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Two-weighted inequality for (p,q)– admissible $B_{k,n}$– potential operators in weighted Lebesgue spaces

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Abstract. *In this paper, we study the boundedness of (p,q)– admissible potential operators, associated with the Laplace-Bessel differential operator* $B_{k,n} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{k} \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}$ *((p,q)– admissible $B_{k,n}$– potential operators) on a weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ including their weak versions. These conditions are satisfied by most of the operators in harmonic analysis, such as $B_{k,n}$– fractional maximal operator, $B_{k,n}$– potential integral operators etc. Sufficient conditions on weight functions ω and ω_1 are given so that (p,q)– admissible $B_{k,n}$– potential operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1<p<q<\infty$ and weak (p,q)– admissible $B_{k,n}$– potential operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1\le p<q<\infty$.*

Keywords. Weighted Lebesgue space, (p,q)–admissible $B_{k,n}$–potential operators, $B_{k,n}$–fractional maximal operator, $B_{k,n}$–potential integral operators, two-weighted inequality.

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1 Introduction

Let $\mathbb{R}^n_{k,+}$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x=(x_1,...,x_n)$ defined by the inequalities $x_1>0,\dots,x_k>0$, $1\le k\le n$. Denote by $L_{p,\gamma}\equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all

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classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{\frac{1}{p}}, \ \ 1 \le p < \infty,$$

where $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$, $\gamma = (\gamma_1, ..., \gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + ... + \gamma_k$.

If $p = \infty$, we assume

$$L_{\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+}) = \{f : \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} |f(x)| < \infty\}.$$

For measurable set $E \subset \mathbb{R}^n_{k,+}$ let $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,k,\gamma) r^{n+|\gamma|}$, where $\omega(n,k,\gamma) = |E(0,1)|_\gamma$. Let $f \in L^{\mathrm{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$ and B be the Laplace-Bessel differential operator:

$$\Delta_B = B + \Delta_{x''}, \ \ B = \sum_{i=1}^k B_i, \ \ B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \ \ \Delta_{x''} = \sum_{j=k+1}^n \frac{\partial^2}{\partial x_j^2}.$$

The $B_{k,n}-$ fractional maximal function (see [9], [10], [12]) is defined by

$$M_{\alpha,\gamma} f(x) = \sup_{r>0} \frac{1}{\omega(n,\gamma) r^{n+|\gamma|}} \int_{E(0,r)} T^y |f(x)| \, (y')^\gamma dy, \ \ 0 \le \alpha < n + |\gamma|$$

and the $B_{k,n}-$ Riesz potential (see [4], [9], [10], [12]) is defined by

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y |x|^{\alpha - n - |\gamma|} \, f(y) \, (y')^\gamma dy, \quad 0 < \alpha < n + |\gamma|.$$

The $B-$ maximal function introduced and L_p boundedness investigated by V.S. Guliyev [9] (in the case $n = k = 1$ by K. Stempak [20], see also [10]-[12]). The fractional integral (potential) operators play an important role in the theory of harmonic analysis, differentiation theory and PDE's. Many mathematicians have dealt with the fractional integrals and related topics associated with the Laplace-Bessel differential operator such as I.A. Kipriyanov [15], L.N. Lyakhov [18], A.D. Gadjiev and I.A. Aliev [4], V.S. Guliyev [9], [10], [11], I.A. Aliev and S. Bayrakci [2], [3], A. Serbetci and I. Ekincioglu [19], and others.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

2 Notations and preliminary results

For $x \in \mathbb{R}^n_{k,+}$ we set $x = (x', x'')$, $x' = (x_1 \ldots, x_k)$, $x'' = (x_{k+1}, \ldots, x_n)$. Let $\mathbb{R}^k_{++} = \{x' \in \mathbb{R}^k : \ x_1 > 0, \ldots, x_k > 0\}$, $1 \le k \le n$, $S_{k,+} = \{x \in \mathbb{R}^n_{k,+} : \ |x| = 1\}$. For $x \in \mathbb{R}^n_{k,+}$ and $r > 0$, we denote by $E(x,r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$ the open ball centered at x of radius r, and by ${}^{\complement}E(x,r) = \mathbb{R}^n_{k,+} \setminus E(x,r)$ denote its complement, $E'(x',r) = \{y' \in \mathbb{R}^k_{++} : \ |x' - y'| < r\}$, ${}^{\complement}E'(x',r) = \mathbb{R}^k_{++} \setminus E'(x',r)$.

Let $1 \le m < k \le n$, we put
$\mathbb{R}^{n-m}_{k-m,+} = \{x_{m+1,n} = (x_{m+1},\ldots,x_n) : x_{m+1},\ldots,x_k > 0\}$, $x' \equiv x_{1,k} = (x_{1,m}, x_{m+1,k})$, where $x_{1,m} = (x_1,...,x_m) \in \mathbb{R}^m_{++}$, $x_{m+1,k} = (x_{m+1},...,x_k) \in \mathbb{R}^{k-m}_{++}$. In this case we write $\gamma = (\gamma_{1,m}, \gamma_{m+1,k})$, $\gamma_{1,m} = (\gamma_1,...,\gamma_m)$, $\gamma_{m+1,k} = (\gamma_{m+1},...,\gamma_k)$, $x_{1,m}^{\gamma_{1,m}} = x_1^{\gamma_1}\ldots x_m^{\gamma_m}$, $x_{m+1,k}^{\gamma_{m+1,k}} = x_{m+1}^{\gamma_{m+1}}\ldots x_k^{\gamma_k}$, $E_m(x_{1,m},r) = \{y_{1,m} \in \mathbb{R}^m_{++} : |x_{1,m} - y_{1,m}| < r\}$. Note that $E_k(x_{1,k},r) = E^{'}(x',r)$ and $E_n(x_{1,n},r) = E(x,r)$.

An almost everywhere positive and locally integrable function $\omega : \mathbb{R}^n_{k,+} \to \mathbb{R}$ will be called a weight. We denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ the set of all measurable function f on $\mathbb{R}^n_{k,+}$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})} \equiv \|f\|_{p,\omega,\gamma;\mathbb{R}^n_{k,+}} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx\right)^{1/p}, \; 1 \le p < \infty$$

is finite. For $\omega = 1$ we get the space $L_{p,\gamma}(\mathbb{R}^n_{k,+})$.

The generalized shift operator ($B-$ shift operator) is defined by, (see [12], [16], [18]):

$$T^y f(x) = C_{\gamma,k} \int_0^\pi ... \int_0^\pi f\big((x',y')_\beta, x'' - y''\big) d\nu(\beta),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma^{-1}\left(\frac{|\gamma|}{2}\right) \prod\limits_{i=1}^{k} \Gamma(\frac{\nu_i+1}{2})$, $(x',y')_\beta$
$= ((x_1,y_1)_{\beta_1}...(x_k,y_k)_{\beta_k})$, $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_iy_i\cos\beta_i + y_i^2)^{1/2}$, $1 \le i \le k$.

Note that $B-$ shift operator is closely connected with $B-$ Laplace-Bessel singular differential operators (see [12], [17]).

In the following lemma we give generalized Hardy inequalities (see [1], [8], [7], [14]) which have an important role in proofs of our main results. This lemma could be directly deduced from results proved by P. Drabek, H. Heinig and A. Kufner (see Theorem 2.1, p. 4 and Theorem 2.2, p. 7 in [13]).

Lemma 2.1 *Suppose that $1 \le p \le q \le \infty$, $p' = p/(p-1)$ and $\omega(x)$ and $v(x)$ are positive functions defined on $\mathbb{R}^n$.*

(1) *For the n-dimensional Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|<|x|/2}|f(y)|\,dy\right)^q \omega(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)|^p v(x)\,dx\right)^{1/p}$$

with a constant C, independent on f, to hold, it is necessary and sufficient that the following condition be satisfied:

$$\sup_{r>0}\left(\int_{|x|>2r}\omega(x)\,dx\right)^{1/q}\left(\int_{|x|<r}v^{1-p'}(x)\,dx\right)^{1/p'} < \infty.$$

(2) *For the n-dimensional (dual) Hardy inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|>2|x|}|f(y)|\,dy\right)^q \omega(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)|^p v(x)\,dx\right)^{1/p}$$

with a constant C, independent on f, to hold, it is necessary and sufficient that the following condition be satisfied:

$$\sup_{r>0}\left(\int_{|x|<r}\omega(x)\,dx\right)^{1/q}\left(\int_{|x|>2r}v^{1-p'}(x)\,dx\right)^{1/p'}<\infty.$$

3 Main results

An operator T is called sublinear, if for all $\lambda,\mu>0$ and for all f and g in the domain of T

$$|T(\lambda f+\mu g)(x)|\le\lambda|Tf(x)|+\mu|Tg(x)|.$$

Definition 3.1 *((p,q)– admissible $B_{k,n}$– potential operator). Let $1<p<\infty$. A sublinear operator $T_{\alpha,\gamma}$, $0<\alpha<n+|\gamma|$ will be called (p,q)– admissible $B_{k,n}$– potential operator, if:*

(1) *$T_{\alpha,\gamma}$ satisfies the size condition of the form*

$$\begin{aligned}&\chi_{E(x,r)}(z)\left|T_{\alpha,\gamma}\left(f\chi_{\mathbb{R}^n_{k,+}\setminus E(x,2r)}\right)(z)\right|\\&\le C\chi_{E(x,r)}(z)\int_{\mathbb{R}^n_{k,+}\setminus E(x,2r)}T^y|x|^{\alpha-n-|\gamma|}\,|f(y)|\,(y')^\gamma dy\end{aligned}\tag{3.1}$$

for $x\in\mathbb{R}^n_{k,+}$ and $r>0$;

(2) *$T_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

Definition 3.2 *(weak (p,q)– admissible $B_{k,n}$– potential operator). Let $1\le p<q<\infty$. A sublinear operator $T_{\alpha,\gamma}$, $0<\alpha<n+|\gamma|$ will be called the weak (p,q)– admissible $B_{k,n}$– potential operator, if:*

(1) *$T_{\alpha,\gamma}$ satisfies the size condition* (3.1).

(2) *$T_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to the weak $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

In the following theorems we give the boundedness of the (p,q)– admissible $B_{k,n}$– potential operator in weighted $L_{p,\gamma}$ spaces.

Theorem 3.1 *Let $1<p<q<\infty$ and $T_{\alpha,\gamma}$, $0<\alpha<n+|\gamma|$ be a (p,q)– admissible $B_{k,n}$– potential operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and the following conditions are satisfied:*

(a) *There exist $C>0$ such that*

$$\sup_{|x|/8<|y|\le 8|x|}\omega_1(y)^{1/q}\le C\,\omega(x)^{1/p}\quad\text{for a.e. } x\in\mathbb{R}^n_{k,+},$$

(b) $\mathcal{A}\equiv$

$$\sup_{r>0}\Big(\int\limits_{\complement E(0,2r)}\omega_1(x)|x|^{-(n+|\gamma|-\alpha)q}(x')^\gamma dx\Big)^{1/q}\Big(\int\limits_{E(0,r)}\omega^{1-p'}(x)(x')^\gamma dx\Big)^{1/p'}<\infty,$$

(c) $\mathcal{B} \equiv$

$$\sup_{r>0} \Big(\int\limits_{E(0,r)} \omega_1(x)(x')^\gamma dx \Big)^{1/q} \Big(\int\limits_{\complement E(0,2r)} \omega^{1-p'}(x)|x|^{-(n+|\gamma|-\alpha)(1-p')}(x')^\gamma dx \Big)^{1/p'} < \infty.$$

Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_\alpha f(x)|^q \omega_1(x)(x')^\gamma dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx \right)^{1/p}. \quad (3.2)$$

Similarly we can write the following weak variant of the Theorem 3.1.

Theorem 3.2 *Let $1 \le p < q < \infty$ and let $T_{\alpha,\gamma}$, $0 < \alpha < n + |\gamma|$ be a weak (p,q)– admissible $B_{k,n}$– potential operators. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied. Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$*

$$\left(\int_{\left\{x \in \mathbb{R}^n_{k,+} : |T_\alpha f(x)| > \lambda\right\}} \omega_1(x)(x')^\gamma dx \right)^{1/q} \le \frac{C}{\lambda^q} \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx \right)^{1/p}. \quad (3.3)$$

Theorem 3.3 *Let $1 \le m < k \le n$, $1 < p < q < \infty$ and $T_{\alpha,\gamma}$, $0 < \alpha < n+|\gamma|$ be a (p,q)– admissible $B_{k,n}$– potential operator. Moreover, let $\omega(x_{1,m})$, $\omega_1(x_{1,m})$ be weight functions on $\mathbb{R}^m_{++}$ and the following conditions are satisfied:*

(a_1) *There exists a constant $C > 0$ such that*

$$\sup_{|x_{1,m}|/8 < |y_{1,m}| < 8|x_{1,m}|} \omega_1(y_{1,m})^{1/q} \le C\, \omega(x_{1,m})^{1/p} \quad \textit{for a.e. } x_{1,m} \in \mathbb{R}^m_{++},$$

(b_1) $\mathcal{A}_1 \equiv$

$$\sup_{r>0} \left(\int_{\complement E_m(0,2r)} \omega_1(x_{1,m}) |x_{1,m}|^{(n-m+|\gamma_{m+1,k}|)(1+q/p')-(n+|\gamma|-\alpha)q} (x_{1,m})^{\gamma_{1,m}} dx_{1,m} \right)^{1/q}$$
$$\times \left(\int_{E_m(0,r)} \omega^{1-p'}(x_{1,m})(x_{1,m})^{\gamma_{1,m}} dx_{1,m} \right)^{1/p'} < \infty,$$

(c_1) $\mathcal{B}_1 \equiv \sup_{r>0} \left(\int_{E_m(0,r)} \omega_1(x_{1,m})(x_{1,m})^{\gamma_{1,m}} dx_{1,m} \right)^{1/q}$

$$\times \left(\int\limits_{\complement E_m(0,2r)} \omega^{1-p'}(x_{1,m}) |x_{1,m}|^{(1-p')(n-m+|\gamma_{m+1,k}|)(1/q+1/p'-n-|\gamma|+\alpha)} (x_{1,m})^{\gamma_{1,m}} dx_{1,m} \right)^{1/p'}$$

$< \infty$.

Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n_{k,+})$

$$\left(\int\limits_{\mathbb{R}^n_{k,+}} |T_\alpha f(x)|^q \omega_1(x_{1,m})(x')^\gamma dx \right)^{1/q} \le C \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m})(x')^\gamma dx \right)^{1/p}. \quad (3.4)$$

Similarly we can write the following weak variant of Theorem 3.3.

Theorem 3.4 *Let $1 \le p < q < \infty$ and let $T_{\alpha,\gamma}$, $0 < \alpha < n+|\gamma|$ be a weak (p,q)– admissible $B_{k,n}$– potential operators. Moreover, let $\omega(x_{1,m})$, $\omega_1(x_{1,m})$ be weight functions on $\mathbb{R}^k_{++}$ and conditions (a_1), (b_1), (c_1) be satisfied. Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$*

$$\left(\int_{\left\{x\in\mathbb{R}^n_{k,+}\,:\,|T_\alpha f(x)|>\lambda\right\}} \omega_1(x_{1,m})(x')^\gamma dx\right)^{1/q}$$
$$\le \frac{C}{\lambda^q}\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m})(x')^\gamma dx\right)^{1/p}. \tag{3.5}$$

Note that, the operators $M_{\alpha,\gamma}$ and $I_{\alpha,\gamma}$ are (p,q)– admissible $B_{k,n}$– potential operator for $1 < p < q < \infty$, $0 < \alpha < n + |\gamma|$ and $1/p - 1/q = \alpha/(n + |\gamma|)$ and weak (p,q)– admissible $B_{k,n}$– potential operators for $1 \le p < q < \infty$, $0 < \alpha < n + |\gamma|$ and $1/p - 1/q = \alpha/(n + |\gamma|)$. Thus, we have the following corollaries.

Corollary 3.1 *Let $1 < p < q < \infty$, $0 < \alpha < n + |\gamma|$ and $1/p - 1/q = \alpha/(n + |\gamma|)$. Moreover, let $\omega(x_{1,m})$, $\omega_1(x_{1,m})$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a_1), (b_1), (c_1) be satisfied. Then the operators $M_{\alpha,\gamma}$ and $I_{\alpha,\gamma}$ are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

Corollary 3.2 *Let $1 \le p < q < \infty$, $0 < \alpha < n + |\gamma|$ and $1/p - 1/q = \alpha/(n + |\gamma|)$. Moreover, let $\omega(x_{1,m})$, $\omega_1(x_{1,m})$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a_1), (b_1), (c_1) be satisfied. Then the operators $M_{\alpha,\gamma}$ and $I_{\alpha,\gamma}$ are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

Theorem 3.5 *Let $1 \le m < k \le n$, $1 < p < q < \infty$ and $T_{\alpha,\gamma}$, $0 < \alpha < n + |\gamma|$ be a (p,q)– admissible $B_{k,n}$– potential operator. Moreover, let $\omega(x_{m+1,n})$, $\omega_1(x_{m+1,n})$ be weight functions on $\mathbb{R}^{n-m}_{k-m,+}$ and the following conditions are satisfied:*

(a_2) There exists a constant $C > 0$ such that

$$\sup_{|x_{m+1,n}|/8<|y_{m+1,n}|<8|x_{m+1,n}|} \omega_1(y_{m+1,n})^{1/q} \le C\,\omega(x_{m+1,n})^{1/p}, \text{for a.e. } x_{m+1,n} \in \mathbb{R}^{n-m},$$

(b_2) $\quad \mathcal{A}_2 \equiv$

$$\sup_{r>0}\left(\int_{\complement E_{n-m}(0,2r)} \omega_1(x_{1,m})|x_{m+1,n}|^{(n-m+|\gamma_{1,m}|)(1+q/p')-(n+|\gamma|-\alpha)q} x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\right)^{1/q}$$
$$\times\left(\int_{E_{n-m}(0,r)} \omega^{1-p'}(x_{m+1,n}) x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\right)^{1/p'} < \infty,$$

(c_2) $\quad \mathcal{B}_2 \equiv \sup_{r>0}\left(\int_{E_{n-m}0,r)} \omega_1(x_{m+1,n}) x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\right)^{1/q}$

$$\times\left(\int\limits_{\complement E_{n-m}(0,2r)} \omega^{1-p'}(x_{m+1,n})|x_{1,m}|^{-((n-m+|\gamma_{1,m}|)(1/q+1/p')-n-|\gamma|+\alpha)(1-p')} x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\right)^{1/p'}$$

$< \infty$.

Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n_{k,+})$

$$\left(\int_{\mathbb{R}^n_{k,+}} |T_\alpha f(x)|^q \omega_1(x_{m+1,n})(x')^\gamma dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})(x')^\gamma dx\right)^{1/p}. \tag{3.6}$$

Similarly we can write the following weak variant of the Theorem 3.5.

Theorem 3.6 *Let $1 \leq p < q < \infty$ and let $T_{\alpha,\gamma}$, $0 < \alpha < n + |\gamma|$ be a weak (p,q)– admissible $B_{k,n}$– potential operators. Moreover, let $\omega(x_{m+1,n})$, $\omega_1(x_{m+1,n})$ be weight functions on $\mathbb{R}^k_{++}$ and conditions (a_2), (b_2), (c_2) be satisfied. Then there exists a constant C, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$*

$$\left(\int_{\left\{x\in\mathbb{R}^n_{k,+}\,:\,|T_\alpha f(x)|>\lambda\right\}} \omega_1(x_{m+1,n})(x')^\gamma dx\right)^{1/q} \leq \frac{C}{\lambda^q}\left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})(x')^\gamma dx\right)^{1/p}. \tag{3.7}$$

Corollary 3.3 *Let $1 < p < q < \infty$, $0 < \alpha < n + |\gamma|$ and $1/p - 1/q = \alpha/(n + |\gamma|)$. Moreover, let $\omega(x_{m+1,n})$, $\omega_1(x_{m+1,n})$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a_2), (b_2), (c_2) be satisfied. Then the operators $M_{\alpha,\gamma}$ and $I_{\alpha,\gamma}$ are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

Corollary 3.4 *Let $1 \leq p < q < \infty$, $0 < \alpha < n + |\gamma|$ and $1/p - 1/q = \alpha/(n + |\gamma|)$. Moreover, let $\omega(x_{m+1,n})$, $\omega_1(x_{m+1,n})$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a_2), (b_2), (c_2) be satisfied. Then the operators $M_{\alpha,\gamma}$ and $I_{\alpha,\gamma}$ are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

4 Proof of the theorems

We begin with the proof of Theorem 3.3 since Theorems 3.1, 3.2 and 3.4 can be proved by the same argument in Theorem 3.3.

Proof of Theorem 3.3. For $l \in Z$ we define $\widetilde{E}_l = \{x \in \mathbb{R}^n_{k,+} :\ 2^l < |x_{1,m}| \leq 2^{l+1}\}$, $\widetilde{E}_{l,1} = \{x \in \mathbb{R}^n_{k,+} :\ |x_{1,m}| \leq 2^{l-1}\}$, $\widetilde{E}_{l,2} = \{x \in \mathbb{R}^n_{k,+} :\ 2^{l-1} < |x_{1,m}| \leq 2^{l+2}\}$,

$\widetilde{E}_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x_{1,m}| > 2^{l+2}\}$. Then $\widetilde{E}_{l,2} = \widetilde{E}_{l-1} \cup \widetilde{E}_l \cup \widetilde{E}_{l+1}$ and the multiplicity of the covering $\left\{\widetilde{E}_{l,2}\right\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$\begin{aligned}|T_\alpha f(x)| &= \sum_{l\in Z} |T_\alpha f(x)|\chi_{\widetilde{E}_l}(x) \le \sum_{l\in Z} |T_\alpha f_{l,1}(x)|\, \chi_{\widetilde{E}_l}(x) \\ &+ \sum_{l\in Z} |T_\alpha f_{l,2}(x)|\, \chi_{\widetilde{E}_l}(x) + \sum_{l\in Z} |T_\alpha f_{l,3}(x)|\, \chi_{\widetilde{E}_l}(x) \\ &\equiv T_{\alpha,1} f(x) + T_{\alpha,2} f(x) + T_{\alpha,3} f(x), \end{aligned} \tag{4.1}$$

where $\chi_{\widetilde{E}_l}$ is the characteristic function of the set $\widetilde{E}_l$, $f_{l,i} = f\chi_{\widetilde{E}_{l,i}}$, $i = 1, 2, 3$. We shall estimate $\|T_{\alpha,1} f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,1}$ we have $|y_{1,m}| \le 2^{l-1} \le |x_{1,m}|/2$. Moreover, $\widetilde{E}_l \cap supp f_{l,1} = \emptyset$ and $|x_{1,m} - y_{1,m}| \ge |x_{1,m}|/2$. Hence, by (3.1)

$$\begin{aligned} T_{\alpha,1} f(x) &\le C \sum_{l\in Z} \left(\int_{\mathbb{R}^n_{k,+}} |f_{l,1}(y)| T^y |x|^{\alpha-n-|\gamma|} dy \right) \chi_{\widetilde{E}_l} \\ &\le C \int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{E_m(0,|x_{1,m}|/2)} T^y |x|^{\alpha-n-|\gamma|} |f(y)| (y')^\gamma dy \\ &\le C \int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{E_m(0,|x_{1,m}|/2)} (|x_{1,m}| + |x_{m+1,n} - y_{m+1,n}|)^{\alpha-n-|\gamma|} \\ &\times (|f(y)| (y')^\gamma dy_{1,m} dy_{m+1,n}) \end{aligned}$$

for any $x \in E_l$.

Note that by using the last inequality we have

$$\begin{aligned} &\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1} f(x)|^q \omega_1(x_{1,m}) (x')^\gamma dx \Big)^{1/q} \\ &\le C \Big(\int\limits_{\mathbb{R}^n_{k,+}} \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \int\limits_{E_m(0,|x_{1,m}|/2)} (|x_{1,m}| + |x_{m+1,n} - y_{m+1,n}|)^{\alpha-n-|\gamma|} \\ &\times |f(y)| y_{1,m}^{\gamma_{1,m}} dy_{1,m} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \Big)^q \omega_1(x_{1,m}) (x')^\gamma dx \Big)^{1/q}. \end{aligned}$$

For $x = (x_{1,m}, x_{m+1,n}) \in \mathbb{R}^n_{k,+}$ let

$$\begin{aligned} I(x_{1,m}) &= \int_{\mathbb{R}^{n-m}_{k-m,+}} \Big(\int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{E_m(0,|x_{1,m}|/2)} \Big(|x_{1,m}| + |x_{m+1,n} - y_{m+1,n}| \Big)^{\alpha-n-|\gamma|} \\ &\times |f(y_{1,m}, y_{m+1,n})| y_{1,m}^{\gamma_{1,m}} dy_{1,m} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \Big)^q (x_{m+1,k})^{\gamma_{m+1,k}} dx_{m+1,n} \\ &= \int_{\mathbb{R}^{n-m}_{k-m,+}} \Big(\int_{E_m(0,|x_{1,m}|/2)} \Big(\int_{\mathbb{R}^{n-m}_{k-m,+}} \Big(|x_{1,m}| + |x_{m+1,n} - y_{m+1,n}| \Big)^{\alpha-n-|\gamma|} \\ &\times |f(y_{1,m}, y_{m+1,n})| y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \Big) y_{1,m}^{\gamma_{1,m}} dy_{1,m} \Big)^q (x_{m+1,k})^{\gamma_{m+1,k}} dx_{m+1,n}. \end{aligned}$$

Note that by using the Minkowski and Young inequalities we obtain

$$I(x_{1,m}) \leq \Big[\int\limits_{E_m(0,|x_{1,m}|/2)} \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} |f(y_{1,m}, y_{m+1,n})|^p y^{\gamma_{m+1,k}}_{m+1,k} dy_{m+1,n} \Big)^{1/p}$$

$$\times \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \frac{(x_{m+1,k})^{\gamma_{m+1,k}} dx_{m+1,n}}{(|x_{1,m}| + |x_{m+1,n}|)^{r(n+|\gamma|-\alpha)}} \Big)^{1/r} y^{\gamma_{1,m}}_{1,m} dy_{1,m} \Big]^q$$

$$= \Big(\int_{E_m(0,|x_{1,m}|/2)} \|f(y_{1,m}, \cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y^{\gamma_{1,m}}_{1,m} dy_{1,m} \Big)^q$$

$$\times \Big(\int_{\mathbb{R}^{n-m}_{k-m,+}} \frac{(x_{m+1,k})^{\gamma_{m+1,k}} dx_{m+1,n}}{(|x_{1,m}| + |x_{m+1,n}|)^{r(n+|\gamma|-\alpha)}} \Big)^{q/r}$$

$$= \Big(\int\limits_{E_m(0,|x_{1,m}|/2)} \|f(y_{1,m}, \cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y^{\gamma_{1,m}}_{1,m} dy_{1,m} \Big)^q$$

$$\times \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \frac{|x_{1,m}|^{n-m+|\gamma_{m+1,k}|-(n+|\gamma|-\alpha)r} (x_{m+1,k})^{\gamma_{m+1,k}} dx_{m+1,n}}{(|x_{m+1,n}| + 1)^{r(n+|\gamma|-\alpha)}} \Big)^{q/r}$$

$$= C|x_{1,m}|^{(n-m+|\gamma_{m+1,k}|)q/r-(n+|\gamma|-\alpha)q} \Big(\int_{E_m(0,|x_{1,m}|/2)} \|f(y_{1,m}, \cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y^{\gamma_{1,m}}_{1,m} dy_{1,m} \Big)^q.$$

Integrating in $\mathbb{R}^m_{++}$ we get

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1} f(x)|^q \omega_1(x_{1,m}(x')^\gamma dx \Big)^{1/q}$$

$$\leq C \Big(\int\limits_{\mathbb{R}^m_{++}} \omega_1(x_{1,m}) |x_{1,m}|^{(n-m+|\gamma_{m+1,k}|)q/r-(n+|\gamma|-\alpha)q}$$

$$\times \Big(\int\limits_{E_m(0,|x_{1,m}|/2)} \|f(y_{1,m}, \cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y^{\gamma_{1,m}}_{1,m} dy_{1,m} \Big)^q x^{\gamma_{1,m}}_{1,m} dx_{1,m} \Big)^{1/q}.$$

Since $\mathcal{A}_1 < \infty$, the Hardy inequality

$$\Big(\int_{\mathbb{R}^m_{++}} \omega_1(x_{1,m}) |x_{1,m}|^{(n-m+|\gamma_{m+1,k}|)q/r-(n+|\gamma|-\alpha)}$$

$$\times \Big(\int_{E_m(0,|x_{1,m}|/2} \|f(y_{1,m}, \cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y^{\gamma_{1,m}}_{1,m} dy_{1,m} \Big)^q x^{\gamma_{1,m}}_{1,m} dx_{1,m} \Big)^{1/q}$$

$$\leq C \Big(\int_{\mathbb{R}^m_{++}} \|f(y_{1,m}, \cdot)\|^p_{\mathbb{R}^{n-m}_{k-m,+}} \omega(x_{1,m}) x^{\gamma_{1,m}}_{1,m} dx_{1,m} \Big)^{1/p}$$

holds and $C \leq c' \mathcal{A}_1$, where c' depends only on n and p. In fact the condition $\mathcal{A}_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [5], [14]). Hence, we obtain

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1} f(x)|^q \omega_1(x_{1,m})(x')^\gamma dx \Big)^{1/q}$$

$$\leq C \Big(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m})(x')^\gamma dx \Big)^{1/p}. \tag{4.2}$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As, easy to verify, for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,3}$ we have $|y_{1,m}| > 2|x_{1,m}|$ and $|x_{1,m} - y_{1,m}| \geq |y_{1,m}|/2$. Since $\widetilde{E}_l \cap supp f_{l,3} = \emptyset$, for $x \in \widetilde{E}_l$ by (3.1) we obtain

$$T_3 f(x) \leq C \int_{\mathbb{R}^{n-m}_{k-m,+}} \int_{\complement_{E_m(0,2|x_{1,m}|)}} |f(y)| \left(|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|\right)^{\alpha-n-|\gamma|} \times (y_{1,m}^{\gamma_{1,m}} dy_{1,m}\, y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}).$$

By using the last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3} f(x)|^q \omega_1(x_{1,m})(x')^\gamma dx \leq C^q \int\limits_{\mathbb{R}^n_{k,+}} \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \int\limits_{\complement_{E_m(0,2|x_{1,m}|)}} |f(y)| \Big(|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|\Big)^{\alpha-n-|\gamma|} \times y_{1,m}^{\gamma_{1,m}} dy_{1,m}\, y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \Big)^q \omega_1(x_{1,m})(x')^\gamma dx. \tag{4.3}$$

For $x = (x_{1,m}, x_{m+1,n}) \in \mathbb{R}^n$ let

$$I_1(x_{1,m}) = \int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \Big(\int\limits_{\complement_{E_m(0,2|x_{1,m}|)}} \int\limits_{\mathbb{R}^{n-m}_{k-m,+}} |f(y)| \Big(|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|\Big)^{\alpha-n-|\gamma|} \times y_{1,m}^{\gamma_{1,m}} dy_{1,m}\, y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \Big)^q (x_{m+1,k})^{\gamma_{m+1,k}}\, dx_{m+1,n}.$$

Note that by using the Minkowski and Young inequalities we obtain

$$I_1(x_{1,m}) \leq \Big[\int_{\complement_{E_m(0,2|x_{1,m}|)}} \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} |f(y)|^p y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n} \Big)^{1/p} \times \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \frac{y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}}{(|y_{1,m}| + |y_{m+1,n}|)^{(n+|\gamma|-\alpha)r}} \Big)^{1/r} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \Big]^q$$
$$= C\Big(\int\limits_{\complement_{E_m(0,2|x_{1,m}|)}} |y_{1,m}|^{(n-m+|\gamma_{m+1,k}|)1/r-n-|\gamma|+\alpha} \|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \Big)^q \times \Big(\int\limits_{\mathbb{R}^{n-m}_{k-m,+}} \frac{y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}}{(|y_{m+1,n}| + 1)^{(n+|\gamma|-\alpha)r}} \Big)^{q/r}$$
$$= C\Big(\int_{\complement_{E_m(0,2|x_{1,m}|)}} |y_{1,m}|^{(n-m+|\gamma_{m+1,k}|)1/r-n-|\gamma|+\alpha} \|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}} y_{1,m}^{\gamma_{1,m}} dy_{1,m} \Big)^q.$$

Integrating over $\mathbb{R}^m_{++}$ we get

$$\Big(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,3}f(x)|^q\omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^m_{++}}\Big(\int\limits_{\complement E_m(0,2|x_{1,m}|)}|y_{1,m}|^{(n-m+|\gamma_{m+1,k}|)1/r-n-|\gamma|+\alpha}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)^q$$
$$\times\,\omega_1(x_{1,m})x_{1,m}^{\gamma_{1,m}}dx_{1,m}\Big)^{1/q}.$$

Since $\mathcal{B}_1<\infty$, the Hardy inequality

$$\Big(\int_{\mathbb{R}^m_{++}}\omega_1(x_{1,m})\Big(\int\limits_{\complement E_m(0,2|x_{1,m}|)}|y_{1,m}|^{n-m-|\gamma|+\alpha}\|f(y_{1,m},\cdot)\|_{p,\mathbb{R}^{n-m}_{k-m,+}}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)^q x_{1,m}^{\gamma_{1,m}}dx_{1,m}\Big)^{1/q}$$
$$\le C\Big(\int\limits_{\mathbb{R}^m_{++}}\|f(y_{1,m},\cdot)\|^p_{p,\mathbb{R}^{n-m}_{k-m,+}}\omega(x_{1,m})|x_{1,m}|^{-((n-m+|\gamma_{m+1,k}|)1/r-n-|\gamma|+\alpha)}x_{1,m}^{\gamma_{1,m}}dx_{1,m}\Big)^{1/p}$$
$$=C\Big(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x_{1,m})(x')^\gamma dx\Big)^{1/p}$$

holds and $C\le c'\mathcal{B}_1$, where c' depends only on n, γ and p. In fact the condition $\mathcal{B}_1<\infty$ is necessary and sufficient for the validity of this inequality (see [5], [14]). Hence, we obtain

$$\Big(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,3}f(x)|^q\omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x_{1,m})(x')^\gamma dx\Big)^{1/p}. \tag{4.4}$$

Finally, we estimate $\|T_{\alpha,2}f\|_{L_{q,\omega_1,\gamma}}$. From $L_{p,\gamma}(\mathbb{R}^n_{k,+})\to L_{q,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of $T_{\alpha,\gamma}$ and condition (a_1) we have

$$\Big(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,2}f(x)|^q\omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$=\Big(\int_{\mathbb{R}^n_{k,+}}\Big(\sum_{l\in Z}\Big|T_\alpha f_{l,2}(x)\Big|\chi_{\widetilde{E}_l}(x)\Big)^q\omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$=\Big(\int_{\mathbb{R}^n_{k,+}}\Big(\sum_{l\in Z}\Big|T_\alpha f_{l,2}(x)\Big|^q\chi_{\widetilde{E}_l}(x)\Big)\omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$=\Big(\sum_{l\in Z}\int_{\widetilde{E}_l}\Big|T_\alpha f_{l,2}(x)\Big|^q\omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$\le\Big(\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y_{1,m})\int_{\mathbb{R}^n}\Big|T_\alpha f_{l,2}(x)\Big|^q(x')^\gamma dx\Big)^{1/q}$$
$$\le\|T_\alpha\|\Big(\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y_{1,m})\Big(\int_{\mathbb{R}^n_{k,+}}\Big|f_{l,2}(x)\Big|^p(x')^\gamma dx\Big)^{q/p}\Big)^{1/q}$$
$$=\|T_\alpha\|\Big(\sum_{l\in Z}\sup_{y\in\widetilde{E}_l}\omega_1(y_{1,m})\Big(\int_{\widetilde{E}_{l,2}}|f(x)|^p(x')^\gamma dx\Big)^{q/p}\Big)^{1/q},$$

where $\|T_\alpha\| \equiv \|T_\alpha\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})\to L_{q,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in \widetilde{E}_{l,2}$, $2^{l-1} < |x_{1,m}| \le 2^{l+2}$, we have by condition (a_1)

$$\sup_{y\in\widetilde{E}_l} (\omega_1(y_{1,m}))^{p/q} = \sup_{2^{l-1}<|y_{1,m}|\le 2^{l+2}} (\omega_1(y_{1,m}))^{p/q}$$
$$\le \sup_{|x_{1,m}|/8<|y_{1,m}|<8|x_{1,m}|} (\omega_1(y_{1,m}))^{p/q} \le b\,\omega(x_{1,m})$$

for almost all $x \in \widetilde{E}_{l,2}$. Therefore

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,2}f(x)|^q \omega_1(x_{1,m})(x')^\gamma dx\Big)^{1/q}$$
$$\le \|T_\alpha\| b \sum_{l\in Z} \Big(\int_{\widetilde{E}_{l,2}} |f(x)|^p \omega(x_{1,m}) dx\Big)^{1/p}$$
$$\le C\Big(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{1,m})(x')^\gamma dx\Big)^{1/p}, \tag{4.5}$$

where $C = 3\|T_\alpha\| b$, since the multiplicity of covering $\left\{\widetilde{E}_{l,2}\right\}_{l\in Z}$ is equal to 3. Inequalities (4.1), (4.2), (4.4), (4.5) imply (3.4) which completes the proof of Theorem 3.3.

Proof of Theorem 3.5. For $l \in Z$ we define $\widetilde{E}_l = \{x \in \mathbb{R}^n_{k,+} : 2^l < |x_{m+1,n}| \le 2^{l+1}\}$, $\widetilde{E}_{l,1} = \{x \in \mathbb{R}^n_{k,+} : |x_{m+1,n}| \le 2^{l-1}\}$, $\widetilde{E}_{l,2} = \{x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x_{m+1,n}| \le 2^{l+2}\}$, $\widetilde{E}_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x_{m+1,n}| > 2^{l+2}\}$. Then $\widetilde{E}_{l,2} = \widetilde{E}_{l-1} \cup \widetilde{E}_l \cup \widetilde{E}_{l+1}$ and the multiplicity of the covering $\left\{\widetilde{E}_{l,2}\right\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$|T_\alpha f(x)| = \sum_{l\in Z} |T_\alpha f(x)| \chi_{\widetilde{E}_l}(x) \le \sum_{l\in Z} \Big|T_\alpha f_{l,1}(x)\Big| \chi_{\widetilde{E}_l}(x)$$
$$+ \sum_{l\in Z} |T_\alpha f_{l,2}(x)|\, \chi_{\widetilde{E}_l}(x) + \sum_{l\in Z} |T_\alpha f_{l,3}(x)|\, \chi_{\widetilde{E}_l}(x)$$
$$\equiv T_{\alpha,1}f(x) + T_{\alpha,2}f(x) + T_{\alpha,3}f(x), \tag{4.6}$$

where $\chi_{\widetilde{E}_l}$ is the characteristic function of the set $\widetilde{E}_l$, $f_{l,i} = f\chi_{\widetilde{E}_{l,i}}$, $i = 1,2,3$. We shall estimate $\|T_{\alpha,1}f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,1}$ we have $|y_{m+1,n}| \le 2^{l-1} \le |x_{m+1,n}|/2$. Moreover, $\widetilde{E}_l \cap supp f_{l,1} = \emptyset$ and $|x_{m+1,n} - y_{m+1,n}| \ge |x_{m+1,n}|/2$. Hence, by (3.1)

$$T_{\alpha,1}f(x) \le C \sum_{l\in Z} \Big(\int_{\mathbb{R}^n_{k,+}} |f_{l,1}(y)| T^y |x|^{\alpha-n-|\gamma|} dy\Big) \chi_{\widetilde{E_l}}$$
$$\le C \int_{\mathbb{R}^m_{++}} \int_{E_{n-m}(0,|x_{m+1,n}|/2)} T^y |x|^{\alpha-n-|\gamma|} |f(y)| (y')^\gamma dy$$
$$\le C \int_{\mathbb{R}^m_{++}} \int_{E_{n-m}(0,|x_{m+1,n}|/2)} (|x_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{\alpha-n-|\gamma|}$$
$$\times (|f(y)|(y')^\gamma dy_{1,m} dy_{m+1,n}) \tag{4.7}$$

for any $x\in E_l$. By using the last inequality we have

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1}f(x)|^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int\limits_{\mathbb{R}^n_{k,+}}\Big(\int\limits_{\mathbb{R}^m_{++}}\int\limits_{E_{n-m}(0,|x_{m+1,n}|/2)}\Big(|x_{m+1,n}|+|x_{1,m}-y_{1,m}|\Big)^{\alpha-n-|\gamma|}$$
$$\times|f(y)|y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)^q\omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}.$$

For $x=(x_{1,m},x_{m+1,n})\in\mathbb{R}^n_{k,+}$ let

$$I(x_{1,m})=\int_{\mathbb{R}^m_{++}}\Big(\int_{\mathbb{R}^m_{++}}\int_{E_{n-m}(0,|x_{m+1,n}|/2)}\Big(|x_{m+1,n}|+|x_{1,m}-y_{1,m}|\Big)^{\alpha-n-|\gamma|}$$
$$\times|f(y_{1,m},y_{m+1,n})|y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)^q(x_{1,m})^{\gamma_{1,m}}dx_{1,m}$$
$$=\int_{\mathbb{R}^m_{++}}\Big(\int_{E_{n-m}(0,|x_{m+1,n}|/2)}\Big(\int_{\mathbb{R}^m_{++}}\Big(|x_{m+1,n}|+|x_{1,m}-y_{1,m}|\Big)^{\alpha-n-|\gamma|}$$
$$\times|f(y_{1,m},y_{m+1,n})|y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q(x_{1,m})^{\gamma_{1,m}}dx_{1,m}.$$

By using the Minkowski and Young inequalities we obtain

$$I(x_{1,m})\le\Big[\int\limits_{E_{n-m}(0,|x_{m+1,n}|/2)}\Big(\int\limits_{\mathbb{R}^m_{++}}|f(y_{1,m},y_{m+1,n})|^p y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)^{1/p}$$
$$\times\Big(\int\limits_{\mathbb{R}^m_{++}}\frac{(x_{1,m})^{\gamma_{1,m}}dx_{1,m}}{(|x_{1,m}|+|x_{m+1,n}|)^{r(n+|\gamma|-\alpha)}}\Big)^{1/r}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big]^q$$
$$=\Big(\int_{E_{n-m}(0,|x_{m+1,n}|/2)}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q$$
$$\times\Big(\int_{\mathbb{R}^m_{++}}\frac{(x_{1,m})^{\gamma_{1,m}}dx_{1,m}}{(|x_{1,m}|+|x_{m+1,n}|)^{r(n+|\gamma|-\alpha)}}\Big)^{q/r}$$
$$=\Big(\int\limits_{E_{n-m}(0,|x_{m+1,n}|/2)}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q$$

$$\times\Big(\int\limits_{\mathbb{R}^m_{++}}|x_{m+1,n}|^{n-m+|\gamma_{1,m}|-(n+|\gamma|-\alpha)r}(x_{1,m})^{\gamma_{1,m}}dx_{1,m}(|x_{1,m}|+1)^{r(n+|\gamma|-\alpha)}\Big)^{q/r}$$
$$=C|x_{m+1,n}|^{(n-m+|\gamma_{1,m}|)q/r-(n+|\gamma|-\alpha)q}$$
$$\times\Big(\int_{E_{n-m}(0,|x_{m+1,n}|/2)}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,n}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q.$$

Integrating in $\mathbb{R}^{n-m}$ we get

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1}f(x)|^q \omega_1(x_{m+1,n}(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int\limits_{\mathbb{R}^{n-m}} \omega_1(x_{m+1,n})|x_{m+1,n}|^{(n-m+|\gamma_{1,m}|)q/r-(n+|\gamma|-\alpha)q}$$
$$\times\Big(\int\limits_{E_{n-m}(0,|x_{m+1,n}|/2)} \|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}\Big)^q x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\Big)^{1/q}.$$

Since $\mathcal{A}_2 < \infty$, the Hardy inequality

$$\Big(\int_{\mathbb{R}^{n-m}} \omega_1(x_{m+1,n})|x_{m+1,n}|^{(n-m+|\gamma_{1,m}|)q/r-(n+|\gamma|-\alpha)}$$
$$\times\Big(\int_{E_{n-m}(0,|x_{m+1,n}|/2} \|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}} y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}\Big)^q x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^{n-m}} \|f(\cdot,y_{m+1,n})\|^p_{\mathbb{R}^m_{++}} \omega(x_{m+1,n}) x_{m+1,k}^{\gamma_{m+1,k}} dx_{m+1,n}\Big)^{1/p}$$

holds and $C \le c'\mathcal{A}_2$, where c' depends only on n and p. In fact the condition $\mathcal{A}_2 < \infty$ is necessary and sufficient for the validity of this inequality (see [5], [14]). Hence, we obtain

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,1}f(x)|^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})(x')^\gamma dx\Big)^{1/p}. \tag{4.8}$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As, easy to verify, for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,3}$ we have $|y_{m+1,n}| > 2|x_{m+1,n}|$ and $|x_{m+1,n} - y_{m+1,n}| \ge |y_{m+1,n}|/2$. Since $\widetilde{E}_l \cap supp f_{l,3} = \emptyset$, for $x \in \widetilde{E}_l$ by (3.1) we obtain

$$T_3 f(x) \le C\int_{\mathbb{R}^m_{++}} \int_{\complement E_{n-m}(0,2|x_{m+1,n}|)} |f(y)|\,(|y_{m+1,n}| + |x_{1,m} - y_{1,m}|)^{\alpha-n-|\gamma|}$$
$$\times\; y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}\, y_{1,m}^{\gamma_{1,m}} dy_{1,m}.$$

Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3}f(x)|^q \omega_1(y_{m+1,n})(x')^\gamma dx$$
$$\le C^q \int\limits_{\mathbb{R}^n_{k,+}} \Big(\int\limits_{\mathbb{R}^m_{++}} \int\limits_{\complement E_{n-m}(0,2|x_{1,m}|)} |f(y)|\Big(|y_{1,m}| + |x_{m+1,n} - y_{m+1,n}|\Big)^{\alpha-n-|\gamma|}$$
$$\times\, y_{1,m}^{\gamma_{1,m}} dy_{1,m}\, y_{m+1,k}^{\gamma_{m+1,k}} dy_{m+1,n}\Big)^q \omega_1(x_{1,m})(x')^\gamma dx.$$

For $x=(x_{1,m},x_{m+1,n})\in\mathbb{R}^n$ let

$$I_1(x_{1,m})=\int\limits_{\mathbb{R}^m_{++}}\Big(\int\limits_{\complement E_{n-m}(0,2|x_{m+1,n}|)}\int\limits_{\mathbb{R}^m_{++}}|f(y)|\Big(|y_{m+1,n}|+|x_{1,m}-y_{1,m}|\Big)^{\alpha-n-|\gamma|}$$
$$\times\, y_{1,m}^{\gamma_{1,m}}dy_{1,m}\,y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q(x_{1,m})^{\gamma_{1,m}}\,dx_{1,m}.$$

Using the Minkowski and Young inequalities we obtain

$$I_1(x_{1,m})\le\Big[\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}\Big(\int\limits_{\mathbb{R}^m_{++}}|f(y)|^p y_{1,m}^{\gamma_{1,m}}dy_{1,m}\Big)^{1/p}$$
$$\times\Big(\int\limits_{\mathbb{R}^m_{++}}\frac{y_{1,m}^{\gamma_{1,m}}dy_{1,m}}{(|y_{1,m}|+|y_{m+1,n}|)^{(n+|\gamma|-\alpha)r}}\Big)^{1/r}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big]^q$$
$$=C\Big(\int\limits_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{(n-m+|\gamma_{1,m}|)1/r-n-|\gamma|+\alpha}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q$$
$$\times\Big(\int\limits_{\mathbb{R}^m_{++}}\frac{y_{1,m}^{\gamma_{1,m}}dy_{1,m}}{(|y_{1,m}|+1)^{(n+|\gamma|-\alpha)r}}\Big)^{q/r}$$
$$=C\Big(\int\limits_{\complement E_{n-m}(0,2|x_{1,m}|)}|y_{m+1,n}|^{(n-m+|\gamma_{1,m}|)1/r-n-|\gamma|+\alpha}\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q.$$

Integrating over $\mathbb{R}^{n-m}$ we get

$$\Big(\int_{\mathbb{R}^n_{k,+}}|T_{\alpha,3}f(x)|^q\omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^{n-m}}\Big(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{(n-m+|\gamma_{1,m}|)1/r-n-|\gamma|+\alpha}$$
$$\times\Big\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q\omega_1(x_{m+1,n})x_{m+1,k}^{\gamma_{m+1,k}}dx_{m+1,n}\Big)^{1/q}.$$

Since $\mathcal{B}_2<\infty$, the Hardy inequality

$$\Big(\int_{\mathbb{R}^{n-m}}\omega_1(x_{m+1,n})\Big(\int_{\complement E_{n-m}(0,2|x_{m+1,n}|)}|y_{m+1,n}|^{(n-m+|\gamma_{1,m}|)1/r-n-|\gamma|+\alpha}$$
$$\times\|f(\cdot,y_{m+1,n})\|_{p,\mathbb{R}^m_{++}}y_{m+1,k}^{\gamma_{m+1,k}}dy_{m+1,n}\Big)^q x_{m+1,k}^{\gamma_{m+1,k}}dx_{m+1,n}\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^{n-m}}\|f(\cdot,y_{m+1,n})\|^p_{p,\mathbb{R}^{n-m}_{k-m,+}}\omega(x_{m+1,n})|x_{m+1,n}|^{-((n-m+|\gamma_{1,m}|)1/r-n-|\gamma|+\alpha)}$$
$$\times\, x_{m+1,k}^{\gamma_{m+1,k}}dx_{m+1,n}\Big)^{1/p}=C\Big(\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(x_{m+1,n})(x')^\gamma dx\Big)^{1/p}$$

holds and $C \le c'\mathcal{B}_1$, where c' depends only on n, γ and p. In fact the condition $\mathcal{B}_2 < \infty$ is necessary and sufficient for the validity of this inequality (see [5], [14]). Hence, we obtain

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,3}f(x)|^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$\le C\Big(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})(x')^\gamma dx\Big)^{1/p}. \tag{4.9}$$

Finally, we estimate $\|T_{\alpha,2}f\|_{L_{q,\omega_1,\gamma}}$. From $L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{q,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of $T_{\alpha,\gamma}$ and condition (a_1) we have

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,2}f(x)|^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$= \Big(\int_{\mathbb{R}^n_{k,+}} \Big(\sum_{l\in Z} \Big|T_\alpha f_{l,2}(x)\Big| \chi_{\widetilde{E}_l}(x)\Big)^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$= \Big(\int_{\mathbb{R}^n_{k,+}} \Big(\sum_{l\in Z} \Big|T_\alpha f_{l,2}(x)\Big|^q \chi_{\widetilde{E}_l}(x)\Big) \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$= \Big(\sum_{l\in Z} \int_{\widetilde{E}_l} \Big|T_\alpha f_{l,2}(x)\Big|^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$\le \Big(\sum_{l\in Z} \sup_{y\in\widetilde{E}_l} \omega_1(x_{m+1,n}) \int_{\mathbb{R}^n} \Big|T_\alpha f_{l,2}(x)\Big|^q (x')^\gamma dx\Big)^{1/q}$$
$$\le \|T_\alpha\| \Big(\sum_{l\in Z} \sup_{y\in\widetilde{E}_l} \omega_1(y_{m+1,n}) \Big(\int_{\mathbb{R}^n_{k,+}} \Big|f_{l,2}(x)\Big|^p (x')^\gamma dx\Big)^{q/p}\Big)^{1/q}$$
$$= \|T_\alpha\| \Big(\sum_{l\in Z} \sup_{y\in\widetilde{E}_l} \omega_1(y_{m+1,n}) \Big(\int_{\widetilde{E}_{l,2}} |f(x)|^p (x')^\gamma dx\Big)^{q/p}\Big)^{1/q},$$

where $\|T_\alpha\| \equiv \|T_\alpha\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{q,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in \widetilde{E}_{l,2}$, $2^{l-1} < |x_{m+1,n}| \le 2^{l+2}$, we have by condition (a_1)

$$\sup_{y\in\widetilde{E}_l} (\omega_1(y_{m+1,n}))^{p/q} = \sup_{2^{l-1}<|y_{m+1,n}|\le 2^{l+2}} (\omega_1(y_{m+1,n}))^{p/q}$$
$$\le \sup_{x_{m+1,n}} |/8 < |y_{m+1,n}| < 8|x_{m+1,n}| (\omega_1(y_{m+1,n}))^{p/q} \le b\omega(y_{m+1,n})$$

for almost all $x \in \widetilde{E}_{l,2}$. Therefore

$$\Big(\int_{\mathbb{R}^n_{k,+}} |T_{\alpha,2}f(x)|^q \omega_1(x_{m+1,n})(x')^\gamma dx\Big)^{1/q}$$
$$\le \|T_\alpha\| b \sum_{l\in Z} \Big(\int_{\widetilde{E}_{l,2}} |f(x)|^p \omega(x_{m+1,n}) dx\Big)^{1/p}$$
$$\le C\Big(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x_{m+1,n})(x')^\gamma dx\Big)^{1/p}, \tag{4.10}$$

where $C = 3\|T_\alpha\| b$, since the multiplicity of covering $\left\{ \widetilde{E}_{l,2} \right\}_{l\in Z}$ is equal to 3. Inequalities (4.6), (4.8), (4.9), (4.10) imply (3.6) which completes the proof of Theorem 3.5.

Similarly we can prove Theorem 3.6 which is the weak variant of Theorem 3.5.

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