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Trace Formula For Second Order Differential Operator Equation

Hajar F. Movsumova

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Abstract. In this paper the regularized trace formula of an operator generated by a differential expression with operator coefficient and eigenvalue dependent boundary condition is obtained.

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In the present paper a regularized trace of Sturm-Liouville's operator equation is studied.

Calculation of the regularized traces is one of the most important problems of the spectral analysis

The formula of the regularized trace of the Sturm-Liouville operator was first obtained by I.M.Gelfand and B.M. Levitan [13].

In 1953, Gelfand and Levitan considered the Sturm-Liouville equation $-y''(x)+q(x)y(x)=\lambda y(x)$, with boundary conditions $y'(0)=0$, $y'(\pi)=0$, where $q(x)\in C^1[0,\pi]$ and derived the formula $\sum\limits_{n=1}^{\infty}(\mu_n-\lambda_n)=\frac{1}{4}(q(0)+q(\pi))$, where μ_n are the eigenvalues of the above operator. For $q(x)\equiv 0$ the eigenvalues of the operator are given by $\lambda_n=n^2$.

After that work numerous papers were devoted to the calculation of regularized trace of scalar differential operators which is the generalization of concept of matrix trace.

The methods used to derive those formulas significantly need asymptotics of eigenvalues, when the remainder forms a convergent series .But for operators associated with boundary value problems for partial differential equations it becomes impossible to use that method. For such operators in many cases one may just find the principal term of asymptotics. By that reason it becomes important to study regularized traces of differential operators with unbounded operator coefficients.

In [18] the regularized trace of operator Sturm-Liouville equation is calculated.

H.F. Movsumova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F. Agayev str., AZ1141, Baku, Azerbaijan
E-mail: movsumovahecer@gmail.com

It should be noted that, variety of studies are devoted to searching a regularized traces of scalar differential operators [8],[9],[12],[13],[15]-[17],[19],[20],[23]as well as differential-operator equations [1]-[4], [6],[7],[18] and discrete abstract operators [10],[14],[21].

Boundary condition of the problem considered in that paper depends on eigenvalue parameter.

In [22], Walter considers a scalar Sturm-Liouville problem with an eigenvalue parameter λ in the boundary conditions. He shows that one can associate a self-adjoint operator with that by finding a suitable Hilbert space.Further,he obtains the expansion theorem by reference to the self-adjointness of that operator. This approach was used by Fulton in [11] later on.

Consider in $L_2(H,(0,1))$ the spectral problem

$$l\,[y] \equiv -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t) \tag{1}$$

$$y'(0) = 0 \tag{2}$$

$$ay(1) + y'(1) = -\lambda y'(1), \tag{3}$$

where A is a self-adjoint positive-definite operator in abstract separable Hilbert space H $(A > E, E$ is an identity operator in $H)$.

Problem (1)-(3) differs from operators in [1]-[3],[6],[7] by boundary conditions.

Denote the eigenvalues and eigen-vectors of the operator A by $\gamma_1 \leq \gamma_2 \leq ...$ and $\varphi_1, \varphi_2, ..$, respectively.

Suppose that the operator-valued function $q(t)$ is weakly measurable, $\|q(t)\|$ is bounded on $[0,1]$ and the following conditions are satisfied:

1) There exists a second order weak derivative of $q(t)$ on $[0,1]$ and for each $t \in [0,1][q^{(k)}(t)]^* = q^{(k)}(t), \quad k = 0,1,2;$

2) $\sum\limits_{k=1}^{\infty} |\, q^{(k)}(\mathrm{t})\varphi_k, \varphi_k)| < \mathrm{const};$

3) $q'(0) = q'(1) = 0;$

4) $\int\limits_0^1 (q\,(t)\,f, f)dt = 0$ for each $f \in H$.

Let $L_2 = L_2(H,(0,1)) \bigoplus H$. Denote a scalar product and the norm in H by $(\cdot,\cdot)$, and $\|\cdot\|$, respectively. Define the scalar product in L_2 as

$$(Y,Z)_{L_2} = \int_0^1 (y\,(t)\,, z\,(t))\,dt + \frac{1}{a}(y_1, z_1) \tag{4}$$

where $Y = \{y(t), y_1\}, \;\; Z = \{z(t), z_1\}, \;\; y(t), z(t) \in L_2(H,(0,1)), \; y_1, z_1 \in H, \; a > 0.$

For $q(t) \equiv 0$ in space L_2 one can associate with problem (1)-(3) in space L_2 a self adjoint operator L_0 defined by

$$\begin{gathered} D(L_0) = \{Y : Y = \{y(t), y_1\} / - y''(t) + Ay(t) \in L_2(H,(0,1)), \\ y'(0) = 0, \, y_1 = -y'(1)\}, \\ L_0 Y = \{-y''(t) + Ay(t), \; ay(1) + y'(1)\}. \end{gathered} \tag{5}$$

The operator corresponding to the case $q(t) \not\equiv 0$ is denoted by $L = L_0 + Q$, where $Q : Q\,\{y\,(t)\,, -y'(1)\} = \{q\,(t)\,y\,(t)\,, 0\}$ is a bounded self-adjoint operator in L_2.

Eigenvalue asymptotics of that operator was studied in [5]. In this paper after the above definitions and the assumptions, the regularized trace of the considered problem will be calculated. Because of the appearance of an eigenvalue parameter in the boundary condition at the end point, the operator associated with problem $(1),(2),(3)$ is not self-adjoint. That is why we will consider in space $L_2(H,(0,1))\bigoplus H$ with the scalar product defined by formula (4). It can easily be shown that in this space, the operator L_0 is self-adjoint.

Note that in $[5]$ for the eigenvalues of the problem

$$-y''(t)+Ay(t)=\lambda y(t)$$

$$y'(0)=0$$

$$ay(1)+by'(1)=\lambda(cy(1)-dy'(1))$$

the following asymptotic formula is obtained:

$$\lambda_{k,n}\sim\gamma_k+\alpha_n^2$$
$$\lambda_k\sim-\frac{b}{d}+\frac{-c^2\pm c\sqrt{c^2+4d\left(b+d\gamma_k\right)}}{2d^2}, \tag{6}$$

where $\alpha_n=\pi n,\ n\in Z.$

Here λ_k are the eigenvalues of the operator L_0. Eigenvalues of L denote by μ_k.

By using [theorem 1, [5]] as in [18] we come to the following statement.

Lemma 1. *If at $k\to\infty,\gamma_k\sim ak^{\alpha},0<a<\infty,2<\alpha<\infty,$ then there exists a subsequence $\lambda_{n_1}<\lambda_{n_2}<\dots$ of the sequence $\lambda_1,\lambda_2,\dots$ such that*

$$\lambda_p-\lambda_{n_m}\geq d_0\left(p^{\frac{2\propto}{2+\propto}}-n_m^{\frac{2\propto}{2+\propto}}\right)\ ,\ p=n_m,n_m+1,\dots,$$

where d_0 is a positive number.

Introduce the following notations

$$\mu^{(i)}=\sum_{k=n_{i-1}+1}^{n_i}\mu_k,\qquad\lambda^{(i)}=\sum_{k=n_{i-1}+1}^{n_i}\lambda_k,\qquad i=1,2,\dots. \tag{7}$$

Call the sum $\sum\limits_{i=1}^{\infty}\left(\lambda^{(i)}-\mu^{(i)}\right)$, a regularized trace of the operator L,since the sum of this series, as it will be shown below, doesn't depend on what way there has been chosen a subsequence $n_1,n_2,\dots$, which satisfies the statement of lemma 1.

In the present work the formula for the sum of this series has been obtained.

Let R_λ^0 and R_λ be the resolvents of the operators $L_0\ and\ L$. The following lemma is true (see [18]):

Lemma 2. *Let $\|q(t)\|$ be bounded on the segment $[0,1]$ and the conditions of lemma 1 be fulfilled. Then at large m the following equality holds:*

$$\sum_{n=1}^{n_m}(\lambda_n-\mu_n)=\sum_{j=1}^{N}(-1)^jM_m^j+\frac{(-1)^N}{2\pi i}\int_{|\lambda|<l_m}\lambda Sp\left[R_\lambda\left(QR_\lambda^0\right)^{N+1}\right]d\lambda, \tag{8}$$

where

$$l_m = \frac{1}{2}\left(\mu_{n_m+1} - \mu_{n_m}\right), \qquad M_m^j = \frac{1}{2\pi i}\int_{|\lambda|=l_m} Sp\ \left[\ \left(QR_\lambda^0\ \right)^j\right] d\lambda$$

$\mu_{n_m}, m = 1,2,3,\ldots$, *is a subsequence, that satisfies the statement of lemma 1* $\big($*N is an arbitarary natural number*$\big)$.

Denote the orthonormal eigenvectors of the operator L_0 by $\{\psi_n\}\ n = 1,2,\ldots$

Then

$$\psi_n = \frac{V}{\|V\|}$$

where V=$\{cos\left(x_{k,n}t\right)\varphi_k, x_{k,n}\sin x_{k,n}\ \varphi_k\}$ and $\{\varphi_k\}$ are the eigen-vectors of the operator A, $x_{k,n}$ is the root of the equation

$$a\cos z - z\sin z - \left(z^2 + \gamma_k\right)z\sin z = 0 \tag{9}$$

and has the following asymptotics (see [5])

$$x_{k,n} \sim \pi n. \tag{10}$$

Denote, for convenience, real and imaginary roots of equation (9), by $x_{k,n}\ \left(k = \overline{1,\infty}\right)$ and $x_{k,0}$, respectively.

$$\|V\|^2 = (V,V)_{L_2} = \int_0^1 cos^2\left(x_{k,n}t\right)dt + \frac{1}{a}(x_{k,n}\sin x_{k,n}\ )^2$$

$$= \int_0^1 \frac{1 + cos\left(2x_{k,n}t\right)}{2}dt + \frac{x_{k,n}^2 sin^2\ x_{k,n}}{a} = \frac{1}{2} + \frac{\sin 2x_{k,n}}{4x_{k,n}} + \frac{x_{k,n}^2 sin^2\ x_{k,n}}{a}$$

$$= \frac{2ax_{k,n} + a\sin 2x_{k,n}\ + 4x_{k,n}^3 sin^2\ x_{k,n}}{4ax_{k,n}}.$$

Therefore, we get

$$\psi_n = \sqrt{\frac{4ax_{k,n}}{2ax_{k,n} + a\sin 2x_{k,n}\ + 4x_{k,n}^3 sin^2\ x_{k,n}}}\{cos\left(x_{k,n}t\right)\varphi_k, x_{k,n}\sin x_{k,n}\ \varphi_k\}$$

$$\begin{pmatrix} n = \overline{0,\infty}, & k = \overline{N,\infty} \\ n = \overline{1,\infty}, & k = \overline{1,N-1} \end{pmatrix} \tag{11}$$

Prove the following lemma.

Lemma 3. Provided that for operator-valued function $q(t)$ hold the conditions 1)-3), then

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2ax_{k,n}\int_0^1\cos(2x_{k,n}t)q_k\left(t\right)dt}{2ax_{k,n} + a\sin 2x_{k,n}\ + 4x_{k,n}^3 sin^2\ x_{k,n}}\right| +$$

$$+\sum_{k=N}^{\infty}\ \left|\frac{2ax_{k,0}\int_0^1\cos(2x_{k,0}t)q_k\left(t\right)dt}{2ax_{k,0} + a\sin 2x_{k,0}\ + 4x_{k,0}^3 sin^2\ x_{k,0}}\right| < \infty$$

where $q_k\left(t\right) = \left(q\left(t\right)\varphi_k, \varphi_k\right)$.

Proof. Observe that, for large k and from (10), we get

$$\frac{2ax_{k,n}}{2ax_{k,n}+a\sin 2x_{k,n}\;+4x_{k,n}^{3}sin^{2}\;x_{k,n}}=\frac{1}{1+\frac{\sin 2x_{k,n}}{2x_{k,n}}+\frac{2x_{k,n}^{2}sin^{2}\;x_{k,n}}{a}}<$$

$$<\frac{1}{1+\frac{\sin 2x_{k,n}}{2x_{k,n}}}=1+O\left(\frac{1}{x_{k,n}}\right). \tag{12}$$

Integrating by parts twice and using condition 3) for q(t) we get

$$\int_{0}^{1}\cos(2x_{k,n}t)q_{k}\left(t\right)dt$$

$$=\frac{1}{2x_{k,n}}\sin\left(2x_{k,n}\right)q_{k}\left(1\right)\;-\frac{1}{\left(2x_{k,n}\right)^{2}}\int_{0}^{1}\cos(2x_{k,n}t)q_{k}^{''}\left(t\right)dt\;. \tag{13}$$

In virtue of asymptotics $x_{k,n}$ and (12), (13) it follows from the last relation,that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2ax_{k,n}\int_{0}^{1}\cos(2x_{k,n}t)q_{k}\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}\;+4x_{k,n}^{3}sin^{2}\;x_{k,n}}\right|=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(1+O\left(\frac{1}{x_{k,n}}\right)\right)\left(O\left(\frac{1}{n^{2}}\right)q_{k}\left(1\right)+\int_{0}^{1}O\left(\frac{1}{n^{2}}\right)\cos(2x_{k,n}t)q_{k}^{''}\left(t\right)dt\right)\le$$

$$\le const\sum_{k=1}^{\infty}\left(q_{k}\left(1\right)+\int_{0}^{1}q_{k}^{''}\left(t\right)dt\right).$$

From condition 2) and the last relation it follows absolute convergence of the series

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{2ax_{k,n}\int_{0}^{1}\cos(2x_{k,n}t)q_{k}\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}\;+4x_{k,n}^{3}sin^{2}\;x_{k,n}}.$$

Consider the series

$$\sum_{k=N}^{\infty}\frac{2ax_{k,0}\int_{0}^{1}\cos(2x_{k,0}t)q_{k}\left(t\right)dt}{2ax_{k,0}+a\sin 2x_{k,0}\;+4x_{k,0}^{3}sin^{2}\;x_{k,0}}$$

From asymptotics $x_{k,0}\sim i\sqrt{\gamma_{k}}$, where $\gamma_{k}\sim ak^{\alpha}$, we get

$$\sum_{k=N}^{\infty}\frac{2ax_{k,0}\int_{0}^{1}\cos(2x_{k,0}t)q_{k}\left(t\right)dt}{2ax_{k,0}+a\sin 2x_{k,0}\;+4x_{k,0}^{3}sin^{2}\;x_{k,0}}=\sum_{k=N}^{\infty}\left(1+O\left(\frac{1}{k^{\propto}}\right)\right)\int_{0}^{1}q_{k}\left(t\right)dt$$

Now absolute convergence of this series follows from condition 2).

The lemma is proved.

Theorem 1. *Let the conditions of lemma 1 be satisfied.Provided operator-valued q(t) satisfies conditions 1)-3),then the formula*

$$\lim_{n\to\infty}M_{m}^{1}\;=-\frac{Sp\;q\left(0\right)+Sp\;q\left(1\right)}{4} \tag{14}$$

is true.

Proof. From (8) at $j=1$ we have

$$M_m^1=\frac{1}{2\pi i}\int_{|\lambda|=l_m} Sp\ \left(QR_\lambda^0\ \right)d\lambda$$

Since QR_λ^0 is a trace class (Q is a bounded operator in L_2 and R_λ^0 is a trace class operator) operator and eigenvectors $\psi_1(x),\psi_2(x),\ldots$ of the operator L_0 form orthonormal basis in the space $L_2(H,(0,1))$, then for large values of m

$$M_m^1=\frac{1}{2\pi i}\int_{|\lambda|<l_m}\sum_{n=1}^{\infty}\left(QR_\lambda^0\,\psi_n,\psi_n\right)_1 d\lambda$$

$$=-\sum_{n=1}^{\infty}\left[(Q\psi_n,\psi_n)_1\ \frac{1}{2\pi i}\int_{|\lambda|=l_m}\frac{d\lambda}{\lambda-\mu_n}\right]=-\sum_{n=1}^{\infty}(Q\psi_n,\psi_n)_1.$$

Taking into account (11) in the last equation we get

$$M_m^1=-\sum_{n=1}^{n_m}(Q\psi_n,\psi_n)=$$

$$=-\sum_{n=1}^{n_m}\frac{4ax_{k,n}}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}}\int_0^1 cos^2\left(x_{k,n}t\right)\left(q\left(t\right)\varphi_{i_n},\varphi_{i_n}\right)dt=$$

$$=-\sum_{n=1}^{n_m}\frac{4ax_{k,n}}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}}\int_0^1\frac{1+cos\left(2x_{k,n}t\right)}{2}\left(q\left(t\right)\varphi_{i_n},\varphi_{i_n}\right)dt.$$

By condition 4)

$$M_m^1=-\sum_{n=1}^{n_m}\frac{2ax_{k,n}}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}}\int_0^1 cos\left(2x_{k,n}t\right)\left(q\left(t\right)\varphi_{i_n},\varphi_{i_n}\right)dt.$$

According to lemma 3 series (13) converges absolutely,so we will have

$$\lim_{m\to\infty}\ M_m^1=\sum_{k=N}^{\infty}\sum_{n=0}^{\infty}\int_0^1\frac{-2ax_{k,n}cos\left(2x_{k,n}t\right)q_k\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}}+$$

$$+\sum_{k=1}^{N-1}\sum_{n=1}^{\infty}\int_0^1\frac{-2ax_{k,n}cos\left(2x_{k,n}t\right)q_k\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}}\equiv I_1+I_2 \tag{15}$$

Compute the value of the series

$$\sum_{n=0}^{\infty}\int_0^1\frac{-2ax_{k,n}cos\left(2x_{k,n}t\right)q_k\left(t\right)dt}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}} \tag{16}$$

For this as $N\to\infty$ we will investigate the asymptotic behavior of the following function

$$T_N\left(t\right)=\sum_{n=0}^{N}\frac{-2ax_{k,n}cos\left(2x_{k,n}t\right)}{2ax_{k,n}+a\sin 2x_{k,n}\ +4x_{k,n}^3 sin^2\ x_{k,n}}.$$

Express the k-th term of the sum $T_N(t)$ as a residue at the pole $x_{k,n}$ of some function of complex variable z:

$$G(z) = \frac{az\cos 2zt}{\left(\frac{a\ ctg\ z}{z} - 1 - (z^2+\gamma_k)\right) z^2 sin^2 z}.$$

This function has simple poles at the points $x_{k,n}$, πn and $z = 0$.

We have

$$\underset{z=x_{k,n}}{\text{res}}\ G(z) = \frac{ax_{k,n}\cos 2x_{k,n}t}{x_{k,n}^2 sin^2 x_{k,n}\left(\frac{a\ ctg\ z}{z} - 1 - (z^2+\gamma_k)\right)'_{z=x_{k,n}}} =$$

$$= -\frac{2ax_{k,n}cos\,(2x_{k,n}t)}{2ax_{k,n} + a\sin 2x_{k,n}\ + 4x_{k,n}^3 sin^2\ x_{k,n}}.$$

Find the residue at πn :

$$\underset{z=\pi n}{\text{res}}\ G(z) = \underset{z=\pi n}{\text{res}} \frac{az\cos 2zt}{\left(\frac{a\ ctg\ z}{z} - 1 - (z^2+\gamma_k)\right) z^2 sin^2 z} =$$

$$= \underset{z=\pi n}{\text{res}} \frac{az\cos 2zt}{\left(\frac{a\cos z}{z} - \sin z - z^2\sin z\ \ - \gamma_k \sin z\ \right)\ z^2 \sin z}$$

$$= \frac{a\pi n\cos 2\pi nt}{a\frac{\cos \pi n}{\pi n}\ (\pi n)^2 \cos \pi n} = \cos 2\pi nt\ .$$

For I_1 take as a contour of integration the rectangle with vertices at $\pm iB$, $A_N \pm iB$,which has cut at $ix_{k,0}$ and will by pass it on the left,and the point -$ix_{k,0}$ on the right. Take also $B > x_{k,0}$. Then B will go to infinity and take $A_N = \pi N + \frac{\pi}{2}$. For such choice of A_N we have $x_{N-1,k} < A_N < x_{N,k}$.

For I_2 take as a contour of integration the rectangle with vertices at $\pm iB, A_N \pm iB$, which bypass the origin on the right hand side of the imaginary axis. B will further go to infinity and take $A_N = \pi N + \frac{\pi}{2}$. For such choice of A_N we have $x_{N-1,k} < A_N < x_{N,k}$.

Since $G(z)$ is an odd function of z,then the integrals along the part of contours on imaginary axis, and the integral along semicircles centered at $\pm ix_{k,0}$ vanish.

If $z = u + iv$, then for large v and for $u \geq 0$ $G(z)$ is of order $O\left(\frac{e^{2|v|(t-1)}}{|v|^3}\right)$ that is why for the given value of A_N the integrals along upper and lower sides of the contour also go to zero when $B \to \infty$.

Hence, we get the formula

$$T_N(t) + S_N(t) = \frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB} G(z)\,dz + \frac{1}{2\pi i}\lim_{r\to 0}\int_{\substack{|z|=r \\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} G(z)\,dz, \qquad (17)$$

where

$$S_N(t) = \sum_{n=1}^{N} \cos 2\pi nt.$$

As $N \to \infty$

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB} G\left(z\right)dz \sim \frac{1}{2\pi i}\int_{A_N-i\infty}^{A_N+i\infty}\frac{\cos 2zt}{z^3 sin^2 z}dz =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\cos\left(2\pi Nt + 2ivt + \pi t\right)}{\left(A_N+iv\right)^3\left(1-\cos\left(2A_N+2iv\right)\right)}dv =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\cos\left((2N+1)\pi t\right)\cos 2ivt - \sin\left((2N+1)\pi t\right)\sin 2ivt}{\left(A_N+iv\right)^3\left(1+\cos 2iv\right)}dv =$$

$$= \frac{1}{2\pi}\cos\left((2N+1)\pi t\right)\int_{-\infty}^{+\infty}\frac{ch\ 2vt}{\left(A_N+iv\right)^3\left(1+\cos 2iv\right)}dv +$$

$$+ \frac{1}{2\pi i}\sin\left((2N+1)\pi t\right)\int_{-\infty}^{+\infty}\frac{sh\ 2vt}{\left(A_N+iv\right)^3\left(1+\cos 2iv\right)}dv. \qquad (18)$$

Denote the integrals on the right hand side of (18) by K_1 and K_2, respectively. Then,

$$|K_1| = \left|\frac{1}{2\pi}\cos\left((2N+1)\pi t\right)\int_{-\infty}^{+\infty}\frac{ch\ 2vt}{\left(A_N+iv\right)^3\left(1+\cos 2iv\right)}dv\right| <$$

$$< \int_{-\infty}^{+\infty}\frac{dv}{\sqrt{\left(A_N^2+v^2\right)^3}} = 2\int_0^{+\infty}\frac{dv}{\sqrt{\left(A_N^2+v^2\right)^3}} < \frac{2}{A_N}\int_0^{+\infty}\frac{dv}{\sqrt{A_N^2+v^2}} =$$

$$= \frac{2}{A_N}\ln\left|\frac{v}{A_N}+\sqrt{\frac{v^2}{A_N^2}+1}\right|_0^{A_N} = \frac{const}{A_N}. \qquad (19)$$

The similar estimate is obtained also for K_2. So, by using (18) and (19) in (17), we get

$$\lim_{N\to\infty}\int_0^1 T_N(t)q_k(t)dt$$

$$= -\lim_{N\to\infty}\int_0^1 S_N(t)q_k(t)dt + \frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 q_k(t)dt \int\limits_{\substack{|z|=r \\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} G(z)dz. \qquad (20)$$

By condition 4) the second term in the right hand side of (20) can be written as

$$\lim_{r\to 0}\int_0^1 q_k(t) \int\limits_{\substack{|z|=r \\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{az\left(1-2\sin^2 zt\right)}{\left(\frac{a\ ctg\ z}{z}-1-\left(z^2+\gamma_k\right)\right)z^2\sin^2 z}dzdt =$$

$$= \lim_{r\to 0}\int_0^1 f_k(t) \int\limits_{\substack{|z|=r \\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{-2az\sin^2 zt}{\left(\frac{a\ ctg\ z}{z}-1-\left(z^2+\gamma_k\right)\right)z^2\sin^2 z}dzdt.$$

Since the numerator of the integrand for small z is of order $O\left(z^3\right)$, and the denominator is of order $O\left(z^2\right)$, then the last one goes to zero.

So, by substitution $\pi t = z$ we have

$$\lim_{N\to\infty}\int_0^1 T_N(t)q_k(t)dt = -\lim_{N\to\infty}\int_0^1 S_N(t)q_k(t)dt = -\sum_{n=1}^{\infty}\int_0^1 \cos 2\pi nt\, q_k(t)dt =$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^1 \cos 2\pi nt\, q_k(t)d\pi t = -\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi} \cos 2nzq_k\left(\frac{z}{\pi}\right)dz =$$

$$= -\frac{1}{4}\frac{2}{\pi}\sum_{n=0}^{\infty}\left[\cos n\cdot 0\int_0^{\pi}\cos nzq_k\left(\frac{z}{\pi}\right)dz + \cos n\cdot\pi\int_0^{\pi}\cos nzq_k\left(\frac{z}{\pi}\right)dz\right] =$$

$$= -\frac{q_k(0)+q_k(1)}{4}. \tag{21}$$

Summing by all k, we get

$$\lim_{n\to\infty} M_m^1 = -\frac{Sp\; q(0) + Sp\; q(1)}{4}.$$

In [7] the following theorem was proved.

Theorem 2. *Let the condition of lemma 1 be fulfilled. If the operator function $q(t)$ satisfies condition 1), 2), 3), then at $n \geq 2$*

$$\lim_{m\to\infty} M_m^n = 0.$$

From lemma 1, theorems 1, 2 it follows that

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\lambda_n - \mu_n) = \frac{Sp\; q(0) + Sp\; q(1)}{4}$$

$$+\frac{(-1)^N}{2\pi i}\lim_{m\to\infty}\int_{|\lambda|=l_m}\lambda Sp\left[R_\lambda\left(QR_\lambda^0\right)^{N+1}\right]d\lambda, \quad N\geq 2.$$

We can show that the limit on the right hand side of the last equation, equals zero. So,using designation (7), we get

$$\sum_{i=1}^{\infty}\left(\lambda^{(i)} - \mu^{(i)}\right) = \frac{Sp\; q(0) + Sp\; q(1)}{4}. \tag{22}$$

Thus,the following theorem is proved.

Theorem 3. *Let the operator function $q(t)$ satisfy conditions 1)-4). Then under the conditions of lemma 1, for the regularized trace formula (22) holds.*

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