (2.12) we have:

$$\left|\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}-(1+\delta_0)^{-k}\right||f_k|(1+\delta_0)^k\le M_f^{(\delta_0)}\varepsilon_n(1+\delta_1)^{-k}g_k;$$

$$\left|\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}\left(f_p(1+\delta_0)^p-f_k(1+\delta_0)^k\right)\right|$$

$$\le M_f^{(\delta_0)}\sum_{\substack{p=0\\p\ne k}}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}\left[g_p+g_k\right]$$

$$\le M_f^{(\delta_0)}\sum_{\substack{p=0\\p\ne k}}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}\left[3g_k+2\left(\sqrt{g_p}-\sqrt{g_k}\right)^2\right]$$

$$\le M_f^{(\delta_0)}4\varepsilon_n(1+\delta_1)^{-k}\left(1+\sqrt{g_k}\right)^2\left[\frac{3g_k}{\Delta_k^2(g)}+2\right].$$

Hence, by virtue of (2.13) and theorem A we obtain that the sequence $T_n f(z)$ tends to $f(z)$ in $A\left(\bar{D}\right)$. Theorem is proved.

Now, we state the following general result on approximation in $A(D)$.

Theorem 2.2 *Let the sequences of positive numbers $b=\{b_k\}_{k=0}^{\infty}$ and $g=\{g_k\}_{k=0}^{\infty}$ satisfy* (2.4) *and $T_n: A\left(\bar{D}\right)\to A\left(\bar{D}\right)$ be a linear operators defined as* (2.6). *If there exist sequence ε_n satisfying* (2.3) *and positive number $\delta_1>0$ such that the inequalities*

$$\left|\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}g_p-(1+\delta_0)^{-k}g_k\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.14}$$

$$\left|\sum_{p=0}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}g_p-(1+\delta_0)^{-k}g_k\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.15}$$

$$\left|\sum_{p=0}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}g_p\sqrt{b_p}-(1+\delta_0)^{-k}g_k\sqrt{b_k}\right|<\varepsilon_n(1+\delta_1)^{-k}, \tag{2.16}$$

$$\left|\sum_{p=-\infty}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}g_pb_p-(1+\delta_0)^{-k}g_kb_k\right|<\varepsilon_n(1+\delta_1)^{-k}. \tag{2.17}$$

hold, then the sequence $T_nf(z)$ tends to $f(z)$ in $A(\bar{D})$ for any function $f\in A_g^{(\delta_0)}(\bar{D})$.

Proof. From (2.15) - (2.17) we have

$$\sum_{p=0}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}\left(\sqrt{b_p}-\sqrt{b_k}\right)^2g_p\le 4\varepsilon_n(1+\delta_1)^{-k}\left(1+\sqrt{b_k}\right)^2. \tag{2.18}$$

Consequently,

$$\sum_{\substack{p=0\\p\ne k}}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}g_p\le\frac{4\varepsilon_n(1+\delta_1)^{-k}\left(1+\sqrt{b_k}\right)^2}{\Delta_k^2(b)}. \tag{2.19}$$

For every $f(z)=\sum_{k=0}^{\infty}f_k\varphi_k(z)\in A_g^{(\delta_0)}(\bar{D})$ we have

$$T_nf(z)-f(z)=\sum_{k=0}^{\infty}\left\{\sum_{p=0}^{\infty}T_{k,p}^{(n)}f_p-f_k\right\}\varphi_k(z)$$

$$=\sum_{k=0}^{\infty}\left\{\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}g_p-(1+\delta_0)^{-k}g_k\right\}\frac{f_k}{g_k}(1+\delta_0)^k\varphi_k(z)$$

$$+\sum_{k=0}^{\infty}\left\{\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}\left(\frac{f_p}{g_p}(1+\delta_0)^p-\frac{f_k}{g_k}(1+\delta_0)^k\right)g_p\right\}\varphi_k(z)$$

$$=J_n^{(1)}(z)+J_n^{(2)}(z). \tag{2.20}$$

Then from (2.5), (2.14), (2.18) and (2.19) we have:

$$\left|\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}g_p-(1+\delta_0)^{-k}g_k\right|\left|\frac{f_k}{g_k}\right|(1+\delta_0)^k\le M_f^{(\delta_0)}\varepsilon_n(1+\delta_1)^{-k}$$

$$\left|\sum_{p=0}^{\infty}T_{k,p}^{(n)}(1+\delta_0)^{-p}\left(\frac{f_p}{g_p}(1+\delta_0)^p-\frac{f_k}{g_k}(1+\delta_0)^k\right)g_p\right|$$

$$\le 2M_f^{(\delta_0)}\sum_{\substack{p=0\\p\ne k}}^{\infty}\left|T_{k,p}^{(n)}\right|(1+\delta_0)^{-p}g_p\le\frac{8M_f^{(\delta_0)}\varepsilon_n(1+\delta_1)^{-k}\left(1+\sqrt{b_k}\right)^2}{\Delta_k^2(b)}.$$

Hence, by virtue of (2.20) and theorem A we obtain that the sequence $T_nf(z)$ tends to $f(z)$ in $A(\bar{D})$. Theorem is proved.

3 Statistical approximation of analytic functions by sequences of linear operators

Using the methods of [12], it is not difficult to obtain statistical analogs of the above theorems. Let us first recall

Definition 3.1 ([3]) *A sequence x_n is said to be statistically convergent to a number x if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{|\{m \le n : |x_m - x| > \varepsilon\}|}{n} = 0,$$

where $|\{m \le n : |x_m - x| > \varepsilon\}|$ is the number of all $m \le n$, for which $|x_m - x| > \varepsilon$. In this case we write $st. - \lim\limits_{n\to\infty} x_n = x$.

We need the following theorem proved in [9] on the statistically convergence to zero in $A\left(\bar{D}\right)$.

Theorem B [9]. *The sequence $f_n(z)$ statistically convergence to zero in $A\left(\bar{D}\right)$ if and only if the coefficients of expansion $f_n(z) = \sum\limits_{k=0}^{\infty} f_k^{(n)} \varphi_k(z)$ satisfy the conditions*

$$\left| f_k^{(n)} \right| < \varepsilon_n (1+\delta)^{-k}$$

for any $n \in N$, $k \in Z_+$, where

$$st. - \lim_{n\to\infty} \varepsilon_n = 0, \delta > 0. \tag{3.1}$$

Theorem 3.1 *Let the sequences of positive numbers $g = \{g_k\}_{k=0}^{\infty}$ satisfy* (2.4) *and $T_n : A\left(\bar{D}\right) \to A\left(\bar{D}\right)$ be linear operators defined by* (2.6). *If there exist sequence ε_n satisfying* (3.1) *and positive number $\delta_1 > 0$ such that the inequalities* (2.7) – (2.10) *hold, then for any function $f \in A_g^{(\delta_0)}\left(\bar{D}\right)$ the sequence $T_n f(z)$ statistically tends to $f(z)$ in $A\left(\bar{D}\right)$.*

The proof is similar to the one of theorem 1 and uses theorem B.

Theorem 3.2 *Let the sequences of positive numbers $b = \{b_k\}_{k=0}^{\infty}$ and $g = \{g_k\}_{k=0}^{\infty}$ satisfy* (2.4) *and $T_n : A\left(\bar{D}\right) \to A\left(\bar{D}\right)$ be linear operators defined by* (2.6). *If there exist sequence ε_n satisfying* (3.1) *and positive number $\delta_1 > 0$ such that the inequalities* (2.14) – (2.17) *hold, then for any function $f \in A_g^{(\delta_0)}\left(\bar{D}\right)$ the sequence $T_n f(z)$ statistically tends to $f(z)$ in $A\left(\bar{D}\right)$.*

The proof is similar to the one of Theorem 2.2 and uses theorem B.

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