Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (1), 83-88 (2016).
Series of Physical-Technical and Mathematical Sciences.

Global bifurcation from zero for some nondifferentiable mappings

Gunay M. Mamedova

Received: 10.07.2015/ Revised: 12.04.2016 / Accepted: 24.06.2016

Abstract. *In this paper we consider global bifurcation from zero for some nonlinear eigenvalue problems in Banach space with are not linearizable. We show the existence of two continua of solutions bifurcating from the interval of the line of trivial solutions. These global continua have properties similar to those of the continua found in Rabonowitz' and Dancer well-known global bifurcation theorems.*

Keywords. bifurcation point, bifurcation interval, global bifurcation, simple eigenvalue, component of solutions

Mathematics Subject Classification (2010): 34B15, 47H11, 47J10, 47J15

1 Introduction

We consider the nonlinear eigenvalue problem

$$Lu = \lambda u + F(\lambda, u) + G(\lambda, u), \tag{1.1}$$

where $L : D(L) \subset E \to E$ is a linear closed operator with compact resolvent, E is a real Banach space and $F : \mathbb{R} \times E \to E$ and $G : \mathbb{R} \times E \to E$ are continuous operators satisfying the following conditions:

$$||F(\lambda, u)|| \le M||u||, \ \forall\lambda \in \mathbb{R}, \ \forall u \in E, \ ||u|| < 1, \tag{1.2}$$

where M is a positive constant and where $||\cdot||$ is the norm in the space E; for any bounded interval $\Lambda \subset \mathbb{R}$,

$$G(\lambda, u) = o(||u||) \ \text{ at } \ u = 0, \tag{1.3}$$

uniformly with respect to $\lambda \in \Lambda$.

As norm in $\mathbb{R} \times E$, we take $||(\lambda, u)|| = \left\{|\lambda|^2 + ||u||^2\right\}^{1/2}$.

It is well-known Krasnoselskii's bifurcation theorem that if $F \equiv 0$ and μ is an eigenvalue of odd multiplicity of the operator L, then $(\mu, 0)$ is an bifurcation point of problem

G.M. Mammadova
Baku State University, Baku, AZ1148, Azerbaijan
E-mail: m.g.m.400@mail.ru

(1.1) and this bifurcation point corresponds to a continuous branch of solutions (see [3, 4]). Rabinowitz [10] has extended this result by showing that in fact the closure of the set of nontrivial solutions of problem (1.1) possesses an continua $\mathcal{C}_\mu$ which contains $(\mu, 0)$ and is either (i) unbounded in $\mathbb{R} \times E$, or (ii) contains $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in \sigma(L)$. Moreover, if $\mu \in \sigma(L)$ is simple, then C_μ can be decomposed into two subcontinua C_μ^+ and C_μ^- such that for some neighborhood U of $(\mu, 0)$, the following implication is true:

$$(\mu, 0) \neq (\lambda, u) \in C_\mu^+ \left(C_\mu^-\right) \cap U \Rightarrow (\lambda, u) = (\lambda, \alpha v + w),$$

where v is an eigenvector corresponding to the eigenvalue μ with $||v|| = 1$, $\alpha > 0$ ($\alpha < 0$) and $||v|| = o(|\alpha|)$ at $\alpha = 0$. Dancer [2] proved a stronger result for these subcontinua, namely either C_μ^+ and C_μ^- both are unbounded or $C_\mu^+ \cap C_\mu^- \neq \{(\mu, 0)\}$.

Because of the presence of the term F problem (1.1) does not in general have a linearization about $u = 0$. For this reason, the set of bifurcation points for (1.1) with respect to the line of trivial solutions need not be discrete (cf. the example from [1, p. 381]). Therefore, to investigate the question of bifurcation for (1.1), one has to consider bifurcation from intervals rather than bifurcation points. We say that bifurcation occurs from an interval if this interval contains at least one bifurcation point. In this framework, in [5] (see also [6-9, 12]) the authors developed an extension of the results of Rabinowitz [10] to the problem (1.1). Let $\mathcal{G}$ denote the closure of the set of nontrivial solutions of (1.1) in $\mathbb{R} \times E$, let **B** denote the set of bifurcation points of problem (1.1) and let $I_\mu = [\mu - M, \mu + M]$. In [5] shows that if μ is an eigenvalue of odd multiplicity of the operator L and $\operatorname{dist}(\mu\,; \sigma(L) \backslash \{\mu\}) > 2M$, then the set **B** is nonempty and furthermore $\mathbf{B} \cap (I_\mu \times \{0\}) \neq \emptyset$ (the interval $I_\mu \times \{0\}$ is called bifurcation interval). Moreover, if $\mathcal{D}_\mu$ is the union of the set $I_\mu \times \{0\}$ and of all the components $\tilde{\mathcal{D}}_{\mu,\lambda}$ of $\mathcal{G}$ which bifurcate from the bifurcation points $(\lambda, 0) \in B \cap (I_\mu \times \{0\})$, then either (i) $\mathcal{D}_\mu$ is unbounded in $\mathbb{R} \times E$, or (ii) $\mathcal{D}_\mu$ contains the set $I_{\hat{\mu}} \times \{0\}$ where $\mu \neq \hat{\mu} \in \sigma(L)$.

In the present paper we obtains stronger results for bifurcation from an interval $I_\mu \times \{0\}$ in the case when μ is an simple eigenvalue of operator L.

2 Preliminary

Assume additionally that

(A) for any sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathbb{R} \times E$ converging to the $(\lambda, 0)$ ($\lambda \in \Lambda$) there exists a subsequence $\{(\lambda_{n_k}, u_{n_k})\}_{n=1}^\infty$ and a number $m \in [-M, M]$ such that

$$\frac{F(\lambda_{n_k}, u_{n_k}) - m u_{n_k}}{||u_{n_k}||} \to 0 \text{ in } E, \text{ as } k \to \infty\,. \tag{2.1}$$

Hence $\mathcal{A} = \{m \in [-M, M]$: there exists sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \mathbb{R} \times E$ converging to the $(\lambda, 0)$ ($\lambda \in \Lambda$) such that the relation (2.1) holds$\} \neq \emptyset$. This condition is satisfied in particular when F Fréchet differentiable at 0 and $F'(0) = \tilde{m} I$, where I is a identity operator in E. Then, it is clear that $\mathcal{A} = \{\tilde{m}\}$.

Lemma 2.1. *Let $\mu \in \sigma(L)$ is of odd multiplicity,* $\operatorname{dist}(\mu\,; \sigma(L) \backslash \{\mu\}) > 2M$ *and F satisfies the condition (A). Then* $\mathbf{B} \cap (I_\mu \times \{0\}) = \{(\mu - m, 0) \,:\, m \in \mathcal{A}\}$.

Proof. Let $(\lambda, 0) \in \mathbf{B} \cap (I_\mu \times \{0\})$. Hence there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset (\mathbb{R} \times E)\backslash(\mathbb{R} \times \{0\})$ which is convergent to $(\lambda, 0)$ in $\mathbb{R} \times E$ and

$$Lu_n = \lambda_n u_n + F(\lambda_n, u_n) + G(\lambda_n, u_n).$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then y_n satisfies the relation

$$Ly_n = \lambda_n y_n + \frac{F(\lambda_n, u_n)}{\|u_n\|} + \frac{G(\lambda_n, u_n)}{\|u_n\|}. \tag{2.2}$$

By condition (A) there exist a subsequence $\{(\lambda_{n_k}, u_{n_k})\}_{k=1}^{\infty}$ of the sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ and a number $m \in [-M, M]$ such that the relation (2.1) holds.

It follows by (2.2) that

$$Ly_{n_k} = (\lambda_{n_k} - m)y_{n_k} + \frac{F(\lambda_{n_k}, u_{n_k}) - mu_{n_k}}{\|u_{n_k}\|} + \frac{G(\lambda_{n_k}, u_{n_k})}{\|u_{n_k}\|}. \tag{2.3}$$

Consequently, by the form and properties of L and the conditions (1.2), (1.3) and (2.1) from (2.3) imply that there exists a subsequence of the sequence $\{y_{n_k}\}_{k=1}^{\infty}$ (which we will relabel as $\{y_{n_k}\}_{k=1}^{\infty}$) which is convergent to y in E and $\|y\| = 1$, $Ly = (\lambda + m)y$. Then $\lambda + m \in \sigma(L)$. By $\lambda \in I_\mu$ and $|m| \leq M$ it follows that $|\lambda + m - \mu| \leq 2M$. Since $\mathrm{dist}\,(\mu\,; \sigma(L)\backslash\{\mu\}) > 2M$, this inequality is only possible when $\lambda = \mu - m$. The proof of Lemma 2.1 is complete.

Let μ be a simple characteristic value of L and let $v \in E$, $l \in E'$ (the dual space of E) be corresponding eigenvectors of L and L^*, the adjoint of L, normalized so that $||v|| = 1$ and $\langle l, v\rangle = 1$, where $\langle \cdot\,, \cdot \rangle$ denotes the duality between E and E'. Let $E_1 = \{u \in E : \langle l, u\rangle = 0\}$. Then $E = \mathbb{R} \oplus E_1$ and each $u \in E$ can be written as $u = \alpha v + w$, where $\alpha = \langle l, u\rangle$ and $w \in E_1$.

Let B_r denote open ball in E of radius r centered at 0.

For each ξ, $\eta \in (0,\, 1)$ define

$$Q_{\xi,\eta} = \{(\lambda, u) \in \mathbb{R} \times E \,:\, \mathrm{dist}(\lambda, I_\mu) < \xi,\, |\langle l, u\rangle| > \eta\, ||u||\}\,.$$

It is clear that $Q_{\xi,\eta}$ is an open subset of $\mathbb{R} \times E$ and consists of two disjoint subsets $Q^+_{\xi,\eta}$ and $Q^-_{\xi,\eta}$, where

$$Q^+_{\xi,\eta} = \{(\lambda, u) \in \mathbb{R} \times E \,:\, \mathrm{dist}(\lambda, I_\mu) < \xi,\, \langle l, u\rangle > \eta\, ||u||\}\,,$$
$$Q^-_{\xi,\eta} = \{(\lambda, u) \in \mathbb{R} \times E \,:\, \mathrm{dist}(\lambda, I_\mu) < \xi,\, \langle l, u\rangle < -\,\eta\, ||u||\}\,.$$

Lemma 2.2. *Let $\mu \in \sigma(L)$ is simple,* $\mathrm{dist}\,(\mu\,; \sigma(L)\backslash\{\mu\}) > 2M$ *and F satisfies the condition (A). Then there exists a $\tau_0 > 0$ such that for all $\tau < \tau_0$*

$$(\mathcal{G}\backslash(I_\mu \times \{0\})) \cap (I_\mu \times B_\tau) \subset Q_{\xi,\eta}.$$

If $(\lambda, u) \in (\mathcal{G}\backslash(I_\mu \times \{0\})) \cap \mathcal{B}_\tau$, then $u = \alpha v + w$ where $\alpha > \eta\, ||u||$ and $w = o(\alpha)$ at $\alpha = 0$.

Proof. If there is no τ_0 as in the statement, then there exist sequences $\tau_n \downarrow 0$ and $(\lambda_n, u_n) \in (\mathcal{G}\backslash(I_\mu \times \{0\})) \cap (I_\mu \times B_{\tau_n})$ such that $\operatorname{dist}(\lambda_n, I_\mu) \leq \tau_n < \xi$ and $u_n \to 0$, but $|\langle l, u_n\rangle| \leq \eta \|u_n\|$. Since $\lambda_n \in [\mu - M - \tau_n, \mu + M + \tau_n] \subset [\mu - M - \xi, \mu + M + \xi]$, there exists a subsequence $\lambda_{n_k} \to \tilde{\lambda} \in I_\mu$ and $\tilde{m} \in \mathcal{A}$ chosen to u_{n_k}. Let $v_{n_k} = \frac{u_{n_k}}{\|u_{n_k}\|}$. Substituting (λ_{n_k}, u_{n_k}) to (1.1) and dividing the resulting equality by $\|u_{n_k}\|$, we have

$$Lv_{n_k} = (\lambda_{n_k} + \tilde{m})v_{n_k} + \frac{F(\lambda_{n_k}, u_{n_k}) - \tilde{m}u_{n_k}}{\|u_{n_k}\|} + \frac{G(\lambda_{n_k}, u_{n_k})}{\|u_{n_k}\|}\,.$$

Then there exists a subsequence of the sequence $\{v_{n_k}\}_{k=1}^{\infty}$ (which we will relabel as $\{v_{n_k}\}_{k=1}^{\infty}$) which is convergent to $\tilde{v}$ in E and $\|\tilde{v}\| = 1,\ L\tilde{v} = (\tilde{\lambda} + \tilde{m})\tilde{v}$. Then $\tilde{v} = v$ or $\tilde{v} = -v$. Hence we have $|\langle l, v_{n_k}\rangle| \to |\langle l,\ v\rangle| \leq \eta\ \|v\| = \eta < 1$ which is impossible in view of condition $\langle l, v\rangle = 1$. Thus there exists τ_0 as above such that if $(\lambda, u) \in (\mathcal{G}\backslash(I_\mu \times \{0\})) \cap (I_\mu \times B_\tau)$ for $\tau < \tau_0$, then $u = \alpha v + w$ with $\alpha > \eta \|u\|$.

It's obvious that $\operatorname{dist}(\lambda, I_\mu) = O(1)$ at $\alpha = 0$. By relation $\|w\| \leq \|u\| + |\alpha| < \left(\frac{1}{\eta} + 1\right)|\alpha|$ it follows that $w = O(|\alpha|)$ at $\alpha = 0$. Consider a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ such that

$$(\lambda_n, u_n) \to (\mu - m, 0) \text{ as } n \to \infty,\ m \in \mathcal{A}$$

and

$$(\lambda_n, u_n) \in (\mathcal{G}\backslash(I_\mu \times \{0\})) \cap (I_\mu \times B_\tau) \cap Q^+_{\xi,\eta}\,.$$

Then

$$\frac{F(\lambda_{n_k}, u_{n_k}) - mu_{n_k}}{\|u_{n_k}\|} \to 0 \text{ and } \frac{G(\lambda_{n_k}, u_{n_k})}{\|u_{n_k}\|} \to 0 \text{ as } n \to \infty$$

and, as above, a subsequence of $\frac{u_n}{\|u_n\|}$ converges to v. Consequently, we have $\left|\langle l, \frac{u_n}{\|u_n\|}\rangle\right| = \frac{\alpha_n}{\|u_n\|} \to 1$. Therefore, setting $u_n = \alpha_n v + w_n$, by this relation we obtain

$$\frac{w_n}{\|u_n\|} = \frac{u_n - \alpha_n v}{\|u_n\|} \to \left(\frac{u_n}{\|u_n\|} - \frac{\alpha_n}{\|u_n\|}v\right) \to 0, \quad n \to \infty,$$

which implies that $\frac{w_n}{\alpha_n} = \frac{w_n}{\|u_n\|}\frac{\|u_n\|}{\alpha_n} \to 0,\ n \to \infty$, along this subsequence. Since this is true for all such subsequences, and likewise with $Q^+_{\xi,\eta}$ replaced by $Q^-_{\xi,\eta}$, it follows that $w = o(\alpha)$ at $\alpha = 0$. The proof of Lemma 2.2 is complete.

Let $B_{\mu,r} = \{(\lambda, u) \in \mathbb{R} \times E\, :\, \operatorname{dist}(\lambda, I_\mu) < r, ||u|| < r\}$.

Corollary 2.1. *Under the conditions of Lemma 2.2 for each* $\lambda \in \{(\mu - m,\ 0)\ :\ m \in A\}$ *the set* $\mathcal{D}_{\mu,\lambda} = \tilde{\mathcal{D}}_{\mu,\lambda} \cup (I_\mu \times \{0\})$ *can be decomposed into two subcontinua* $\mathcal{D}^-_{\mu,\lambda}$ *and* $D^+_{\mu\lambda}$ *such that* $I_\mu \times \{0\} \subset \mathcal{D}^{\mp}_{\mu,\lambda}$ *and*

$$\mathcal{D}^-_{\mu,\lambda} \subset Q^-_{\xi,\eta} \cup (I_\mu \times \{0\}) \text{ and } \mathcal{D}^+_{\mu,\lambda} \subset Q^+_{\xi,\eta} \cup (I_\mu \times \{0\})\,.$$

Moreover, if $(\lambda, u) \in \mathcal{D}^-_{\mu,\lambda}\left(\mathcal{D}^+_{\mu,\lambda}\right) \cap (B_{\mu,\tau}\backslash(\mathbb{R} \times \{0\}))$, *then* $u = \alpha v + w$, *where* $w = o(\alpha)$ *at* $\alpha = 0$.

It follows by Corollary 2.1 that the set $\mathcal{D}_{\mu,}$ can be decomposed into two subcontinua $\mathcal{D}_{\mu}^{-}$ and $\mathcal{D}_{\mu}^{+}$ where

$$\mathcal{D}_{\mu}^{-}=\bigcup_{\mathbf{B}\cap(I_\mu\times\{0\})}\mathcal{D}_{\mu,\lambda}^{-}\ \text{ and }\ D_{\mu}^{+}=\bigcup_{\mathbf{B}\cap(I_\mu\times\{0\})}\mathcal{D}_{\mu,\lambda}^{+}.$$

The main result of this paper is the following

Theorem 2.1. *Let $\mu\in\sigma(L)$ is simple,* $\operatorname{dist}(\mu;\sigma(L)\backslash\{\mu\})>2M$ *and F satisfies the condition (A). Then either $\mathcal{D}_{\mu}^{-}$ and $\mathcal{D}_{\mu}^{+}$ are both unbounded in $\mathbb{R}\times E$, or $\mathcal{D}_{\mu}^{-}\cap D_{\mu}^{+}\neq I_\mu\times\{0\}$.*

Proof. Remark to Theorem 2 from [2] and Lemmas 2.1, 2.2 and Corollary 2.1 implies that for each $\lambda\in\{(\mu-m,\,0)\,:\,m\in A\}$ either (i) subcontinua $\mathcal{D}_{\mu,\lambda}^{-}$ is unbounded in $\mathbb{R}\times E$, or (ii) $\mathcal{D}_{\mu,\lambda}^{-}$ meets $\mathcal{D}_{\mu,\lambda}^{+}$ outside of a neighborhood of $(\lambda,0)$. On the other hand by Theorem from [5] either (i) $\mathcal{D}_{\mu}$ is unbounded in $\mathbb{R}\times E$, or (ii) $\mathcal{D}_{\mu}$ contains the set $I_{\hat\mu}\times\{0\}$ where $\mu\neq\hat\mu\in\sigma(L)$. Hence, either (i) $\mathcal{D}_{\mu}^{-}$ is unbounded in $\mathbb{R}\times E$, or (ii) D_{μ}^{-} meets $\mathcal{D}_{\mu}^{+}$ outside of a neighborhood of $I_\mu\times\{0\}$. It also shows that a similar result holds for $\mathcal{D}_{\mu}^{+}$. The proof of Theorem 2.2 is completed.

3 Application to some nonlinear eigenvalue problem for ordinary differential equations of second order

Consider the following problem

$$-y''=\lambda y+m|y|+o(|y|),\ \ 0<m\leq 1, \tag{3.1}$$

$$y(0)=0=y(\pi). \tag{3.2}$$

Let $H=L_2(0,\pi)$. Define the operators $L\,:\,D(L)\subset H\to H,\ F\,:\,H\to H$ and $G:\mathbb{R}\times H\to H$ as follows:

$$D(L)=\left\{y\in H\mid y\in W_2^2(0,\pi),\ -y''\in L_2(0,\pi),\ y(0)=0=y(\pi)\right\},$$

$$Ly=-y'',\ \ F(\lambda,y)=my,\ \ G(\lambda,y)=0(||y||)\text{ at } y=0. \tag{3.3}$$

Then the problem (3.1)-(3.2) can be written as an operator equation in the form of (1.1), i.e.

$$Ly=\lambda y+F(\lambda,y)+G(\lambda,y).$$

It is known that L is a self-adjoint operator in H and possesses infinitely many eigenvalues $\mu_k=k^2,\ k=1,2,\,\ldots$, all of which are simple. By (3.3) it follows that the conditions (1.2) and (1.3) are satisfied, and $M=m\leq 1$. Accordingly, $I_k=I_{\mu_k}=[k^2-m,k^2+m]$ for our problem. Hence $\inf\limits_{k\in\mathbb{N}}\operatorname{dist}(\mu_k,\sigma(L)\backslash\{\mu_k\})=3\geq 2\geq 2m$. Then by Theorem 2.1 for each $k\in\mathbb{N}$ the component $\mathcal{D}_k^{-}\equiv\mathcal{D}_{\mu_k}^{-}$ ($\mathcal{D}_k^{+}\equiv\mathcal{D}_{\mu_k}^{+}$) of solutions of problem (3.1)-(3.2), containing $I_k\times\{0\}$, either (i) is unbounded in $\mathbb{R}\times H$, or (ii) $\mathcal{D}_k^{-}$ meets $\mathcal{D}_k^{+}$ outside of a neighborhood of $[k^2-m,k^2+m]\times\{0\}$.

Note that any eigenfunction $y(x) > 0\,(y(x) < 0)$, $x \in (0, \pi)$, of problem (3.1)-(3.2) is also an eigenfunction of the nonlinear problem

$$-y'' = (\lambda - m(-m))y + o(y), \;\; 0 < m \leq 1,$$
$$y(0) = y(\pi) = 0.$$

Hence by [10, Theorem 2.3] the set $\mathcal{D}_1^-$ $(\mathcal{D}_1^+)$ is unbounded in $\mathbb{R} \times H$.

References

1. Berestycki, H.: *On some nonlinear Sturm-Liouville problems*. J. Diff. Equat., **26**, 375–390 (1977).
2. Dancer, E.N.: *On the structure of solutions of non-linear eigenvalue problems*. Indiana Univ. Math. J., **23**, 1069–1076 (1974).
3. Gaines, R.E., Mawhin, J.L.: Coincidence degree and nonlinear differential equations. *Springer-Verlag, Berlin, Heidelberg, New-York* (1977).
4. Krasnoselskii, M.A.: Topological methods in the theory of nonlinear integral equations. *Pergamon, London* (1974).
5. Makhmudov, A.P., Aliev, Z.S.: *Global bifurcation of solutions of certain nonlinearizable eigenvalue problems*. Diff. Equat., **25**, 71–76 (1989).
6. Makhmudov, A.P., Aliev, Z.S.: *Nondifferentiable perturbations of spectral problems for a pair of self-adjoint operators and global bifurcation*. Soviet Math., **34** (1), 51–60 (1990).
7. Mamedova, G.M.: *Local and global bifurcation for some nonlinearizable eigenvalue problems*. Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan, **40** (2), 45–51 (2014).
8. Przybycin, J.: *On bifurcation intervals for nonlinear eigenvalue problems*. Ann. Polon. Math. , **71** (1), 39–46 (1999).
9. Przybycin, J.: *Some theorems of Rabinowitz type for nonlinearizable eigenvalue problems*. Opuscula Mathematica, **24** (1), 115–121 (2004).
10. Rabinowitz, P.H.: *Some global results for nonlinear eigenvalue problems*. J. Funct. Anal., **7**, 487–513 (1971).
11. Rynne, B.P.: *Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable*. J. Math. Ann. Apll., **228**, 141–156 (1998).
12. Schmitt, K., Smith, H.L.: *On eigenvalue problems for nondifferentiable mappings*. J. Diff. Equat., **33**, 294-319 (1979).