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On the structure of some real algebraic varieties

Ilgar Sh. Jabbarov

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Abstract. *In the considered article we investigate the structure of real algebraic sets, i. e. sets which are defined by the system of algebraic equations with real coefficients. In the paper one made a new approach for this purpose. Our considerations based on the methods of mathematical analysis.*

Keywords. Real algebraic set, Banach algebra, Jacobean, Lagrange theorem.

Mathematics Subject Classification (2010): 14P05, 13J30.

1 Introduction

The structure of the real algebraic varieties is important, up to the end not studied question of Algebra and Analysis. Many problems of Mathematical Analysis lead to the study of such varieties ([3-7]). The case delivers many difficulties for the methods of modern algebraic geometry ([10-12]). These difficulties are two types. First of them consisted in the fact that the field of real numbers is not algebraically closed. The second one is more substantive and consisted in the use of Zarisskiy topology (see [11]).

Mentioned above difficulties we overcome by consideration of differential calculus in the Banach algebras. We will consider the field of complex numbers as a Banach algebra $R \times R$ in which the norm defined as a modulus of a complex number. For us at first it is necessarily to show that the differentiation over complex pares is agreed with the differentiation over real parameters. Note that the addition and multiplication of pairs is defined by a standard way:

$$(\xi', \eta') + (\xi'', \eta'') = (\xi' + \xi'', \eta' + \eta''), (\xi', \eta') \cdot (\xi'', \eta'') = (\xi'\xi'' - \eta'\eta'', \xi'\eta'' + \xi''\eta').$$

The norm is defined as $\|(\xi, \eta)\| = \sqrt{\xi^2 + \eta^2}$. Let $\lambda = (\xi, \eta)$, $\mu = (\sigma, \omega)$. Then, we have:

$$\frac{d(\lambda\mu)}{d\lambda} = \lim_{h \to 0} \frac{(\lambda + h)\mu - \lambda\mu}{h} = \mu.$$

Ilgar Sh. Jabbarov,
Ganja State University 187, Sh.I.Xatai str., AZ2000 Ganja, Azerbaijan
E-mail: ilgar_j@rambler.ru

$$\frac{d(\lambda\mu)}{d\xi} = \frac{\partial(\xi\sigma - \eta\omega, \xi\omega + \eta\sigma)}{\partial\xi} = (\sigma, \omega) = \mu.$$

It is obvious that

$$\frac{d(\lambda\mu)}{d\xi} = \frac{d(\lambda\mu)}{d\lambda}\frac{d\lambda}{d\xi} = \mu \cdot (1, 0) = \mu.$$

From the formulas (3.1) now it is clear that at the calculation of derivatives real and complex variables are equal in rights.

In the present article the solution of this problem in a sufficient generality is given by attraction of methods of the mathematical analysis ([13, p. 80]).

2 Preliminary results

In [13, p. 80] it was shown that the algebraic equation

$$z^n - a_1 z^{n-1} + a_2 z^{n-2} + \cdots + (-1)^n a_n = 0$$

with complex coefficients in every domain of change of a vector $\bar{a} = (a_1, ..., a_n)$ in which the discriminant of the equation is non-zero defines roots as smooth functions of coefficients. A real analogue of this statement takes place. For proving our main results we need some preliminary results from Algebra and Analysis. We bring them as auxiliary lemmas.

Lemma 2.1 *Let the function $f(x)$ be continuous on the segment $[a, b]$ and differentiable on (a, b). Then there exist a point $c \in (a, b)$ for which $f(a) - f(b) = f'(c)(a - b)$.*

This theorem is known as a theorem of Lagrange on finite increments and its proof can be found in [1,9].

The following theorem is an easy consequence of the lemma 2.1.

Lemma 2.2 *Let the function $f(x)$ be differentiable on (a, b) and its derivateve be bounded on this interval $|f'(c)| \leq m$ when $c \in (a, b)$. Then there exist the limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to b} f(x)$.*

The following lemma is known as a theorem on implicit functions. We use the formulation of this result as in [3, p. 68].

Lemma 2.3 *Let a function $z = \Phi(x, y) : (E \subset X) \times (F \subset Y) \to Z$ mapping a neighborhood $W = \{(x, y) : |x - a| < r, |y - b| < \rho\}$ of the point $\{a, b\}$, $a \in E \subset X$, $b \in F \subset Y$ from the direct product of normed spaces X, Y into the normed space Z be given. Let $\Phi(a, b) = 0$ and the operator $\frac{\partial\Phi(x,y)}{\partial y}$ (of the partial derivative through the space Y) exists, is continuous and invertible in indicated neighborhood W. Then there exist a ball $U_\delta = \{x \in E :, y) : |x - a| \leq \delta \leq r\}$ and a function $y = f(x) : E \to F$, defined and continuous in the ball U_δ such that $b = f(a)$ and $\Phi(x, f(x)) \equiv 0$ for all $x \in U_\delta$. The required function $y = f(x)$ is unique in the sence: if there exist two functions $f_1(x)$ and $f_2(x)$ satisfying the indicated above conditions, and defined correspondingly on neighborhoods U_{δ_1} and $U_{\delta_2}$of the point a then in the intersection $U = U_{\delta_1} \bigcap U_{\delta_2}$ they are coincident.*

3 Basic results

Theorem 3.1 *Let we are given with an algebraic equation*

$$z^n - a_1 z^{n-1} + a_2 z^{n-2} + \cdots + (-1)^n a_n = 0,$$

with real coefficients and discriminant $\Delta(\bar{a})$. Then in every one-connected domain $D \subset \{\bar{a} \in R^n | \Delta(\bar{a}) \neq 0\}$ where the discriminant of the equation differs from zero and there exists $a_0 \in D$ such that the considered algebraic equation has $s(1 \leq s \leq n)$ real roots the given equation uniquely defines exactly s real smooth implicit functions of coefficients satisfying this equation.

Proof. The proof of the theorem based on a method used in [13, p. 80] applied to algebraic equations with real coefficients. We will prove it in the form: under theorem's conditions the given algebraic equation defines $n = s + 2r$ number of such real smooth functions of coefficients that the first s from them will be roots of the given equation.

Suppose that the given equation at some $\bar{a}_0 \in D$ has real roots $\lambda_1^0, ..., \lambda_s^0$, $s \geq 0$. Under the basic theorem of algebra it has in addition r ($r \geq 0$) complex roots $\xi_1^0 \pm i\eta_1^0, ..., \xi_r^0 \pm \eta_r^0$ so that $n = s + 2r$. Nonsider Viet formulas

$$a_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$a_1 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \cdots + \lambda_{n-1}\lambda_n, \tag{3.1}$$

$$\cdots \quad \cdots \quad \cdots$$

$$a_n = \lambda_1\lambda_2 \cdots \lambda_n,$$

where $\lambda_{s+2j-1} = \xi_j + i\eta_j, \lambda_{s+2j} = \xi_j - i\eta_j; j = 1, ..., r$. On the given roots these formulas uniquely define the point $\bar{a} = (a_1, ..., a_n) \in R^n$. Conversely, each point $\bar{a} = (a_1, ..., a_n) \in R^n$ on which the discriminant of the equation is distinct from zero (i.e. roots are distinct in pairs) defines the roots uniquely (not counting their order). Hence, if to allocate small area around some root $\bar{\lambda} = (\lambda_1, ..., \lambda_s, \xi_1, \eta_1, ..., \xi_r, \eta_r)$, correspondence will be one to one. Following the work specified above, we should establish that the formulas (3.1) define in some neighborhood of the point $\bar{a}_0$ one to one mapping

$$(a_1, ..., a_n) \leftrightarrow \bar{\lambda} = (\lambda_1, ..., \lambda_s, \xi_1, \eta_1, ..., \xi_r, \eta_r)$$

with nonzero Jacobean

$$\frac{\partial\,(a_1, ..., a_n)}{\partial\,(\lambda_1, ..., \lambda_s, \xi_1, ..., \eta_r)}.$$

(at the point $\bar{a}_0$ we put $\bar{\lambda} = \bar{\lambda}_0 = \left(\lambda_1^0, ..., \lambda_s^0, \xi_1^0, \eta_1^0, ..., \xi_r^0, \eta_r^0\right)$). Thus, the equalities (3.1) we consider as a mapping of the normed space $R^s \times R^{2r}$ to real space R^n considering the complex number $\xi + i\eta$ as a pare(ξ, η), and we will use the lemma 2.3.

Let's apply now a method of a mathematical induction. At $n = 1$ the theorem is true. Let it is fair for all polynomials of degree $\leq n - 1$. Consider now the case of polynomials of degree n.

Let's separately consider case $s = 0$ when everywhere in D the equation has not real roots. But there are complex roots $\xi_1^0 \pm i\eta_1^0, ..., \xi_r^0 \pm i\eta_r^0$. By an induction we will prove that in some neighborhood of the point $\bar{\lambda}_0 = \left(\xi_1^0, \eta_1^0, ..., \xi_r^0, \eta_r^0\right)$ we have:

$$\frac{\partial\left(a_1, ..., a_n\right)}{\partial\left(\xi_1, ..., \eta_r\right)} \neq 0.$$

For $r = 2$ this statement is known from an elementary course of algebra. Let our statement be fair for polynomials of even degree $2r - 2$. From the Viet formulas we derive:

$$a_1 = \lambda_1 + \lambda_2 + c_1,$$

$$a_2 = \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)c_1 + c_2,$$

$$a_3 = \lambda_1\lambda_2 c_1 + (\lambda_1 + \lambda_2)c_2 + c_3,$$

$$\cdots \quad \cdots \quad \cdots$$

$$a_{n-1} = \lambda_1\lambda_2 c_{n-3} + (\lambda_1 + \lambda_2)c_{n-2},$$

$$a_n = \lambda_1\lambda_2 c_{n-2},$$

Where real numbers $c_1, ..., c_{n-2}$ are defined by equalities:

$$c_1 = \lambda_3 + \cdots + \lambda_n,$$

$$c_2 = \lambda_3\lambda_4 + \cdots + \lambda_{n-1}\lambda_n,$$

$$\cdots \quad \cdots \quad \cdots$$

$$c_{n-2} = \lambda_3 \cdots \lambda_n.$$

We have:

$$\frac{\partial\left(a_1, ..., a_n\right)}{\partial\left(\xi_1, ..., \eta_r\right)} = \frac{\partial\left(a_1, ..., a_n\right)}{\partial\left(\lambda_1, \lambda_2, c_1, ..., c_{n-2}\right)} \frac{\partial\left(\lambda_1, \lambda_2\right)}{\partial\left(\xi_1, \eta_1\right)} \cdot \frac{\partial\left(\lambda_1, \lambda_2, c_1, ..., c_{n-2}\right)}{\partial\left(\lambda_1, \lambda_2, \xi_3, ..., \eta_r\right)}.$$

Due to the inductive assumption

$$\frac{\partial\left(\lambda_1, \lambda_2, c_1, ..., c_{n-2}\right)}{\partial\left(\lambda_1, \lambda_2, \xi_3, ..., \eta_r\right)} = \frac{\partial\left(c_1, ..., c_{n-2}\right)}{\partial\left(\xi_3, ..., \eta_r\right)} \neq 0,$$

and the first Jacobean on the right part we will write in as follows

$$\frac{\partial\left(a_1, ..., a_n\right)}{\partial\left(\lambda_1, \lambda_2, c_1, ..., c_{n-2}\right)} = \frac{\partial\left(a_1, ..., a_n\right)}{\partial\left(\lambda_1, b_1, b_2, ..., b_{n-1}\right)} \cdot \frac{\partial\left(\lambda_1, b_1, b_2, ..., b_{n-1}\right)}{\partial\left(\lambda_1, \lambda_2, c_1, ..., c_{n-2}\right)},$$

where complex numbers $b_1, b_2, ..., b_{n-1}$ are already defined by equalities (*) below. The first determinant is equal

$$\lambda_1^{n-1} - b_1\lambda_1^{n-2} + \cdots + (-1)^{n-1}b_{n-1},$$

and the second looks like

$$\lambda_2^{n-2} - c_1\lambda_2^{n-3} + \cdots + (-1)^{n-2}c_{n-2}.$$

As roots are in pairs different, both determinants on the right part are distinct from zero. Clearly, that $\frac{\partial(\lambda_1,\lambda_2)}{\partial(\xi_1,\eta_1)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \neq 0$. So, we have shown, that $\frac{\partial(a_1,\dots,a_n)}{\partial(\xi_1,\dots,\eta_r)} \neq 0$. Then the vector $\bar{a}_0 \in D$ with real components defines, under the lemma 2.3, $n = 2r$ complex functions of a kind $\xi_1 + i\eta_1, \dots, \xi_r + i\eta_r$. Owing to a continuity, probably increasing of a number of real roots during continuous variation of a real vector $\bar{a}_0 \in D$ occurs only when some of functions $\eta = \eta_1, \dots, \eta_r$ continuously varying vanishes in some real point $\bar{a}_0$. Then, in the considered point we get a multiple zero $\xi = \xi \pm i\eta$ that contradicts theorem's condition. Hence, the given equation has no real roots at all real $\bar{a}_0 \in D$. Thus, consideration of the case $s = 0$ is finished.

Let's consider now a case when there is at least one real root. By lemma's condition, at some $\bar{a}_0 \in D$ the equation has real roots$\lambda_1^0, \dots, \lambda_s^0$, thus, $s \geq 1$, and in addition r $(r \geq 0)$ complex roots $\xi_1^0 \pm i\eta_1^0, \dots, \xi_:^0 \pm \eta_r^0$. Define numbers $b_1^0 = \lambda_2^0 + \cdots + \lambda_n^0$, $b_2^0 = \lambda_2^0\lambda_3^0 + \cdots + \lambda_{n-1}^0\lambda_n^0, \dots$, $b_{n-1}^0 = \lambda_2^0 \cdots \lambda_n^0$. These numbers are real, because their complex conjugate leads to permuting of numbers on the right side. As the polynomials standing on the right part are symmetric then they do not change. Hence, the numbers $\lambda_2^0, \dots, \lambda_n^0$ are roots of the polynomial

$$z^{n-1} - b_1^0 z^{n-2} + b_2^0 z^{n-3} - \cdots + (-1)^{n-1} b_{n-1}^0 = 0,$$

with real coefficients. All of these roots are different, i.e. the discriminant of last equation is distinct from zero. It remains unchanged in some neighborhood of the point $(b_1^0, \dots, b_{n-1}^0)$. In this neighborhood we will consider the system:

$$\begin{aligned} b_1 &= \lambda_2 + \cdots + \lambda_n, \\ b_2 &= \lambda_2\lambda_3 + \cdots + \lambda_{n-1}\lambda_n, (*) \\ &\cdots \quad \cdots \quad \cdots \\ b_{n-1} &= \lambda_2 \cdots \lambda_n. \end{aligned}$$

Under the inductive assumption, the following determinant is distinct from zero in some neighborhood of the point $(b_1^0, \dots, b_{n-1}^0)$:

$$\frac{\partial(b_1, \dots, b_{n-1})}{\partial(\lambda_2, \dots, \eta_r)} \neq 0.$$

We have [3, p. 82]:

$$\begin{aligned} a_1 &= \lambda_1 + b_1, \\ a_2 &= \lambda_1 b_1 + b_2, \\ &\cdots \quad \cdots \quad \cdots \\ a_n &= \lambda_1 b_{n-1}. \end{aligned} \tag{3.2}$$

Further, as λ_1^0 is distinct from all other zeros we have:

$$\det \frac{\partial(a_1, \dots, a_n)}{\partial(\lambda_1, b_1 \dots, b_{n-1})} = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ b_1 & \lambda_1 & 1 & \cdots & 0 \\ b_2 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & \lambda_1 \end{vmatrix} =$$

$$= \lambda_1^{n-1} - b_1\lambda_1^{n-2} + \cdots + (-1)^{n-1}b_{n-1} \neq 0$$

at the point $(\lambda_1^0, b_1^0, ..., b_{n-1}^0)$. This relation remains unchanged in some neighborhood of the point $(\lambda_1^0, b_1^0, ..., b_{n-1}^0)$ where the following relation will be carried out also:

$$\det \frac{\partial (a_1, ..., a_n)}{\partial (\lambda_1, \lambda_2, ..., \eta_r)} = \det \frac{\partial (a_1, ..., a_n)}{\partial (\lambda_1, b_1, ..., b_{n-1})} \det \frac{\partial (\lambda_1, b_1, ..., b_{n-1})}{\partial (\lambda_1, \lambda_2, ..., \eta_r)} =$$

$$= \det \frac{\partial (a_1, ..., a_n)}{\partial (\lambda_1, b_1, ..., b_{n-1})} \det \frac{\partial (b_2, ..., b_{n-1})}{\partial (\lambda_2, ..., \eta_{n-1})} \neq 0.$$

By the theorem on implicit functions, in some neighborhood of the point $\bar{a}_0 \in D$, the system (2) defines numbers $\lambda_1, b_1, ..., b_{n-1}$ as smooth functions of numbers $a_1, a_2, ..., a_n$. Further, under the inductive assumption the algebraic equation of degree $n-1$

$$z^{n-1} - b_1 z^{n-2} + b_2 z^{n-3} - \cdots + (-1)^{n-1}b_{n-1} = 0$$

defines the roots as smooth functions of coefficients.

Hence, the given equation defines n real smooth functions of coefficients from which the first s will be demanded solutions of the equation.

Now we take any other point $\bar{a}' \in D$ and connect the points $\bar{a}_0$ and $\bar{a}'$ by some continuous curve in the domain D. It is best known the way of one-valued continuation of these solutions along this curve into a smooth function. As any two curve connecting these points are homotopic (see [2, p. 90] and [2, p. 496]), the continued function is unique in the domain D.

As above, we notice, that the number s can vary (to increase or decrease) in D only by even number when for some j, coefficients of imaginary parts - numbers η_j - continuously varying, will vanish at some point $\bar{a} \in D$. Then ξ_j stands a multiple root that contradicts the condition $\Delta(\bar{a}) \neq 0$. Hence, s does not vary in D. The theorem 1 is proved.

Theorem 3.2 *Consider in the space R^n any polynomial equation*

$$f(\bar{x}) = 0$$

of degree k in any bounded parallelepiped. Then the set of solutions of this equation (if is not empty) consists in the union of finite number of the closed surfaces of a dimension $k-1$, and the number of these surfaces bounded by a number depending only on equation's degree.

Proof. We will consider at first a case of two independent variables. The theorem's statement is true for a polynomial of the first degree. We will apply the method of mathematical induction with respect to the degree of polynomial. Let we are given with any polynomial

$$f(x,y) = a_0(x)y^r + a_1(x)y^{r-1} + \cdots + a_r(x)$$

of degree d in some rectangular area $C = \{(x,y) \in R^2 | u_1 \leq x \leq u_2, v_1 \leq y \leq v_2\}$, and the statement of the theorem is satisfied for all bivariate polynomials of two variable of smaller degrees. The discriminant of the polynomial $f(x,y)$ considered as a polynomial on y is some polynomial $\Delta(x)$ of the variable x, where $(-1)^{r(r-1)/2}a_0(x)\Delta(x) = R(f, f'_y)$, of a degree not exceeding d^{2r-1} (here $R(f, f'_y)$ means the resultan of polynomials f and

f'_y). Let $\xi_1, ..., \xi_l$ - be various real roots of $\Delta(x)$ and $a_0(x)$. The segment $[u_1, u_2]$ will be dissected by these roots into no more than $l + 1$ segments in the interior of which $\Delta(x) \neq 0$. We will consider one of such open intervals δ. Let at some $x_0 \in \delta$, the equation $f(x_0, y) = 0$ has s the real roots (with respect to y). Then, by the theorem 2.1, the algebraic equation $y^r + a_0^{-1}a_1y^{r-1} + \cdots + a_0^{-1}a_r = 0$ in some one-connected domain D of changing of coefficients (the high coefficient is fixed) defines r smooth real functions $\varphi_i(a_0^{-1}a_1, ..., (-1)^r a_0^{-1}a_r), i = 1, ..., r$ say the first s of which will be real roots of the equation. Hence, the vector $\bar{a}(x) = (a_1(x)/a_0(x), ..., (-1)^r a_r(x)/a_0(x))$ defines a one-dimensional subvariety in D, for $x \in \delta$ (owing to simple connectivity of area D), i.e. a line which is passing through the point $\bar{a}(x_0)$, and we receive a one-valued solutions

$$y_i = \varphi_i \left(a_1(x)/a_0(x), ..., (-1)^r a_r(x)/a_0(x)\right), i = 1, ..., s$$

of the equation $f(x, y) = 0$. Since the roots are in pairs various, different solutions y_i, owing to uniqueness, cannot intersect each other in the interval (v_1, v_2) and cannot have self-intersection, because an interval is passed by x only once. Thus, according to the lemma 1, s does not vary with changing of x in δ.

Let now $f(x, y) = 0$ at some fixed $x \in \delta$. Then, at the point y where the pair (x, y) satisfies the considered equation we should have $f'_y(x, y) \neq 0$. Otherwise y should be a multiple root that contradicts the condition. Under the inductive assumption the lemma is true for polynomials $f'_x(x, y) \pm f'_y(x, y)$. Therefore, the considered area of changing of the pair (x, y) always can be dissected into closed Jordan subsets having intersections only by their boundaries in each of which the absolute value of some partial derivative of the function $f(x, y)$ takes on maximal values among them. In each such subarea it is possible to accept that the derivative $y'_i(x) = -f'_x/f'_C$ of implicit function $y_i(x)$ is bounded with its absolute value on a considered open interval. The lemma 2.2 show that the function $y_i(x)$ tends to a limit on the interval ends that makes possible to receive connected solution of the given equation in the considered domain beginning from given piece $y_i(x)$ of the solution.

Straight lines $y = v_1, v_2$ can intersect the graph of each solution y_i in no more than $d - r$ points. Therefore, in the rectangle $\delta \times [v_1, v_2]$ there are no more than finite (depending only on$d$) number of the graph of solutions. Thus, the formulated statement is true for the system containing one equation of two variables.

The general case of n variables easily can be considered now by means of a method of mathematical induction with respect to the number of components of the variable x considering it as a vector in the equation $f(x, y) = 0$: $x = (x_1, ..., x_{n-1}), y = x_n$. It is necessarily to notice only that the mapping $y : W \to Y \subset R$ of one-connected domain W of change of the variable x (defined by the inductive assumption) defined by the fixed solution y, and where the discriminant of equation $f(x, y) = 0$ does not vanish, is continuous and, hence, an image of this domain also is one-connected. Therefore, there exists s one-valued smooth solutions

$$y_i = \varphi_i \left(a_1(x)/a_0(x), ..., (-1)^r a_r(x)/a_0(x)\right)$$

of the equation $f(x, y) = 0$, $x = (x_1, ..., x_{n-1}), y = x_n$ which have no intersections (and dissects the cylinder domain $\left\{(x, y) | x \in \bar{W}, y \in [v_1, v_2]\right\}$ into no more than finite number of one-connected closed subdomains). The theorem 3.2 is proved.

Consequence 3.1 Let in the space R^n the system of polynomial equations

$$f_1(\bar{x}) = 0, ..., f_k(\bar{x}) = 0; \ k < n \tag{3.3}$$

be given. Then in each closed rectangular domain $\Pi \subset R^n$ located in one-connected open domain where Jacoby matrix of the system (3.1) has maximal rank the set of solutions of the system (3.1) consists of the unuion of finite number of closed surfaces of a dimension $n - k$.

The consequence's proof can be spent by an induction with respect to the number of equations of the system (3.1). Let's consider, for example, system containing two equations

$$f_1(x, y, z) = 0,$$
$$f_2(x, y, z) = 0.$$

Not breaking a generality it is possible to assume that some of minors of Jacoby matrix of the given system, for example the minor $\begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{vmatrix}$ accepts everywhere the maximal values for its module among all minors of the second order. Then, under the theorem of implicit functions the given system in some neighborhood of the given root has a solution $x = \varphi_1(z)$, $y = \varphi_2(z)$, and derivatives of these functions are bounded. Then, as it noted above, from the Lagrange theorem it follows that these functions have limit values at the ends of an interval of continuous changing of z.

Let's consider the area where the maximal minor is defined as above the resultant of the polynomials of the system (3.3) with respect to z (i.e. polynomials of the system are considered as polynomials of z). Then we receive a condition of compatibility of a kind $R(x, y) = 0$, where $R$is a resultant. In the considered parallelepiped the pare (x, y) varies in a rectangle. According to, the theorem 1, we find a finite number of solutions of the equation $R(x, y) = 0$ of a kind $x = u(y)$ or $y = v(x)$. Then, the set of solutions of the system (3.3) found above represents the finite number of connected solutions of the system. The consequence is proved.

References

1. Arkhipov, G.I., Sadovnichiy, V.A., Chubarikov, V.N.: Lectures on Mathematical Analysis. *M: High Scool*, (1999) (Russian).
2. Dubrovin, B.A., Novikov, S.P., Fomenko, A.T.: Modern Geometry. *M: Nauka, 2-d ed.*, (1968).
3. Dzhabbarov, I.Sh.: *On an identity of Harmonic Analysis and its application*. Dokl. AS USSR, **314** (5) (1990).
4. Dzhabbarov, I.Sh.: *On estimates of Trigonometric Integrals*. Trans. RAS, **207**, 82–92 (1994).
5. Dzhabbarov, I.Sh.: *On estimates of Trigonometric Integrals*. Chebishevskii sbornik, **11**, issue 1(33), 85–108 (2010).
6. Dzhabbarov, I.Sh.: *On the exponent of convergence of two dimensional Terry's problem. Scientific notes of Orlov State University,* **6** (50), 80–89 (2012).

7. Dzhabbarov, I.Sh.: *On the exponent of convergence of multi-dimensional Terry's problem. Chebishevskii sbornik,* **14**, issue 2, 74–103 (2013).
8. Evgrafov, M.A.: Analytic Functions. *M: Nauka* (1991) (Russian).
9. Fihtenholz, G.M.: *Cours of differential and integral calculus.* (**1**), M: Fizmatlit (2001).
10. Hodge, W.V.D., Pedoe, D.: Methods of Algebraic Geometry. (I), I, (III), *M.* (1955) (Russian).
11. Shafarevich, I.R. *Topics of Algebraic Geometry*. M:MCCMS, (2007).
12. Shafarevich, I.R., Averbukh, B.G., Vainberg, Y.R., Jijchenko, A.B., Manin, Y.I., Moishezon, B.G., Turina, G.N., Turin, A.N.: *Algebraic surfaces*. Trans. MIAS USSR, **75**, 3–215 (1965).
13. Shilov, G.E.: Mathematical Analysis. Functions of several variables. *M: Nauka* (1972) (Russian).