Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (1), 64-73 (2016).
Series of Physical-Technical and Mathematical Sciences.

The initial-boundary value problem for one fouth order hyperbolic equation with memory operator

Sevda E. Isayeva · Gulnara D. Shukurova

Received: 30.09.2015/ Revised: 22.02.2016 / Accepted: 14.04.2016

Abstract. *In this work we consider the initial-boundary value problem for one fourth order semilinear hyperbolic equation with memory operator. We prove the existence of a bounded absorbing set for this problem.*

Keywords. Hysteresis · Memory operator · Semigroup · A bounded absorbing set · Fourth order hyperbolic equation

Mathematics Subject Classification (2010): 35L10, 35L15, 35L20

1 Introduction

The equations with memory operator, especially the equations with hysteresis have a great importance among the nonlinear partial differential equations. Hysteresis relations appear in friction, ferromagnetism, superconductivity. The research of solutions of partial differential equations with hysteresis nonlinearities is a nontrivial problem. The equations, when hysteresis operator is under the operator of differentiation with respect to the time variable, have special difficulties.

From a practical point of view, the research of an asymptotic behavior of the dynamic system which is originated by the corresponding initial-boundary value problem, have a special significance. Such problems were researched, for example, in [10].

In this work the initial-boundary value problem for one semilinear fourth order hyperbolic equation with memory operator is considered and the existence of a bounded absorbing set for this problem is proved.

Sevda E. Isayeva
Baku State University
E-mail: isayevasevda@rambler.ru

Gulnara D. Shukurova
Baku State University

2 Problem statement. Basic results

Let $\Omega \subset R^N \, (N \geq 1)$ be a bounded, connected set with a smooth boundary Γ. We consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left[u + F(u)\right] + \Delta^2 u + |u|^p u = h \text{ in } Q = \Omega \times (0,T), \tag{2.1}$$

$$u = 0, \ \ \Delta u = 0, \ \ (x,t) \in \Gamma \times [0,T], \tag{2.2}$$

$$\left[u + F(u)\right]|_{t=0} = u^{(0)} + w^{(0)}, \ \frac{\partial u}{\partial t}\Big|_{t=0} = u^{(1)} \ \text{ in } \Omega, \tag{2.3}$$

where $p > 0$ and nonlinear operator F acts from $M\left(\Omega; C^0([0,T])\right)$ to $M\left(\Omega; C^0([0,T])\right)$. Here $M\left(\Omega; C^0([0,T])\right)$ is a space of measurable functions, which act from Ω to $C^0([0,T])$. We assume that the operator F is a memory operator, which is applied at each point $x \in \Omega$ independently, that is the output $\left[F(u(x,\cdot))\right](t)$ depends on $u(x,\cdot)|_{[0,t]}$, but not on $u(y,\cdot)|_{[0,t]}$ for any $y \neq x$ (see [10]).

We assume that, it holds the following conditions for operator F:

$$\begin{cases} \text{if for arbitrary } \upsilon_1, \ \upsilon_2 \in M\left(\Omega; C^0([0,T])\right) \ \text{ and for arbitrary } t \in [0,T] \\ \upsilon_1 = \upsilon_2 \ \text{ in } \ [0,t], \ \text{ then } \left[F(\upsilon_1)\right](\cdot,t) = \left[F(\upsilon_2)\right](\cdot,t) \ \text{ a.e. in } \Omega; \end{cases} \tag{2.4}$$

$$\begin{cases} \text{if } \upsilon_n \in M\left(\Omega; C^0([0,T])\right) \ \text{ and } \upsilon_n \to \upsilon \text{ uniformly,} \\ \text{then } \ F(\upsilon_n) \to F(\upsilon) \ \text{ uniformly in } [0,T] \ , \text{ a.e. in } \Omega; \end{cases} \tag{2.5}$$

$$\begin{cases} \text{there exist } L > 0, \ g \in L^2(\Omega) \text{ such that , for arbitrary } \upsilon \in M\left(\Omega; C^0([0,T])\right) \\ \left\|\left[F(\upsilon)\right](x,\cdot)\right\|_{C^0([0,T])} \leq L \left\|\upsilon(x,\cdot)\right\|_{C^0([0,T])} + g(x), \ \text{ a.e. in } \Omega; \end{cases} \tag{2.6}$$

$$\begin{cases} \text{if } \upsilon \in M\left(\Omega; C^0([0,T])\right) \text{ and for arbitrary } [t_1,t_2] \subset [0,T] \\ \upsilon(x,\cdot) \text{ is affine in } [t_1,t_2], \text{ a.e. in } \Omega, \\ \text{then } \left\{\left[F(\upsilon)\right](x,t_2) - \left[F(\upsilon)\right](x,t_1)\right\}\left[\upsilon(x,t_2) - \upsilon(x,t_1)\right] \geq 0, \ \text{ a.e. in } \Omega, \end{cases} \tag{2.7}$$

$$\begin{cases} \text{there exists } 0 < L_1 < 1 \text{ such that , for arbitrary } \upsilon \in M\left(\Omega; C^0([0,T])\right) \text{ and for} \\ \forall\, [t_1,t_2] \subset [0,T], \ \text{ if } \upsilon(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \text{ then} \\ \left|\left[F(\upsilon)\right](x,t_2) - \left[F(\upsilon)\right](x,t_1)\right| \leq L_1 \left|\upsilon(x,t_2) - \upsilon(x,t_1)\right| \ \text{ a.e. in } \Omega. \end{cases} \tag{2.8}$$

As an example we can represent the Bouc operator (see [10] or [2]).

Let $V = H_0^2(\Omega) \bigcap L^{p+2}(\Omega)$. We assume that

$$u^{(0)} \in V, \ w^{(0)} \in L^2(\Omega), u^{(1)} \in L^2(\Omega), \tag{2.9}$$

$$h \in L^2(\Omega). \tag{2.10}$$

Definition 2.1 *A function* $u \in L^2(0,T;V) \bigcap H^1\left(0,T;L^2(\Omega)\right)$ *is said to be a solution of problem* (2.1)-(2.3) *if* $F(u) \in L^2(Q)$ *and*

$$\int_Q \left\{ -\frac{\partial u}{\partial t} \cdot \frac{\partial \upsilon}{\partial t} - \left[u + F(u)\right]\frac{\partial \upsilon}{\partial t} + \Delta u \cdot \Delta \upsilon + |u|^p u\upsilon \right\} dxdt$$

$$= \int_Q h\upsilon dxdt + \int_\Omega \left[u^{(0)}(x) + w^{(0)}(x) + u^{(1)}(x)\right] \ \upsilon(x,0)\, dx,$$

for every $\upsilon \in L^2(0,T;V) \bigcap H^1\left(0,T;L^2(\Omega)\right)$ $(\upsilon(\cdot,T) = 0$ *a.e. in* $\Omega)$.

Well posedness of problem (2.1)-(2.3) without $F(u)$, was studied by different authors (see, for example [9]). The corresponding problem for the parabolic equation without nonlinear term $|u|^p u$ and with Δu was studied in [10]. Analogous problems were investigated in [3]-[6].

In this work, we study the existence of a bounded absorbing set for problem (2.1)-(2.3). It is proved (see [4]) the theorem about the existence and uniqueness of solutions of problem (2.1)-(2.3) under conditions (2.4)-(2.10),

$$p \le \frac{2}{N-2}, N \ge 3, \tag{2.11}$$

and for $\forall u, \upsilon \in M\left(\Omega; W^{1,1}(0,T)\right)$

$$\frac{\partial}{\partial t}\left[F(u) - F(\upsilon)\right] \le L_2 \frac{\partial}{\partial t}(u-\upsilon). \tag{2.12}$$

By the condition (2.11): $V = H_0^2(\Omega) \bigcap L^{p+2}(\Omega) = H_0^2(\Omega)$. We set $E = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Since under the conditions (2.4)-(2.12), the problem (2.1)-(2.3) has a unique solution, by well-known scheme (see, for example [1]) we can prove that the problem (2.1)-(2.3) generates the semigroup $\{S(t)\}_{t\ge 0}$ in E by the formula:

$$S(t)\left(u^{(0)}, u^{(1)}, w^{(0)}\right) = (u, u_t, w),$$

where u is a unique solution of this problem.

We introduce the following functional

$$\Phi_\eta(y) = \frac{1}{2}\|q\|^2 + \frac{1}{2}\|\Delta u\|^2 - (h,u) + \frac{1}{p+2}\left(|u|^{p+2}, 1\right) + \eta\left[(u,q) + \frac{1}{2}\|u\|^2 (F(u),1)\right],$$

where $y = (u, F(u), q)$, η is some positive constant. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and scalar product in $L^2(\Omega)$.

We divide $[0,T]$ by points $t_n = nk,\ n = 0,1,...,m$ into m parts and introduce the following notations:

$$u_m^0 = u^{(0)},\ w_m^0 = w^{(0)},\ u_m^1 = u^{(0)} + ku^{(1)}, u_m^{-1} = u^{(0)} - ku^{(1)},$$

$$u_m^n(x) = u(x, nk),\ n = 2,...,m,$$

$$w_m^n(x) = [F(u_m)](x, nk),\ n = 1,...,m, \quad \text{a.e. in } \Omega,$$

where

$u_m(x,\cdot) =$ linear time interpolate of $u(x,nk)$ for $n = 1,...,m$ a.e. in Ω.

We define $w_m(x,\cdot)$ similarly. Setting

$$\Phi_{\eta m}^n = \Phi_\eta\left(u_m^n, w_m^n, \frac{u_m^n - u_m^{n-1}}{k}\right),\ n = 1,...,m, \text{ a.e. in } \Omega,$$

consider the problem

$$\frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2} + \frac{u_m^n - u_m^{n-1}}{k}$$

$$+\frac{w_m^n - w_m^{n-1}}{k} + \Delta^2 u_m^n + |u_m^n|^p u_m^n = h \text{ in } V',\ n = 1,...,m, \tag{2.13}$$

$$u_m^0 = u^{(0)}, \;\; w_m^0 = w^{(0)}, u_m^1 = u^{(0)} + ku^{(1)}, u_m^{-1} = u^{(0)} - ku^{(1)}, \tag{2.14}$$

and functional

$$\begin{aligned} \varPhi_{\eta m}^n = \; & \tfrac{1}{2}\left\|\tfrac{u_m^n - u_m^{n-1}}{k}\right\|^2 + \tfrac{1}{2}\left\|\varDelta^2 u_m^n\right\|^2 - (h, u_m^n) + \tfrac{1}{p+2}\left(|u_m^n|^{p+2}, 1\right) + \\ & +\eta\left[\left(u_m^n, \tfrac{u_m^n - u_m^{n-1}}{k}\right) + \tfrac{1}{2}\left\|u_m^n\right\|^2 + (w_m^n, 1)\right] \; . \end{aligned} \tag{2.15}$$

Lemma 2.1 *Assume that* (2.4)-(2.12) *hold and let* $u_m^n(x)$ *be a solution of problem* (2.13)-(2.14). *Then there exists a natural number* m_1 *such that, for arbitrary* $m > m_1$ *it holds the following inequality*

$$\frac{\varPhi_{\eta m}^n - \varPhi_{\eta m}^{n-1}}{k} + \delta \varPhi_{\eta m}^n \le C\,,\; n = 1, 2, ..., m\,, \tag{2.16}$$

where C *is a positive constant independent of* m*.*

Proof. By (2.13), (2.15) we have

$$\begin{aligned} \frac{\varPhi_{\eta m}^n - \varPhi_{\eta m}^{n-1}}{k} &= \frac{1}{2k}\left(\frac{u_m^n - u_m^{n-1}}{k} - \frac{u_m^{n-1} - u_m^{n-2}}{k}, \frac{u_m^n - u_m^{n-1}}{k} + \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) \\ & + \frac{1}{2k}\left(\varDelta u_m^n - \varDelta u_m^{n-1}, \varDelta u_m^n + \varDelta u_m^{n-1}\right) - \frac{1}{k}\left(h, u_m^n - u_m^{n-1}\right) + \frac{1}{(p+2)k} \\ & \times\left(|u_m^n|^{p+2} - \left|u_m^{n-1}\right|^{p+2}, 1\right) + \frac{\eta}{k}\left[\left(u_m^n, \frac{u_m^n - u_m^{n-1}}{k}\right) - \left(u_m^{n-1}, \frac{u_m^{n-1} - u_m^{n-2}}{k}\right)\right. \\ & \left. + \frac{1}{2}\left(u_m^n - u_m^{n-1}, u_m^n + u_m^{n-1}\right) + \left(\frac{w_m^n - w_m^{n-1}}{k}, 1\right)\right] \\ & = \frac{1}{2}\left(\frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}, 2\frac{u_m^n - u_m^{n-1}}{k} - \frac{u_m^n - u_m^{n-1}}{k} + \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) \\ & + \frac{1}{2}\left(\frac{\varDelta u_m^n - \varDelta u_m^{n-1}}{k}, 2\varDelta u_m^n - \varDelta u_m^n + \varDelta u_m^{n-1}\right) \\ & - \left(h, \frac{u_m^n - u_m^{n-1}}{k}\right) + \frac{1}{p+2}\left(\frac{|u_m^n|^{p+2} - \left|u_m^{n-1}\right|^{p+2}}{k}, 1\right) \\ & + \eta\left[\left(u_m^n, \frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}\right) + \frac{1}{k}\left(u_m^n, \frac{u_m^{n-1} - u_m^{n-2}}{k}\right)\right. \\ & \left. - \frac{1}{k}\left(u_m^{n-1}, \frac{u_m^{n-1} - u_m^{n-2}}{k}\right) + \frac{1}{2}\left(\frac{u_m^n - u_m^{n-1}}{k}, u_m^n + u_m^{n-1}\right) + \left(\frac{w_m^n - w_m^{n-1}}{k}, 1\right)\right] \\ & = \left(\frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}, \frac{u_m^n - u_m^{n-1}}{k}\right) \\ & - \frac{1}{2}\left(\frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k^2}, \frac{u_m^n - 2u_m^{n-1} + u_m^{n-2}}{k}\right) \\ & + \left(\varDelta\left(\frac{u_m^n - u_m^{n-1}}{k}\right), \varDelta u_m^n\right) - \frac{1}{2}\left(\varDelta\left(\frac{u_m^n - u_m^{n-1}}{k}\right), \varDelta\left(u_m^n - u_m^{n-1}\right)\right) \end{aligned}$$

$$-\left(h,\frac{u_m^n-u_m^{n-1}}{k}\right)+\frac{1}{p+2}\left(\frac{|u_m^n|^{p+2}-\left|u_m^{n-1}\right|^{p+2}}{k},1\right)+\eta\left(u_m^n,\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2}\right)$$
$$+\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^{n-1}-u_m^{n-2}}{k}\right)+\frac{\eta}{2}\left(\frac{u_m^n-u_m^{n-1}}{k},2u_m^n-u_m^n+u_m^{n-1}\right)$$
$$+\eta\left(\frac{w_m^n-w_m^{n-1}}{k},1\right)=\left(\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2},\frac{u_m^n-u_m^{n-1}}{k}\right)$$
$$+\left(\Delta\left(\frac{u_m^n-u_m^{n-1}}{k}\right),\Delta u_m^n\right)-\left(h,\frac{u_m^n-u_m^{n-1}}{k}\right)$$
$$+\left(|u_m^n|^p u_m^n,\frac{u_m^n-u_m^{n-1}}{k}\right)-\frac{1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2-\frac{1}{2k}\left\|\Delta u_m^n-\Delta u_m^{n-1}\right\|^2$$
$$+\eta\left(u_m^n,\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2}\right)+\eta\left(\frac{u_m^n-u_m^{n-1}}{k},u_m^n\right)$$
$$+\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^n-u_m^{n-1}}{k}-\frac{u_m^n-u_m^{n-1}}{k}+\frac{u_m^{n-1}-u_m^{n-2}}{k}\right)$$
$$-\frac{\eta}{2k}\left\|u_m^n-u_m^{n-1}\right\|^2+\eta\left(\frac{w_m^n-w_m^{n-1}}{k},1\right)$$
$$=-\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\left(\frac{w_m^n-w_m^{n-1}}{k},\frac{u_m^n-u_m^{n-1}}{k}\right)-\frac{1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2$$
$$-\frac{1}{2k}\left\|\Delta u_m^n-\Delta u_m^{n-1}\right\|^2-\eta\left(\frac{w_m^n-w_m^{n-1}}{k},u_m^n\right)-\eta\left\|\Delta u_m^n\right\|^2-\eta\left(|u_m^n|^{p+2},1\right)$$
$$+\eta\left(h,u_m^n\right)+\eta\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k}\right)$$
$$-\frac{\eta}{2k}\left\|u_m^n-u_m^{n-1}\right\|^2+\eta\left(\frac{w_m^n-w_m^{n-1}}{k},1\right). \tag{2.17}$$

Let

$$\delta<\eta\,. \tag{2.18}$$

Then by (2.7), (2.8) we obtain from (2.17):

$$\frac{\Phi_{\eta m}^n-\Phi_{\eta m}^{n-1}}{k}+\delta\Phi_{\eta m}^n=-\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\left(\frac{w_m^n-w_m^{n-1}}{k},\frac{u_m^n-u_m^{n-1}}{k}\right)$$
$$-\frac{1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2-\frac{1}{2k}\left\|\Delta u_m^n-\Delta u_m^{n-1}\right\|^2$$
$$-\eta\left(\frac{w_m^n-w_m^{n-1}}{k},u_m^n-1\right)-\eta\left\|\Delta u_m^n\right\|^2-\eta\left(|u_m^n|^{p+2},1\right)+\eta\left(h,u_m^n\right)$$
$$+\eta\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2-\eta\left(\frac{u_m^n-u_m^{n-1}}{k},\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k}\right)-\frac{\eta}{2k}\left\|u_m^n-u_m^{n-1}\right\|^2$$
$$+\frac{\delta}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+\frac{\delta}{2}\left\|\Delta u_m^n\right\|^2-\delta\left(h,u_m^n\right)+\frac{\delta}{p+2}\left(|u_m^n|^{p+2},1\right)$$

$$+\delta\eta\left(u_m^n, \frac{u_m^n-u_m^{n-1}}{k}\right)+\frac{\delta\eta}{2}\left\|u_m^n\right\|^2+\delta\eta\left(w_m^n,1\right)\le(-1+\eta+\frac{\delta}{2})\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2$$

$$+(-\frac{1}{2k^3}+\frac{\eta}{2k^2})\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2+\frac{\eta L_1\nu_0}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2$$

$$+\left(\frac{4\eta}{\nu_0}+\frac{\delta\eta L}{2}\right)\left\|u_m^n\right\|^2+(-\eta+\frac{\delta}{2})\left\|\Delta u_m^n\right\|^2+(-\eta+\frac{\delta}{p+2})\left(\left|u_m^n\right|^{p+2},1\right)$$

$$+(\eta-\delta)\left(h,u_m^n\right)+\frac{\eta}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2+\frac{\delta\eta}{2}\left\|u_m^n\right\|^2+\frac{\delta\eta}{2}\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2$$

$$+\frac{\delta\eta}{2}\left\|u_m^n\right\|^2+\frac{\delta\eta}{2}\|g\|^2+\left(2\eta+\frac{\delta\eta}{2}\right)\|1\|^2$$

$$\le\ (-1+\eta+\frac{\delta}{2}+\frac{\eta L_1\nu_0}{2}+\frac{\eta}{2}+\frac{\delta\eta}{2})\left\|\frac{u_m^n-u_m^{n-1}}{k}\right\|^2$$

$$+\frac{\eta k-1}{2k^3}\left\|u_m^n-2u_m^{n-1}+u_m^{n-2}\right\|^2+(\left(\frac{4\eta}{\nu_0}+\frac{\delta\eta L}{2}\right)c_\Omega^2-\eta+\frac{\delta}{2}$$

$$\frac{\nu(\eta-\delta)c_\Omega^2}{2}+\delta\eta c_\Omega^2)\left\|\Delta u_m^n\right\|^2+(-\eta+\frac{\delta}{p+2})\left(\left|u_m^n\right|^{p+2},1\right)$$

$$+\frac{\eta-\delta}{2\nu}\|h\|^2+\frac{\delta\eta}{2}\|g\|^2+\left(2+\frac{\delta}{2}\right)\eta mes\Omega. \tag{2.19}$$

We choose the numbers ν_0, η, δ, ν such that, it holds the following inequalities (we add them inequality (18)):

$$-1+\eta+\tfrac{\delta}{2}+\tfrac{\eta L_1\nu_0}{2}+\tfrac{\eta}{2}+\tfrac{\delta\eta}{2}\le 0\,,$$
$$\eta k-1\le 0\,,\ \ \eta-\delta>0\,,$$

$$\left(\frac{4\eta}{\nu_0}+\frac{\delta\eta L}{2}\right)c_\Omega^2-\eta+\frac{\delta}{2}+\frac{\nu(\eta-\delta)c_\Omega^2}{2}+\delta\eta c_\Omega^2\le 0\,,$$

$$-\eta+\frac{\delta}{p+2}\le 0.$$

After elementary transformations in last inequalities we have:

$$\nu_0>2c_\Omega^2\,,$$

$$\eta<\min\left\{\frac{2}{3+L_1\nu_0},\ \frac{1}{k}\right\}\,,$$

$$\delta<\min\left\{\eta,\ \frac{2-\eta\left(3+L_1\nu_0\right)}{\eta+1},\ \frac{4\eta\left(\nu_0-2c_\Omega^2\right)}{L\nu_0\eta c_\Omega^2+2\eta\nu_0\left(1+2\eta c_\Omega^2\right)}\right\}\,,$$

$$\nu<\frac{\eta\left(4\nu_0+(8+\delta L\nu_0)c_\Omega^2-2\delta\nu_0-4\eta\delta\nu_0c_\Omega^2\right)}{2\left(\eta-\delta\right)c_\Omega^2\nu_0}\,.$$

Thus from (2.19) we obtain that, for arbitrary $m\ >m_1$ it holds the inequality

$$\frac{\Phi_{\eta m}^n-\Phi_{\eta m}^{n-1}}{k}+\delta\Phi_{\eta m}^n\le\frac{\eta-\delta}{2\nu}\|h\|^2+\frac{\delta\eta}{2}\|g\|^2+\left(2+\frac{\delta}{2}\right)\eta mes\Omega,\ n=1,2,...,m\,,$$

where $m_1 = \frac{T}{k_1}$, $k_1 = \frac{1}{\eta}$.

Let $\|h\|^2 \le \bar{m}$ and $\|g\|^2 \le \bar{\bar{m}}$. Then

$$\frac{\Phi_{\eta m}^{n} - \Phi_{\eta m}^{n-1}}{k} + \delta\Phi_{\eta m}^{n} \le C\,,\; n = 1, 2, ..., m\,,$$

where $C = \frac{\eta-\delta}{2\nu}\bar{m} + \dfrac{\delta\eta}{2}\bar{\bar{m}} + \left(2 + \frac{\delta}{2}\right)\eta mes\Omega$.

Lemma 2.1 is proved.

Now we consider the existence of a bounded absorbing set for problem (2.1)-(2.3).

Note that, a bounded set $B_0 \subset E$ is said to be absorbing, if for arbitrary bounded set $B \subset E$, there exists $t_1(B)$ such that $S(t)\,B \subset B_0$ for all $t \ge t_1(B)$ (see. [7]).

Theorem 2.1 *Problem* (2.1)-(2.3) *has a bounded absorbing set* $B_0 \subset E$ *when the conditions* (2.4)-(2.12) *hold.*

Proof. Let

$$B_0 = \left[\left\{ y = (u, F(u), q) \in E : \Phi_\eta(y) \le \frac{2C}{\delta} \right\}\right],$$

where $[M]$ denotes a closure of set M.

1. We prove at first that a set B_0 is bounded.

$$\begin{aligned}\Phi_\eta(y) &= \frac{1}{2}\|q\|^2 + \frac{1}{2}\|\Delta u\|^2 - (h,u) + \frac{1}{p+2}\left(|u|^{p+2}, 1\right) + \eta\left[(u,q) + \frac{1}{2}\|u\|^2\right] \\ &\ge \frac{1}{2}\|q\|^2 + \frac{1}{2}\|\Delta u\|^2 - \frac{1}{2\nu_1}\|h\|^2 - \frac{\nu_1}{2}\|u\|^2 + \frac{1}{p+2}\left(|u|^{p+2}, 1\right) - \frac{\eta}{2\nu_2}\|u\|^2 \\ &\quad - \frac{\eta\nu_2}{2}\|q\|^2 + \frac{\eta}{2}\|u\|^2 = \frac{1}{2}(1-\eta\nu_2)\|q\|^2 + \frac{1}{2}\|\Delta u\|^2 \\ &\quad + \frac{1}{2}\left(-\nu_1 - \frac{\eta}{\nu_2} + \eta\right)\|u\|^2 - \frac{1}{2\nu_1}\|h\|^2 + \frac{1}{p+2}\left(|u|^{p+2}, 1\right). \end{aligned} \tag{2.20}$$

We choose $\nu_1,\ \nu_2$ such that:

$$1 - \eta\nu_2 > 0\,,\;\; -\nu_1 - \frac{\eta}{\nu_2} + \eta > 0\,,$$

that is

$$1 < \nu_2 < \frac{1}{\eta}\,,\;\; \nu_1 < \eta\left(1 - \frac{1}{\nu_2}\right).$$

Let

$$\nu_3 = \frac{1}{2}\min\left\{1 - \eta\nu_2;\ -\nu_1 - \frac{\eta}{\nu_2} + \eta\right\}.$$

Then by (2.20) we have

$$\Phi_\eta(y) \ge \nu_3\left(\|q\|^2 + \|\Delta u\|^2 + \|u\|^2\right) - \frac{1}{2\nu_1}\|h\|^2 \ge \nu_3\|y\|_E^2 - \frac{1}{2\nu_1}\bar{m}^2\,,$$

whence we obtain that,

$$\|y\|_E^2 \le \frac{1}{\nu_3}\Phi_\eta(y) + \frac{\bar{m}^2}{2\nu_1\nu_3} \le \frac{1}{\nu_3}\cdot\frac{2C}{\delta} + \frac{\bar{m}^2}{2\nu_1\nu_3},$$

that is B_0 is bounded.

2. Now we prove that B_0 is absorbing. We put an arbitrary bounded set $B \subset E: \ B = \{y \in E: \ \|y\|_E \le \chi\}$. Let $y^0 = \left(u^{(0)}, \ w^{(0)}, \ u^{(1)}\right) \in B$. We have to find $t_1(B) = t_1(\chi)$ such that, $y = S(t)\, y^0$ or $(u, F(u), u_t) = S(t)\left(u^{(0)}, w^{(0)}, \ u^{(1)}\right)$ belongs to set B_0 for arbitrary $t \ge t_1(\chi)$. Since u is a solution of problem (2.1)-(2.3) with initial data y^0, then it holds inequality (2.16) for solution $u_m^n(x) = u(x, nk)$ of problem (2.13)-(2.14), by multiplying which by $e^{\delta\, n\, k}$, we have

$$\frac{\Phi_{\eta m}^{n} - \Phi_{\eta m}^{n-1}}{k} e^{\delta\, n\, k} + \delta \Phi_{\eta m}^{n} e^{\delta\, n\, k} \le C e^{\delta\, n\, k}$$

or

$$\frac{\Phi_{\eta m}^{n} e^{\delta\, n\, k} - \Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k}}{k} + \frac{\Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k}}{k} - \frac{\Phi_{\eta m}^{n-1} e^{\delta\, n\, k}}{k} + \delta \Phi_{\eta m}^{n} e^{\delta\, n\, k} \le C e^{\delta\, n\, k}$$

or

$$\frac{\Phi_{\eta m}^{n} e^{\delta\, n\, k} - \Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k}}{k} - \Phi_{\eta m}^{n-1} \delta \frac{e^{\delta\, n\, k} - e^{\delta\,(n-1)\, k}}{\delta k}$$
$$+\delta \Phi_{\eta m}^{n} e^{\delta\, n\, k} \le C e^{\delta\, k} e^{\delta (n-1)\, k}. \tag{2.21}$$

It is evident that

$$e^{\delta\,(n-1)\, k} = \frac{e^{\delta\, n\, k} - e^{\delta\,(n-1)\, k}}{\delta k} + \alpha(k) \ ,$$

where $\alpha(k) \to 0$ as $k \to 0$.

By last relation we have from inequality (2.21):

$$\frac{\Phi_{\eta m}^{n} e^{\delta\, n\, k} - \Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k}}{k} + \delta\left(\Phi_{\eta m}^{n} e^{\delta\, n\, k} - \Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k}\right) + \delta \alpha(k)\, \Phi_{\eta m}^{n-1}$$
$$\le C e^{\delta\, k}\left(\frac{e^{\delta\, n\, k} - e^{\delta\,(n-1)\, k}}{\delta k} + \alpha(k)\right)$$

or

$$(1 + \delta k)\, \frac{\Phi_{\eta m}^{n} e^{\delta\, n\, k} - \Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k}}{k} + \delta \alpha(k)\, \Phi_{\eta m}^{n-1}$$
$$\le C e^{\delta\, k} \frac{e^{\delta\, n\, k} - e^{\delta\,(n-1)\, k}}{\delta k} + C e^{\delta\, k} \alpha(k),$$

whence we obtain that,

$$\Phi_{\eta m}^{n} e^{\delta\, n\, k} - \Phi_{\eta m}^{n-1} e^{\delta\,(n-1)\, k} \le \frac{C}{\delta}\left(e^{\delta\, n\, k} - e^{\delta\,(n-1)\, k}\right) + \frac{C - \delta \Phi_{\eta m}^{n-1}}{1 + \delta k} k \alpha(k).$$

We sum the last inequality for $n = 1, ..., l$ for arbitrary $l \in \{\, 1, ..., m \,\}$. Then we have

$$\Phi_{\eta m}^{l} e^{\delta\, l\, k} - \Phi_{\eta m}^{0} \le \frac{C}{\delta}\left(e^{\delta\, l\, k} - 1\right) + \frac{k \alpha(k)}{1 + \delta k} \sum_{n=1}^{l} \left(C - \delta \Phi_{\eta m}^{n-1}\right),$$

whence

$$\Phi_{\eta m}^{l} \le \frac{C}{\delta} + \left(\Phi_{\eta m}^{0} - \frac{C}{\delta} + \frac{k \alpha(k)}{1 + \delta k} \sum_{n=1}^{l} \left(C - \delta \Phi_{\eta m}^{n-1}\right)\right) e^{-\,\delta\, l\, k}.$$

Since $\left\|y^0\right\|_E \leq \chi$, it is evident that, $\Phi_{\eta m}^0 \leq c(\chi)$, where $c(\chi)$ is a positive constant which depends on χ. Therefore from the last inequality we have

$$\Phi_{\eta m}^l \leq \frac{C}{\delta} + \left(c(\chi) - \frac{C}{\delta} + \frac{k\alpha(k)}{1+\delta k} \sum_{n=1}^{l} \left(C - \delta \Phi_{\eta m}^{n-1} \right) \right) e^{-\delta l k}. \tag{2.22}$$

We choose l such that,

$$\left(c(\chi) - \frac{C}{\delta} + \frac{k\alpha(k)}{1+\delta k} \sum_{n=1}^{l} \left(C - \delta \Phi_{\eta m}^{n-1} \right) \right) e^{-\delta l k} \leq \frac{C}{\delta} \tag{2.23}$$

or

$$\left(c(\chi) - \frac{C}{\delta} \right) e^{-\delta l k} \leq \frac{C}{\delta} - o(k) \ .$$

Since $C = \frac{\eta-\delta}{2\nu}\bar{m}$, then we choose ν such that,

$$c(\chi) - \frac{C}{\delta} \leq 0$$

that is

$$\nu \leq \frac{\bar{m}}{2c(\chi)} \left(\frac{\eta}{\delta} - 1 \right) \ .$$

Then (2.23) holds for arbitrary $l \in \{1, ..., m\}$. Therefore from (2.22), (2.23) we obtain that,

$$\Phi_{\eta m}^l \leq \frac{2C}{\delta} \quad \text{for arbitrary } l \in \{1, ..., m\} \ ,$$

that is

$$\Phi_{\eta m}^l = \frac{1}{2} \left\| \frac{u_m^l - u_m^{l-1}}{k} \right\|^2 + \frac{1}{2} \left\| \Delta u_m^l \right\|^2 - \left(h, u_m^l \right) + \frac{1}{p+2} \left(\left| u_m^l \right|^{p+2}, 1 \right)$$

$$+\eta \left[\left(u_m^l, \frac{u_m^l - u_m^{l-1}}{k} \right) + \frac{1}{2} \left\| u_m^l \right\|^2 \right] \leq \frac{2C}{\delta}, \tag{2.24}$$

for arbitrary $l \in \{1, ..., m\}$.

Let

$$\tilde{u}_m(x,t) = u_m^n(x), \text{ if } (n-1)k < t \leq nk, \ \ n = 1, 2, ..., m; \text{ a.i. in } \Omega,$$

and define $\tilde{w}_m$ similarly. Then from (2.24) we have

$$\frac{1}{2} \left\| \frac{\partial u_m}{\partial t} \right\|^2 + \frac{1}{2} \left\| \Delta \tilde{u}_m \right\|^2 - (h, \tilde{u}_m) + \frac{1}{p+2} \left(\left| \tilde{u}_m \right|^{p+2}, 1 \right)$$

$$+\eta \left[\left(\tilde{u}_m, \frac{\partial u_m}{\partial t} \right) + \frac{1}{2} \left\| \tilde{u}_m \right\|^2 \right] \leq \frac{2C}{\delta} \tag{2.25}$$

Since as $m \to \infty$

$$u_m \to u \text{ weakly star in } H^1\left(0, T; L^2(\Omega)\right) \bigcap L^\infty\left(0, T; H_0^1(\Omega)\right),$$

$$\tilde{u}_m \to u \text{ weakly star in } L^\infty\left(0,T;H_0^1(\Omega)\right),$$

then passing to the limit in inequality (2.25), when $m \to \infty$, we have

$$\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\|\Delta u\|^2 - (h,u) + \frac{1}{p+2}\left(|u|^{p+2},1\right) + \eta\left[\left(u,\frac{\partial u}{\partial t}\right) + \frac{1}{2}\|u\|^2\right] \le \frac{2C}{\delta}$$

or

$$\Phi_\eta\left(u,F(u),u_t\right) \le \frac{2C}{\delta},$$

that is

$$y \in B_0 .$$

Theorem 2.1 is proved.

References

1. Babin, A.V., Vishik, M.I.: Attractors for evolution equations, *Nauka, Moscow* (1989) (Russian)
2. Bouc, R.: Modele mathematique d'hysteresis et application aux systemes a un degre de liberte. *These, Marseille* (1966).
3. Chueshov, I., Lasiecka, I.: *Attractors for second-order evolution equations with a non-linear damping*. J. Dynam. Differ. Equ., **16** (2), 469–512 (2004).
4. Isayeva, S.: *Initial-boundary value problem for one fourth order hyperbolic equation with memory operator*. News of Baku University, series of physico-mathematical sciences, (3), 88–98 (2015).
5. Khanmamedov, A.Kh.: *Global attractors for von Karman equations with nonlinear interior dissipation*. J. Math. Anal. Appl., **318**, 92–101 (2006).
6. Khanmamedov, A.Kh.: *Global attractors for wave equations with nonlinear interior damping and critical exponents*. J. Differential Equations, **230**, 7020–719 (2006).
7. Ladyzhenskaya, O.A.: *On the determination of minimal global attractors for the Navier-Stokes equations and other partial differential equations*. Uspekhi Mat. Nauk, **42** (6), 25–60 (1987); English translation: Russian Math. Surveys, **42** (6), 27–73 (1987).
8. Lasiecka, I., Ruzmaikina, A.R.: *Finite dimensionality and regularity of attractors for 2-D semilinear wave equation with nonlinear dissipation*. J. Math. Anal. Appl., **270**, 16–50 (2002).
9. Lions, J.L.: Some solution methods for nonlinear boundary value problems, *Mir, Moscow* (1972) (Russian).
10. Visintin, A.: Differential Models of Hysteresis, *Springer* (1993).