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On the properties of the orthogonal polynomials with weight along the piecewise smooth contour

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Abstract. *In this work, we investigate the order of the height of the modulus of orthogonal polynomials over a piecewise smooth contour and also arbitrary algebraic polynomials in the weighted Lebesgue space, where the contour and weight function have some singularities. We analyze the different conditions with respect to the degree of singularity of weight and contour.*

Keywords. Orthogonal polynomials, Algebraic polynomials, Conformal mapping, Quasicircle, Smooth curve.

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1 Introduction

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $L \subset \mathbb{C}$ be a closed rectufiable Jordan curve, $G := intL$, with $0 \in G$, $\Omega := extL$. Let $h(z)$ nonnegative, summable on a L and nonzero except possible on a set of measure zero function. The systems of polynomials $\{K_n(z)\}$, $K_n(z) = a_n z^n + ...,\ \deg K_n = n$, $n = 0, 1, 2, ...$, satisfyung the condition

$$\int_L h(z) K_n(z) \overline{K_m(z)}\, |dz| = \begin{cases} 1, \, n = m, \\ 0, \, n \neq m, \end{cases}$$

are called orthonormal polynomials for the pair (L, h). These polynomials are determined uniquely if the coefficient $a_n > 0$.

These polynomials were first studied in [34], [35]. In [33], [21], [18] [38] [39], [22] and [16] was investigated some properties of the polynomials $K_n(z)$ under the various conditions on the weight function $h(z)$ and contour L. In particular, obtained some estimates for the rate of growth of the polynomials $K_n(z)$ on the contour L, depending of the singularites of the weight function $h(z)$ on L and of the contour L.

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Lets $w = \Phi(z)$ be a univalent conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$, $\Phi(\infty) = \infty$, $\lim\limits_{z\to\infty} \frac{\Phi(z)}{z} > 0$, and $\Psi := \Phi^{-1}$. For $t \geq 1$, a exteriror level curves for region G defined as:

$$L_t := \{z : \ |\Phi(z)| = t\}, \ L_1 \equiv L, \ G_t := intL_t, \ \Omega_t := extL_t.$$

Let $\{z_j\}_{j=1}^{m}$ be the fixed system of distinct points on curve L. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0} \backslash G$, consider generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \tag{1.1}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., m$, and h_0 is uniformly separated from zero in L, i.e. there exists a constant $c_0(L) > 0$ such that $h_0(z) \geq c_0(L) > 0$ is fullfiled for all $z \in G_{R_0}$.

Let a rectifiable Jordan curve be L, has a natural parametrization $z = z(s), 0 \leq s \leq l := mesL$. It is said to be $L \in C(1, \alpha)$, $0 < \alpha < 1$, if $z(s)$ is continuously differentiable and $z'(s) \in Lip\alpha$.Let L belong to $C(1, \alpha)$ everywhere except for a single point $z_1 \in L$, i.e., the derivative $z'(s)$ satisfies the Lipschitz condition on the $[0, l]$ and $z(0) = z(l) = z_1$, but $z'(0) \neq z'(l)$. Assume that L has a corner at z_1 with exterior angle $\nu_1\pi$, $0 < \nu_1 \leq 2$, and denote the set of such curves by $C(1, \alpha, \nu_1)$. In [39], author was investigated this problem for $K_n(z)$ with the weight function $h(z)$ defined as in (1.1) and for the curve $L \in C(1, \alpha, \nu_1)$ and is shown that the condition of "pay off" singularity curve and weight function at the points z_1 can be given as follows:

$$(1 + \gamma_1)\, \nu_1 = 1, \tag{1.2}$$

and, under this conditions, for $K_n(z)$ provided the following estimation:

$$|K_n(z)| \leq c(L)\sqrt{n+1}, \ z \in L, \tag{1.3}$$

where $c(L) > 0$ is a constant independent on n.

In [39] also shown, if the singularity of a curve and weight function at the points z_1 satisfy the condition:

$$(1 + \gamma_1)\, \nu_1 > 1, \tag{1.4}$$

then

$$|z - z_1|^{\mu_1} |K_n(z)| \leq c_1(L)\sqrt{n+1}, \ z \in L, \tag{1.5}$$

$$|K_n(z_1)| \leq c_2(L)\, (n+1)^{s_1}, \tag{1.6}$$

where

$$s_1 = \frac{1}{2}\, (1 + \gamma_1)\, \nu_1, \ \mu_1 = \frac{1}{2}\left(1 + \gamma_1 - \frac{1}{\nu_1}\right),$$

and $c_1(L) > 0$, $c_2(L) > 0$ are the constants independent on n.

In this work we study the estimations of the (1.5) and (1.6)-type, under the condition (1.4), for piecewise smooth contours of the complex plane and we obtain the analog of the estimations (1.5) and (1.6). In parallel we study this problem for arbitrary algebraic polynomials in the weighted Lebesgue space.

2 Definitions and Main Results

Throughout this paper, $c, c_{0,}c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, ...\} \cup \{0\}$.

Without loss of generality, the number R_0 in the definition of the weight functions, we can take $R_0 = 2$. Otherwise the natural number n can be choosen $n \geq \left[\frac{\varepsilon_0}{R_0-1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we denote

$$\|P_n\|_{L_p} := \|P_n\|_{L_p(h,L)} := \left(\int\limits_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p}, \ 0 < p < \infty,$$

$$\|P_n\|_{L_\infty} := \|P_n\|_{L_\infty(1,L)} := \max_{z \in L} |P_n(z)| \, , \ p = \infty.$$

For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k + 1, ..., m$.

Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w : |w| < 1\}$ onto the $G = intL$ normalized by $\psi(0) = 0$, $\psi'(0) > 0$. By [29, pp.286-294], we say a bounded Jordan region G is called κ *-quasidisk,* $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, the homeomorphism of the plane $\overline{\mathbb{C}}$ on plane $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a κ *-quasicircle*. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is κ*-quasidisk (κ -quasicircle)* for some $0 \leq \kappa < 1$.

We denoted the class of κ *-quasicircle* by $Q(\kappa)$, $0 \leq \kappa < 1$, and denote by $L \in Q$, if $L \in Q(\kappa)$, for some $0 \leq \kappa < 1$. It is well-known that the quasicircle may not even be locally rectifiable in [23, p.104].

We say that $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $L \in Q(\kappa)$ and L is rectifiable. Analogously, $L \in \widetilde{Q}$, if $L \in \widetilde{Q}(\kappa)$, for some $0 \leq \kappa < 1$.

Definition 2.1 *We say that $L \in Q_\alpha$, $0 < \alpha \leq 1$, if $L \in Q$ and $\Phi \in Lip\alpha$, $z \in \overline{\Omega}$.*

The class Q_α is sufficiently wide. In accordance with the results given as [30], [24], [40] and references cited therein, it is possible to conduct a detailed analysis of this class. Here we consider only some cases:

Remark 2.1 a) If $L = \partial G$ is a Dini-smooth curve [30, p.48], then $L \in Q_1$.

b)If $L = \partial G$ is a piecewise Dini-smooth curve and largest exterior angle at L has opening $\alpha\pi, 0 < \alpha \leq 1$, [30, p.52], then $L \in Q_\alpha$.

c)If $L = \partial G$ is a smooth curve having continuous tangent line, then $L \in Q_\alpha$ for all $0 < \alpha < 1$.

d) If L is quasismooth (in the sense of Lavrentiev), that is, for every pair $z_1,\ z_2 \in L$, if $s(z_1, z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on L, there exists a constant $c > 1$ such that $s(z_1, z_2) \leq c\,|z_1 - z_2|$, then $\Phi \in Lip\ \alpha$ for $\alpha = \frac{1}{2}(1 - \frac{1}{\pi} \arcsin \frac{1}{c})^{-1}$ [40].

e) If L is "c-quasiconformal" (see, for example, [24]), then $\Phi \in Lip\ \alpha$ for $\alpha = \frac{\pi}{2(\pi - \arcsin\frac{1}{c})}$. Also, if L is an asymptotic conformal curve, then $\Phi \in Lip\ \alpha$ for all $0 < \alpha < 1$ [24].

Definition 2.2 *It is said that $L \in \widetilde{Q}_\alpha, 0 < \alpha \leq 1$, if $L \in Q_\alpha$ and L is rectifiable.*

In this case, we have the following:

Theorem A. [28] *Let $p > 0$. Suppose that $L \in \widetilde{Q}_\alpha$, for some $0 < \alpha \leq 1$ and $h(z)$ defined as in (1.1) with $\gamma_j = 0$, for all $j = \overline{1,m}$. Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, there exists $c_1 = c_1(L,p) > 0$ such that*

$$\|P_n\|_{L_\infty} \leq c_1 \|P_n\|_{L_p(h_0,L)} \begin{cases} (n+1)^{\frac{1}{\alpha p}}, & \frac{1}{2} \leq \alpha \leq 1, \\ (n+1)^{\frac{\delta}{p}}, & 0 < \alpha < \frac{1}{2}, \end{cases} \tag{2.1}$$

where $\delta = \delta(L)$, $\delta \in [1,2]$, is a certain number.

Theorem A provides an opportunity to observe the growth of $|P_n(z)|$ on the curve L. A similar results for different Jordan curves and weight functions was investigated in [20], [36], [37], [25], [26], [27, pp.122-133], [31], [15, Theorem 6], [2]-[10] and others (see also the references cited therein).

The 2.1 makes it possible to calculate α in the right parts of estimations (2.1). In addition, for $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, the estimation (2.1) is satisfied for $\alpha = \frac{1}{1+\kappa}$ [9].

Now, lets introduce "special" singular points on the curve L.

Definition 2.3 *We say that $L \in \widetilde{Q}\left[\nu\right], 0 < \nu < 2$, if*

a) $L \in \widetilde{Q}$,

b) For $\forall z \in L$, there exists a $r := r(L,z) > 0$ and $\nu := \nu(L,z),\ 0 < \nu < 2$, such that for some $0 \leq \theta_0 < 2$ a closed maximal circular sector

$$S(z;r,\nu) := \left\{\zeta : \zeta = z + re^{i\theta\pi},\ \theta_0 < \theta < \theta_0 + \nu\right\}$$

of radius r and opening $\nu\pi$ lies in $\overline{G} = \overline{intL}$ with vetrex at z.

Definition 2.4 *We say that $L \in \widetilde{Q}_\alpha\left[\nu_1,...,\nu_m\right], 0 < \nu_1,...,\nu_m < 2,\ 0 < \alpha \leq 1$, if there exists a system of points $\{\zeta_i\} \in L$, $i = \overline{1,m}$, such that $L \in \widetilde{Q}\left[\nu_i\right]$ for any points $\zeta_i \in L$, $i = \overline{1,m}$, and $\Phi \in Lip\alpha$, $0 < \alpha \leq 1,\ z \in \overline{\Omega} \backslash \left\{\zeta_i\right\}$.*

It is clear from Definition 2.3 (2.4), that each contour $L \in \widetilde{Q}_\alpha\left[\nu_1,...,\nu_m\right], 0 < \nu_1,...,\nu_m < 2,\ 0 < \alpha \leq 1, i = \overline{1,m}$, may have "singularities" at the points $\{\zeta_i\}_{i=1}^m \in L$. If a contour L does not have such "singularities", i.e. if $\nu_i = 1$, $i = \overline{1,m}$, then it is written as $L \in \widetilde{Q}_\alpha$, $0 < \alpha \leq 1$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^m \in L$ are defined in (1.1) and $\{\zeta_i\}_{i=1}^m \in L$ are defined in Definitions 2.2 coincides. Without the loss of generality, we also will assume that the points $\{z_i\}_{i=1}^m$ are ordered in the positive direction on the curve L.

In [28], shown the condition of "pay off" of singularity of curve and weight function at the points $\{z_i\}_{i=1}^m$:

Theorem B. *Let* $p>0$. *Suppose that* $L\in \widetilde{Q}_{\alpha}\left[\nu_1,...,\nu_m\right]$, *for some* $0<\nu_1,...,\nu_m<1,\ \frac{1}{2-\nu_i}\leq\alpha\leq 1$; $h(z)$ *defined as in (1.1) and*

$$\gamma_i+1=\frac{1}{\alpha(2-\nu_i)}, \tag{2.2}$$

for each points $\{z_i\}_{i=1}^m$. *Then, for any* $P_n\in\wp_n,\ n\in\mathbb{N}$, *there exists* $c_2=c_2(L,p,\gamma_i)>0$ *such that*

$$\|P_n\|_{L_\infty}\leq c_2(n+1)^{\frac{1}{\alpha p}}\|P_n\|_{L_p(h,L)}. \tag{2.3}$$

Corollary B. *Suppose that* $L\in \widetilde{Q}_{\alpha}\left[\nu_1,...,\nu_m\right]$, *for some* $0<\nu_1,...,\nu_m<1,\ \frac{1}{2-\nu_i}\leq \alpha\leq 1$; $h(z)$ *defined as in (1.1). Then, under the conditions (2.2),*

$$\|K_n\|_{L_\infty}\leq c_2(n+1)^{\frac{1}{2\alpha}}. \tag{2.4}$$

Theorem B (Corollary B) show that, if the equality (2.2) is satisfied, then the growth of rate of the polynomials $P_n(z)$ ($K_n(z)$) on L does not depend on whether the weight function $h(z)$ and the boundary contour L have singularity or not. The condition (2.2) is called the condition of "interference of singularities" of weight function h and contour L at the "singular" points $\{z_i\}_{i=1}^m$.

Now, let the equality (2.2) does not hold for each singular point $\{z_i\}_{i=1}^m$. In [11] and [12] was investigated the cases $\gamma_i+1<\frac{1}{\alpha(2-\nu_i)}$ and $\gamma_i+1>\frac{1}{\alpha(2-\nu_i)}$ respectuvely, and obtain analogous results to the (1.5) and (1.6) for the case $0<\nu_i<1$. In this work, we investigate case $0\leq\nu_i\leq 1$ for piesewise smooth contour. For this we will begin with definition.

Let S be rectifiable Jordan curve or arc and let $z=z(s),\ s\in[0,\ |S|],\ |S|:=mes\ S$, denote the natural representation of S.

Definition 2.5 *We say that a Jordan curve or arc* $S\in C_\theta$, *if* S *has a continuous tangent* $\theta(z):=\theta(z(s))$ *at every point* $z(s)$.

Now, we shall define a new class of curves L, which have a exterior corners (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^m\in L$.

Definition 2.6 *We say that a Jordan region* $L\in PC_\theta(\lambda_1,\lambda_2,...,\lambda_m)$, $0<\lambda_i\leq 2,\ i=\overline{1,m}$, *if* $L=\partial G$ *consists of the union of finite* C_θ*(smooth)- arcs* $\{L_i\}_{i=1}^m$, *such that they have exterior (with respect to* $\overline{G}$*) angles* $\lambda_i\pi,\ 0<\lambda_i\leq 2$, *at the corner points* $\{z_i\}_{i=1}^m\in L$, *where two arcs meet.*

We have the following main result:

Theorem 2.1 *Let* $p>0$. *Suppose that* $L\in PC_\theta(\lambda_1,...,\lambda_m)$, *for some* $0<\lambda_i\leq 2,\ i=\overline{1,m}$; $h(z)$ *defined as in (1.1) and*

$$\gamma_i+1>\frac{1}{\lambda_i}, \tag{2.5}$$

for each point $\{z_i\}_{i=1}^m$. *Then there exists* $c_j=c_j(L,p,\gamma_i,\varepsilon)>0,\ j=3,4$, *such that, for any* $P_n\in\wp_n,\ n\in\mathbb{N}$, *we have:*

$$\max_{z\in L}\left(\prod_{i=1}^m|z-z_i|^{\mu_i}|P_n(z)|\right)\leq c_3 n^{\frac{1}{p}+\varepsilon}\|P_n\|_{L_p(h,L)}, \tag{2.6}$$

$$|P_n(z_i)| \leq c_4 n^{s_i} \|P_n\|_{L_p(h,L)}, \tag{2.7}$$

where $\mu_i := \frac{1}{p}\left(\gamma_i + 1 - \frac{1}{\lambda_i}\right)$, $s_i = \frac{\gamma_i+1}{p}\widetilde{\lambda}_i,\ i = \overline{1,m};\ \widetilde{\lambda}_i := \begin{cases} \lambda_i + \varepsilon, \text{ if } 0 < \lambda_i < 2, \\ 2, \quad \text{ if } \lambda_i = 2, \end{cases}$ *for arbitrary small* $\varepsilon > 0$.

Corollary 2.1 *Under the comditions of Theorem 2.1, we have:*

$$\max_{z\in L}\left(\prod_{i=1}^{m}|z - z_i|^{\mu_i}|K_n(z)|\right) \leq c_3 n^{\frac{1}{2}+\varepsilon}, \tag{2.8}$$

$$|K_n(z_i)| \leq c_4 n^{s_i}, \tag{2.9}$$

where μ_i *and* s_i *defines as above for* $p = 2$.

Note that, $C(1,\alpha,\lambda_1) \subset PC_\theta(\lambda_1)$ for each fixed $0 < \lambda_1 \leq 2$ and $PC_\theta(\lambda_1) \subset \widetilde{Q}_\alpha[\lambda_1]$, for each fixed $0 < \lambda_1 < 1$. So, (2.8) and (2.9) coincides with (1.5) and (1.6) to within an arbitrary small $\varepsilon > 0$. Thus, the Corollary 2.1 generalizes the corresponding result in [39].

Remark 2.2 a) The inequality (2.7) are sharp to within an arbitrary small $\varepsilon > 0$. For the polynomials $P_n^*(z) = 1 + 2z + ... + (n+1)z^n$, $h^*(z) = h_0(z)$ and $L := \{z : |z| = 1\}$, there exists a constant $c_5 = c_5(h_0) > 0$ such that:

$$\|P_n^*\|_{L_\infty} \geq c_5\sqrt{n}\,\|P_n^*\|_{L_2(h^*,\, L)}.$$

b) The inequalities (2.6) is sharp in the sense that for the arbitrary polynomial $P_n \in \wp_n$, $L \in PC_\theta(\lambda_1)$ and arbitrary small ϵ, $0 < \epsilon < \mu_1$, the following is true:

$$|z - z_1|^{\mu_1-\epsilon}|P_n(z)| \leq c_6 n^{\frac{1}{p}+\epsilon}\|P_n\|_{L_p(h,L)},$$

where

$$\mu_1 := \frac{1}{p}\left(\gamma_1 + 1 - \frac{1}{\lambda_1}\right),\ c_6 = c_6(\gamma_1, \lambda_1, \epsilon).$$

3 Some auxiliary results

For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well-known (see, for example, [13], [23, p.97] and [32]):

Definition 3.1 *The Jordan arc (or curve) L is called* $K-$*quasiconformal* ($K \geq 1$), *if there is a* $K-$*quasi-conformal mapping* f *of the region* $D \supset L$ *such that* $f(L)$ *is a line segment (or circle).*

Let $F(L)$ *denotes the set of all sense preserving plane homeomorphisms* f *of the region* $D \supset L$ *such that* $f(L)$ *is a line segment (or circle) and lets define*

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ *is the maximal dilatation of a such mapping* f. *L is a quasiconformal curve, if* $K_L < \infty$, *and L is a* $K-$*quasiconformal curve, if* $K_L \leq K$.

According to [32], we have the following fact:

Corollary 3.1 *If $S \in C_\theta$, then S is $(1+\varepsilon)-$quasiconformal for arbitrary small $\varepsilon > 0$.*

Remark 3.1 It is well-known that, if we are not interested with the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are identical. However, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider that if the curve L is $K-$quasiconformal, then it is $\kappa-$quasicircle with $\kappa = \frac{K^2-1}{K^2+1}$.

By the following Remark 3.1, for simplicity, we will use both terms, depending on the situation.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set

$$d(z, M) = dist(z, M) := \inf\{|z-\zeta| : \zeta \in M\}.$$

For $\delta > 0$ and $z \in \mathbb{C}$ let us set: $B(z,\delta) := \{\zeta : |\zeta - z| < \delta\}$, $\Omega(z,\delta) := \Omega \cap B(z,\delta)$.

Lemma 3.1 *[1] Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1,2,3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. and similarly so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1-w_3}{w_1-w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1-z_3}{z_1-z_2}\right| \preceq \left|\frac{w_1-w_3}{w_1-w_2}\right|^{c},$$

where $\varepsilon = \varepsilon(L) < 1,\ c = c(L) > 1,\ 0 < r_0 < 1$ are constants, depending on L and $L_{r_0} := \{z = \psi(w) : |w| = r_0\}$.

Corollary 3.2 *Under the assumptions of Lemma 3.1, if $z_3 \in L_{r_0}$ ($z_3 \in L_{Rr_0}$), then*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}$$

Corollary 3.3 *If $L \in C_\theta$, then*

$$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon},$$

for all $\varepsilon > 0$.

Let $\{z_j\}_{j=1}^m$ be a fixed the system of the points on L and the weight function $h(z)$ defined as (1.1).

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_i - z_j| : i,j = 1,2,...,m,\ i \neq j\}$, we put $\Omega(z_j,\ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min_{1\leq j\leq m} \delta_j$, $\Omega(\delta) := \bigcup_{j=1}^{m} \Omega(z_j,\ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta) := \bigcup_{j=1}^{m} \Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$.

Throungoht this work, we will take $R = 1 + \frac{\varepsilon_0}{n+1}$, for some fixed $0 < \varepsilon_0 < 1$. Further, for any $t > 1$ and $j = \overline{1,m}$, we introduce:

$$w_j : = \Phi(z_j), \varphi_j := \arg w_j,\ L_t^j := L_t \cap \overline{\Omega}^j;\ F_t^j := \Phi(L_t^j) \tag{3.1}$$

where $\Omega_t^j := \Psi(\Delta_{t,j}^{'})$;

$$\Delta_{t,1}^{'} : = \left\{ w = te^{i\theta} : t > 1,\ \frac{\varphi_m + \varphi_1}{2} \le \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$

$$\Delta_{t,m}^{'} : = \left\{ w = te^{i\theta} : t > 1,\ \frac{\varphi_{m-1} + \varphi_m}{2} \le \theta < \frac{\varphi_m + \varphi_1}{2} \right\},$$

and, for $j = \overline{2, m-1}$

$$\Delta_{t,j}^{'} := \left\{ w = te^{i\theta} : t > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \le \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}.$$

$L = \bigcup\limits_{j=1}^{m} L^j$; $L_t = \bigcup\limits_{j=1}^{m} L_t^j$.

We will use the well known estimation for the $\Psi^{'}$ (see, for example, [14, Th.2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi\left(\tau\right), L)}{|\tau| - 1}. \tag{3.2}$$

The following lemma is a consequence of the results given in [17], [40].

Lemma 3.2 *Let $L \in C_\theta(\lambda_1, ..., \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, 2, ..., m$,. Then, for arbitrary smaal $\varepsilon > 0$*
i) for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j + \varepsilon} \preceq |\Psi(w) - \Psi(w_j)| \preceq |w - w_j|^{\lambda_j - \varepsilon}$, $|w - w_j|^{\lambda_j - 1 + \varepsilon} \preceq |\Psi^{'}(w)| \preceq |w - w_j|^{\lambda_j - 1 - \varepsilon}$,

ii) for any $w \in \overline{\Delta} \backslash \Delta_j$, $(|w| - 1)^{1+\varepsilon} \preceq d(\Psi(w), L)| \preceq (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \preceq |\Psi^{'}(w)| \preceq (|w| - 1)^{-\varepsilon}$.

Lemma 3.3 *[7] Let L be a rectifiable Jordan curve, $h(z)$ defined as in (1.1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have:*

$$\|P_n\|_{L_p(h, L_R)} \le R^{n + \frac{1+\gamma *}{p}} \|P_n\|_{L_p(h, L)},\ p > 0. \tag{3.3}$$

Remark 3.2 In case of $h(z) \equiv 1$, the estimation (3.3) has been proved in [19].

Lemma 3.4 *Let L be a rectifiable Jordan curve, $h(z)$ defined as in (1.1); $R = 1 + \frac{1}{n}$, $1 \le R_1 < R$. Then, for arbitrary $P_n(z) \in \wp_n$, $z \in L_R$ and any numbers $\mu_j \ge 0$, $j = 1, 2, ..., m$, we have:*

$$\prod_{j=1}^{m} |z - z_j|^{\mu_j} |P_n(z)| \tag{3.4}$$

$$\preceq \|P_n\|_{L_p} \left(\int\limits_{L_{R_1}} \prod_{j=1}^{m} |\zeta - z_j|^{p\mu_j - \gamma_j} \frac{|d\zeta|}{|\zeta - z|^2} \right)^{1/p},\ p > 0.$$

Proof. Let $\{\xi_j\}$, $1 \le j \le m \le n$, be a zeros of $P_n(z)$ lying on Ω. Lets define the function Blashke with respect to the zeros of the polynomial $P_n(z)$ [41, p.120]:

$$B_m(z) := \prod_{j=1}^{m} B^j(z) := \prod_{j=1}^{m} \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)} \ , \ z \in \Omega,$$

It is easy that the $B_m(\xi_j) = 0$ and $|B_m(z)| \equiv 1$ at $z \in L$. For any $p > 0$ and $z \in \Omega$ let us set:

$$G_n(z) := \prod_{j=1}^{m} \left[\frac{z - z_j}{\Phi(z)}\right]^{p\mu_j/2} \left[\frac{P_n(z)}{\Phi^{n+1}(z) B_m(z)}\right]^{p/2}.$$

The function $G_n(z)$ is analytic in Ω, continious on $\overline{\Omega}$, $G_n(\infty) = 0$ and does not have zeros in Ω_{R_1}. We take an arbitrary continious branch of the $G_n(z)$ and for this branch, we maintain the same designation. Then, the Cauchy integral representation for the $G_n(z)$ given:

$$G_n(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} G_n(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in L_R.$$

Therefore,

$$\left| \prod_{j=1}^{m} \left[\frac{z - z_j}{\Phi(z)}\right]^{p\mu_j/2} \left[\frac{P_n(z)}{\Phi^{n+1}(z) B^j(z)}\right]^{p/2} \right|$$
$$\le \frac{1}{2\pi} \int\limits_{L_{R_1}} \prod_{j=1}^{m} \left|\frac{\zeta - z_j}{\Phi(\zeta)}\right|^{p\mu_j/2} \left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta) B^j(\zeta)}\right|^{p/2} \frac{|d\zeta|}{|\zeta - z|},$$

or

$$J_n \ : = \prod_{j=1}^{m} [|z - z_j|]^{p\mu_j/2} |P_n(z)|^{p/2} \tag{3.5}$$
$$\le \frac{1}{2\pi} \prod_{j=1}^{m} \frac{\max\limits_{z \in L_R} |\Phi(z)|^{p\mu_j/2} \left|\Phi^{n+1}(z) B^j(z)\right|^{p/2}}{\min\limits_{z \in L_{R_1}} |\Phi(\zeta)|^{p\mu_j/2} |\Phi^{n+1}(\zeta) B^j(\zeta)|^{p/2}}$$
$$\times \int\limits_{L_{R_1}} \prod_{j=1}^{m} |\zeta - z_j|^{p\mu_j/2} |P_n(\zeta)|^{p/2} \frac{|d\zeta|}{|\zeta - z|}.$$

Since $\left|B^j(\zeta)\right| = 1$ for $\zeta \in L$, then for arbitrary $\varepsilon_1, \ 0 < \varepsilon_1 < 1$, there exists a $L_{R^*} := \left\{z : |\Phi(z)| = 1 + \frac{\varepsilon_1}{n}\right\}$, such that for any $j = 1, 2, ..., m$, the following are satisfied:

$$\left|B^j(\zeta)\right| > 1 - \varepsilon_1, \ z \in L_{R^*}.$$

Then,

$$|B_m(\zeta)| > (1 - \varepsilon_1)^m \succeq 1,$$

for $\varepsilon_1 \le n^{-1}$ and $\zeta \in L_{R_1}$. Later

$$|\Phi(\zeta)| = R_1 > 1, \ |\Phi(\zeta)|^{n+1} = R_1^{n+1} \succeq 1,$$

for $\zeta \in L_{R_1}$. On the other hand, we obtain

$$|\Phi(z)|^{p\mu_j/2} \preceq 1, \quad \left|\Phi^{n+1}(z)B_m(z)\right|^{p/2} \preceq 1, \ z \in L_R.$$

According to this estimations, from (3.5), we have:

$$J_n \preceq \int\limits_{L_{R_1}} \prod_{j=1}^{m} |\zeta - z_j|^{p\mu_j/2} \left|P_n(\zeta)\right|^{p/2} \frac{|d\zeta|}{|\zeta - z|}.$$

Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$ and applying the Hölder inequality, we obtain:

$$J_n \preceq \left(\int\limits_{L_{R_1}} h(\zeta) \left|P_n(\zeta)\right|^p |d\zeta| \right)^{1/2} \times \left(\int\limits_{L_{R_1}} \prod_{j=1}^{m} |\zeta - z_j|^{p\mu_j - \gamma_j} \frac{|d\zeta|}{|\zeta - z|^2} \right)^{1/2}.$$

According to Lemma 3.3, from (3.5), we get:

$$\prod_{j=1}^{m} |z - z_j|^{\mu_j} \left|P_n(z)\right| \tag{3.6}$$

$$\preceq \|P_n\|_{L_p} \left(\int\limits_{L_{R_1}} \prod_{j=1}^{m} |\zeta - z_j|^{p\mu_j - \gamma_j} \frac{|d\zeta|}{|\zeta - z|^2} \right)^{1/p}.$$

and we complete the proof.

4 Proof of Theorems

Throungoht proofs of all theorems, we will take $n \geq \left[\frac{\varepsilon_0}{R_0 - 1}\right]$, where ε_0, $0 < \varepsilon_0 < 1$, some fixed small constant. In addition, in case when $n = 0$, the number n, participating in the all inequalities below will be changed to $(n+1)$.

41 Proof of Theorem 2.1

Proof. Suppose that $L \in PC_\theta(\lambda_1, ..., \lambda_m)$, for some $0 < \lambda_i \leq 2, \ i = \overline{1, m}$, be given and $h(z)$ defined in (1.1).

For $R > 1$, let $w = \varphi_R(z)$ denotes the univalent conformal mapping of G_R onto B normalized by $\varphi_R(0) = 0, \ \varphi_R'(0) > 0$, and let $\left\{\zeta_j\right\}, \ 1 \leq j \leq m \leq n$, zeros of $P_n(z)$, lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} B_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)} \tag{4.1}$$

denotes a Blashke function with respect to zeros $\left\{\zeta_j\right\}, 1 \leq j \leq m \leq n$, of $P_n(z)$. Clearly,

$$|B_{m,R}(z)| \equiv 1, \ z \in L_R; \ \ |B_{m,R}(z)| < 1, \ z \in G_R.$$

For each $R > 1, p > 0$ and $z \in G_R$, let us set

$$T_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2} ,$$

The function $T_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_n(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i}\int\limits_{L_R} T_n(\zeta)\frac{d\zeta}{\zeta - z},\ z \in G_R,$$

or

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}\right| \le \frac{1}{2\pi}\int\limits_{L_R}\left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta - z|} \le \int\limits_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta - z|},$$

since $|B_{m,R}(\zeta)| = 1$, for $\zeta \in L_R$. Lets now $z \in L$. Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$, by the Hölder inequality, we obtain

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2} \le \frac{1}{2\pi}\left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2} \times \left(\int\limits_{L_R}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta - z_j|^{\gamma_j}\,|\zeta - z|^2}\right)^{1/2}$$

$$=: \frac{1}{2\pi} I_{n,1} \times I_{n,2}(z).$$

Then, since $|B_{m,R}(z)| < 1$, for $z \in L$, from Lemma 3.3, we have:

$$|P_n(z)| \preceq (I_{n,1}\cdot I_{n,2}(z))^{2/p} \preceq \|P_n\|_p \cdot (I_{n,2}(z))^{2/p},\ z \in L.$$

By taking $z = z_j$, we get:

$$|P_n(z_j)| \preceq \|P_n\|_p \cdot (I_{n,2}(z_j))^{2/p},\ \ z \in L. \tag{4.2}$$

where

$$(I_{n,2}(z_j))^2 = \sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}|\zeta - z_j|^{2+\gamma_j}} \tag{4.3}$$

$$\asymp \sum_{i=1}^{m}\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}} \asymp \int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}},\ i = \overline{1,m},$$

since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct. Therefore, for the proof of (4.2) sufficiently to evaluate the following integral for each $i = \overline{1,m}$:

$$I_{n,2}^i(z_i) := \int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}}.$$

For simplicity of our next calculations, we assume that $i=1$. We denote that,

$$L^1_{R,1}:=L^1_R\cap\Omega(z_1,\delta),\ L^1_{R,2}:=L_R\backslash L^1_{R,1};\ \ F^1_{R,i}:=\Phi(L^1_{R,i}); \qquad (4.4)$$
$$L^1_1:=L^1\cap B(z_1,\delta),\ L^1_2:=\ \ L^1\backslash L^1_1;\ F^1_i:=\Phi(L^1_i),\ i=1,2.$$

By using this notations and setting $\delta := c_1 d_{1,R}$ for some $c_1>1$, where $d_{1,R}:=d(z_1,L^1_R)$, $\left|L^1_{R,i}\right|:=mesL^1_{R,i},\ i=1,2$, we have:

$$I^1_{n,2}(z_1):=\int\limits_{L^1_R}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}=\int\limits_{L^1_{R,1}}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}+\int\limits_{L^1_{R,2}}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}},$$

where

$$\int\limits_{L^1_{R,1}}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}\preceq\int\limits_{d_{1,R}}^{c_1d_{1,R}}\frac{ds}{s^{2+\gamma_1}}\preceq\frac{1}{d^{1+\gamma_1}_{1,R}};\ \int\limits_{L^1_{R,2}}\frac{|d\zeta|}{|\zeta-z_1|^{2+\gamma_1}}\preceq\int\limits_{c_1d_{1,R}}^{|L^1_{R,2}|}\frac{ds}{s^{2+\gamma_1}}\preceq\frac{1}{d^{1+\gamma_1}_{1,R}}.$$

According these estimations, from (4.2) and (4.3), we get:

$$|P_n(z_1)|\preceq\frac{1}{d^{1+\gamma_1}_{1,R}}\left\|P_n\right\|_{L_p}. \qquad (4.5)$$

On the other hand, by Lemma 3.2, for $0<\lambda_1<2$, and [14], for arbitrary continuum with simple connected complement, we have:

$$d_{1,R}\succeq\frac{1}{n^{\widetilde{\lambda}_1}}, \qquad (4.6)$$

where $\widetilde{\lambda}_1:=\begin{cases}\lambda_1+\varepsilon, \text{ if } 0<\lambda_1<2,\\ 2, \quad\quad \text{if } \lambda_1=2,\end{cases}$ for arbitrary small $\varepsilon>0$. From (4.5) and (4.6), we get the proof of (2.7) .

Lets now, under the condition (2.5) we will show estimation (2.6). For given $R=1+\frac{\varepsilon_0}{n}$, lets $R_1=1$.Then, from Lemma 3.4, we get:

$$\prod_{j=1}^m|z-z_j|^{\mu_j}|P_n(z)| \qquad (4.7)$$
$$\preceq\left\|P_n\right\|_{L_p}\left(\int\limits_L\prod_{j=1}^m|\zeta-z_j|^{p\mu_j-\gamma_j}\frac{|d\zeta|}{|\zeta-z|^2}\right)^{1/p},\ z\in L_R.$$

By denoting last integral as

$$J_{n,m}(L):=\left(\int\limits_L\prod_{j=1}^m|\zeta-z_j|^{p\mu_j-\gamma_j}\frac{|d\zeta|}{|\zeta-z|^2}\right)^{1/p}, \qquad (4.8)$$

we see that, to prove sufficiently estimation of the integral $J_{n,m}(L)$. Since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct, according to notations (3.1), for arbitrary fixed $j,\ 1 \leq j \leq m$, we get:

$$(J_{n,m}(L))^p = \sum_{i=1}^{m} \int\limits_{L^i} \prod_{j=1}^{m} |\zeta - z_j|^{p\mu_j - \gamma_j} \frac{|d\zeta|}{|\zeta - z|^2} \tag{4.9}$$

$$\asymp \sum_{i=1}^{m} \int\limits_{L^i} |\zeta - z_j|^{p\mu_j - \gamma_j} \frac{|d\zeta|}{|\zeta - z|^2} =: \sum_{i=1}^{m} J_{n,j}^{i}(L^i),$$

where, for each subarc $l \subset L_R^i,\ J_{n,j}^i(l)$ is denoted by

$$J_{n,j}^{i}(l) := \int\limits_{l} |\zeta - z_j|^{p\mu_j - \gamma_j} \frac{|d\zeta|}{|\zeta - z|^2}. \tag{4.10}$$

It remains to estimate the integrals $J_{n,j}^i(L^i)$ for each $i = \overline{1,m}$. For simplicity of our next calculations, we assume that

$$m = 1,\ j = 1,\ \mu := \mu_1;\ s := s_1,\ \gamma := \gamma_1,\ \lambda := \lambda_1. \tag{4.11}$$

In this situation, the integral $J_{n,j}^i(L^1)$ can be written as:

$$J_{n,1}^{1}(L^1) := \int\limits_{L^1} |\zeta - z_1|^{p\mu - \gamma} \frac{|d\zeta|}{|\zeta - z|^2}. \tag{4.12}$$

Lets denote by:

$$l_{R,1}^1 := L_R^1 \cap \Omega(z_1, c_1 d_{1,R}),\ c_1 > 1, \tag{4.13}$$

$$l_{R,2}^1 := L_R^1 \cap (\Omega(z_1,\delta_1) \setminus \Omega(z_1, c_1 d_{1,R})),\ l_{R,3}^1 := L_R^1 \backslash (l_{R,1}^1 \cup l_{R,2}^1);\ F_{R,j}^1 := \Phi(l_{R,j}^1),$$
$$l_1^1 := L^1 \cap B(z_1, c_1 d_{1,R}),\ l_2^1 := L^1 \cap (B(z_1,\delta_1) \backslash B(z_1, c_1 d_{1,R})),$$
$$l_3^1 := L^1 \backslash \left(l_1^1 \cup l_2^1\right);\ F_j^1 := \Phi(l_j^1),\ j = 1,2,3.$$

Then

$$J_{n,1}^1 = \sum_{j=1}^{3} \int\limits_{l_j^1} |\zeta - z_1|^{p\mu - \gamma} \frac{|d\zeta|}{|\zeta - z|^2} = \sum_{j=1}^{3} J_{n,1}^1(l_j^1),$$

where

$$J_{n,1}^1(l_j^1) := \int\limits_{l_j^1} |\zeta - z_1|^{p\mu - \gamma} \frac{|d\zeta|}{|\zeta - z|^2},\ \ j = 1,2,3. \tag{4.14}$$

So, we need to evaluate the integrals $J_{n,1}^1(l_i^1)$ for each $i = 1,2,3$.

1) Suppose first that $z \in l_{R,1}^1$.

1.1) According Lemma 3.2, we get:

a) If $1 \leq \lambda_1 \leq 2$, then

$$J_{n,1}^1(l_1^1) = \int\limits_{l_1^1} \frac{|\zeta - z_1|^{1 - \frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.15}$$

$$\preceq d_{1,R}^{1-\frac{1}{\lambda_1}} \int\limits_{l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z|^2} \preceq d_{1,R}^{1-\frac{1}{\lambda_1}} \int\limits_{d(z,\, l_{R,1}^1)}^{c_1 d(z,\, l_{R,1}^1)} \frac{ds}{s^2} \preceq \frac{d_{1,R}^{1-\frac{1}{\lambda_1}}}{d(z,\, l_1^1)} \preceq n^{1+\varepsilon};$$

b) If $0 < \lambda_1 < 1$, then

$$J_{n,1}^1(l_1^1) = \int\limits_{l_1^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.16}$$

$$= \int\limits_{l_1^1 \cap \{\zeta : |\zeta - z_1| \geq |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l_1^1 \cap \{\zeta : |\zeta - z_1| < |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{d(z,\, L)}^{c_2 d(z,\, L)} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\, L_R)}^{c_3 d(z_1,\, L_R)} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \preceq d^{-\frac{1}{\lambda_1}}(z,\, L) + d^{-\frac{1}{\lambda_1}}(z_1,\, L_R) \preceq n^{1+\varepsilon}.$$

1.2) a) If $1 \leq \lambda_1 \leq 2$, then let us remember that $z \in l_{R,1}^1$, and consequently, $|z_1 - z| \leq c_1 d_{1,R}$ for some $c_1 > 1$. Then, $|\zeta - z_1| \leq |\zeta - z| + |z - z_1| \leq |\zeta - z| + c_1 d_{1,R}$, and, according to well-known inequality [41, p.121]

$$|A + B|^p \leq |A|^p + |B|^p,\ 0 < p \leq 1,\ A > 0,\ B > 0,$$

we get:

$$|\zeta - z_1|^{1-\frac{1}{\lambda_1}} \preceq |\zeta - z|^{1-\frac{1}{\lambda_1}} + (d_{1,R})^{1-\frac{1}{\lambda_1}}.$$

Therefore, applyng Lemma 3.2, we obtain:

$$J_{n,1}^1(l_2^1) := \int\limits_{l_2^1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.17}$$

$$\preceq \int\limits_{l_2^1 \cup l_1^1} \frac{|d\zeta|}{|\zeta - z|^{1+\frac{1}{\lambda_1}}} + d_{1,R}^{1-\frac{1}{\lambda_1}} \cdot \int\limits_{l_2^1 \cup l_1^1} \frac{|d\zeta|}{|\zeta - z|^2} \preceq \int\limits_{d(z,\, L)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + d_{1,R}^{1-\frac{1}{\lambda_1}} \cdot \int\limits_{d(z,\, L)}^{c_2} \frac{ds}{s^2}$$

$$\preceq d^{-\frac{1}{\lambda_1}}(z,\, L) + d_{1,R}^{1-\frac{1}{\lambda_1}} \cdot d^{-1}(z,\, L) \preceq n^{1+\varepsilon}.$$

b) If $0 < \lambda_1 < 1$, then

$$J_{n,2}^1(l_2^1) := \int\limits_{l_2^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.18}$$

$$= \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| \geq |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| < |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{d(z,\, L)}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\, L_R)}^{\delta_1} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \preceq d^{-\frac{1}{\lambda_1}}(z,\, L) + d^{-\frac{1}{\lambda_1}}(z_1,\, L_R) \preceq n^{1+\varepsilon}.$$

1.3) a) If $1 \leq \lambda_1 \leq 2$, then

$$J^1_{n,1}(l^1_3) = \int\limits_{l^1_3} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.19}$$

$$\preceq (diamL)^{1-\frac{1}{\lambda_1}} \int\limits_{l^1_3} \frac{|d\zeta|}{|\zeta - z|^2} \preceq \frac{(diamL)^{1-\frac{1}{\lambda_1}}}{(\delta_1 - c_1 d_{1,R})^2} \left|l^1_3\right| \preceq 1.$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,1}(l^1_3) := \int\limits_{l^1_3} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z'|^2} \preceq \frac{\left|l^1_{R,3}\right|}{\delta_1^{1-\frac{1}{\lambda_1}} (\delta_1 - c_1 d_{1,R})^2} \preceq 1. \tag{4.20}$$

2) Let $z \in l^1_{R,2}$.

2.1) According Lemma 3.2, we get:

a) If $1 \leq \lambda_1 \leq 2$, then

$$J^1_{n,1}(l^1_1) = \int\limits_{l^1_1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.21}$$

$$= \int\limits_{l^1_1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \preceq d_{1,R}^{1-\frac{1}{\lambda_1}} \int\limits_{l^1_1} \frac{|d\zeta|}{|\zeta - z|^2} \preceq d_{1,R}^{1-\frac{1}{\lambda_1}} \int\limits_{d(z,\, L^1)}^{c_2} \frac{ds}{s^2} \preceq \frac{d_{1,R}^{1-\frac{1}{\lambda_1}}}{d(z,\, L^1)} \preceq n^{1+\varepsilon};$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,1}(l^1_1) = \int\limits_{l^1_1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.22}$$

$$\begin{aligned} &= \int\limits_{l^1_1 \cap \{\zeta : |\zeta - z_1| \geq |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^1_1 \cap \{\zeta : |\zeta - z_1| < |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \\ &\leq \int\limits_{l^1_1 \cup l^1_2} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l^1_1 \cup l^1_2} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \\ &\preceq \int\limits_{d(z,\, L)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} \preceq d^{-\frac{1}{\lambda_1}}(z,\, L) + d^{-\frac{1}{\lambda_1}}(z_1,\, L_R) \preceq n^{1+\varepsilon}. \end{aligned}$$

2.2) a) If $1 \leq \lambda_1 \leq 2$, then

$$J^1_{n,1}(l^1_2) = \int\limits_{l^1_2} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.23}$$

$$= \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| > |\zeta - z|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} + \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| \le |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| > |\zeta - z|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} + \int\limits_{d(z_1, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| > |\zeta - z|\}} \frac{|\zeta - z_1| \, |d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}} |\zeta - z|^2} + n^{1+\varepsilon}.$$

Lets denote by $I_{n,2}(z_1, z)$ the last integral and let $F := \Phi(l_2^1 \cap \{\zeta : |\zeta - z_1| > |\zeta - z|\})$. For the estimation $I_{n,2}(z_1, z)$, first of all, replacing the variable $\tau = \Phi(\zeta)$, we obtain:

$$I_{n,2}(z_1, z) \ : = \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| > |\zeta - z|\}} \frac{|\zeta - z_1| \, |d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}} |\zeta - z|^2}$$

$$= \int\limits_F \frac{|\Psi(\tau) - \Psi(w_1)| \, |\Psi'(\tau)| \, |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\frac{1}{\lambda_1}} |\Psi(\tau) - \Psi(w)|^2}$$

$$= \int\limits_F \left| \frac{\Psi(\tau) - \Psi(w_1)}{\Psi(\tau) - \Psi(w)} \right| \frac{|\Psi'(\tau)| \, |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\frac{1}{\lambda_1}} |\Psi(\tau) - \Psi(w)|}$$

Since $z \in l_2^1$, then $c_1 d_{1,R} \le |\zeta - z| < |\zeta - z_1| \le \delta_1$. According to Lemma 3.1, in this case we have $|\tau| - 1 \preceq |\tau - w| \preceq |\tau - w_1| \preceq 1$. Assume that $|\tau - w| < |\tau - w_1|$ (the inverse is trivial). We set $\varepsilon_0 := |\tau| - 1$. In this case, we take the discs centered at the point w_1, and radius $2^s \varepsilon_0$, $s = 1, 2, ...N$, where we choose a number N such that the circle is $Q_N = \left\{ \tau : |\tau - w_1| = 2^N \varepsilon_0 \right\}$, that satisfies the conditions $Q_N \cap \{t : |t| = R\} \ne \varnothing$, $Q_{N+1} \cap \{t : |t| = R\} = \varnothing$. Then, setting $F^s := F \cap \left\{ t : 2^{s-1} \varepsilon_o \le |t - w_1| \le 2^s \varepsilon_o \right\}$, and applyung Lemma 3.1 and Lemma 3.2, we have:

$$I_{n,2}(z_1, z) \tag{4.24}$$

$$= \int\limits_F \left| \frac{\Psi(\tau) - \Psi(w_1)}{\Psi(\tau) - \Psi(w)} \right| \frac{|\Psi'(\tau)| \, |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\frac{1}{\lambda_1}} |\Psi(\tau) - \Psi(w)|}$$

$$\preceq \sum_{s=1}^{\infty} \int\limits_{F^s} \left[\frac{|\tau - w_1|}{|\tau| - 1} \right]^{\varepsilon(L)} \frac{|d\tau|}{|\tau - w_1|^{1+\varepsilon}} \frac{\varepsilon_0^{\lambda_1 - 1 - \varepsilon} |d\tau|}{|\tau - w|^{\lambda_1 + \varepsilon}}$$

$$\preceq \sum_{s=1}^{\infty} \left(\frac{2^s \varepsilon_o}{\varepsilon_o} \right)^{\varepsilon(L)} \frac{\varepsilon_0^{\lambda_1 - 1 - \varepsilon}}{(2^{s-1} \varepsilon_o)^{1+\varepsilon}} \int\limits_{F^s} \frac{|d\tau|}{|\tau - w|^{\lambda_1 + \varepsilon}}$$

$$\preceq 2^{1+\varepsilon} \varepsilon_o^{\lambda_1 - 2 - \varepsilon} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon(L)}}{2^{1+\varepsilon}} \right)^s \int\limits_{F^s_{R_1,1}} \frac{|d\tau|}{|\tau - w|^{\lambda_1 + \varepsilon}}$$

$$\preceq \varepsilon_o^{\lambda_1 - 2 - \varepsilon} \int\limits_{n^{-1}}^{c_4} \frac{ds}{s^{\lambda_1 + \varepsilon}} \sum_{s=1}^{\infty} \left(\frac{2^{\varepsilon(L)}}{2^{1+\varepsilon}} \right)^s \preceq \left(\frac{1}{n} \right)^{\lambda_1 - 2 - \varepsilon} \cdot n^{\lambda_1 - 1 + \varepsilon} \preceq n^{1+\varepsilon},$$

where $\varepsilon(L),\ 0<\varepsilon(L)<1$, taken from Lemma 3.1.

b) If $0<\lambda_1<1$, then

$$J^1_{n,1}(l^1_2)=\int\limits_{l^1_2}\frac{|d\zeta|}{|\zeta-z_1|^{\frac{1}{\lambda_1}-1}|\zeta-z|^2} \tag{4.25}$$

$$=\int\limits_{l^1_2\cap\{\zeta:|\zeta-z_1|\geq|\zeta-z|\}}\frac{|d\zeta|}{|\zeta-z|^{\frac{1}{\lambda_1}+1}}+\int\limits_{l^1_2\cap\{\zeta:|\zeta-z_1|<|\zeta-z|\}}\frac{|d\zeta|}{|\zeta-z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\leq\int\limits_{L^1}\frac{|d\zeta|}{|\zeta-z|^{\frac{1}{\lambda_1}+1}}+\int\limits_{L^1}\frac{|d\zeta|}{|\zeta-z_1|^{\frac{1}{\lambda_1}+1}}\preceq\int\limits_{d(z,\,L)}^{c_2}\frac{ds}{s^{\frac{1}{\lambda_1}+1}}+\int\limits_{d(z_1,\,L_R)}^{c_3}\frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq d^{-\frac{1}{\lambda_1}}(z,\ L)+d^{-\frac{1}{\lambda_1}}(z_1,\ L_R)\preceq n^{1+\varepsilon}.$$

2.3) a) If $1\leq\lambda_1\leq 2$, then

$$J^1_{n,1}(l^1_3)=\int\limits_{l^1_3}\frac{|\zeta-z_1|^{1-\frac{1}{\lambda_1}}|d\zeta|}{|\zeta-z|^2} \tag{4.26}$$

$$=\int\limits_{l^1_3\cap\{\zeta:|\zeta-z_1|>|\zeta-z|\}}\frac{|\zeta-z_1|^{1-\frac{1}{\lambda_1}}|d\zeta|}{|\zeta-z|^2}+\int\limits_{l^1_3\cap\{\zeta:|\zeta-z_1|\leq|\zeta-z|\}}\frac{|d\zeta|}{|\zeta-z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq\int\limits_{l^1_3\cap\{\zeta:|\zeta-z_1|>|\zeta-z|\}}\frac{|\zeta-z_1|^{1-\frac{1}{\lambda_1}}|d\zeta|}{|\zeta-z|^2}+\int\limits_{d(z_1,\,L_R)}^{c_3}\frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq\int\limits_{l^1_3\cap\{\zeta:|\zeta-z_1|>|\zeta-z|\}}\frac{|\zeta-z_1|\,|d\zeta|}{|\zeta-z_1|^{\frac{1}{\lambda_1}}|\zeta-z|^2}+n^{1+\varepsilon}.$$

Denote by $I_{n,3}(z_1,z)$ the last integral. We estimate this integral analogously to integral $I_{n,2}(z_1,z)$. Consequently, in this case for $J^1_{n,1}(l^1_3)$ we will obtain:

$$J^1_{n,2}(l^1_3)\preceq n^{1+\varepsilon},\ \forall\varepsilon>0. \tag{4.27}$$

b) If $0<\lambda_1<1$, then

$$J^1_{n,2}(l^1_3)=\int\limits_{l^1_3}\frac{|d\zeta|}{|\zeta-z_1|^{\frac{1}{\lambda_1}-1}|\zeta-z|^2} \tag{4.28}$$

$$\leq\int\limits_{l^1_3}\frac{|d\zeta|}{|\zeta-z|^{\frac{1}{\lambda_1}+1}}\preceq\int\limits_{d(z,\,L)}^{c_3}\frac{ds}{s^{\frac{1}{\lambda_1}+1}}\preceq\frac{1}{d^{\frac{1}{\lambda_1}}(z,\ L)}\preceq n^{1+\varepsilon}.$$

3) Now, lets $z\in l^1_3$. Note that, this case includes also the case of $z\in L\backslash L^1$, since the point z more removed from the z_1.

3.1) a) If $1 \leq \lambda_1 \leq 2$, then

$$J^1_{n,2}(l^1_1) = \int\limits_{l^1_1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.29}$$

$$= \int\limits_{l^1_1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \preceq \frac{(diamL)^{1-\frac{1}{\lambda_1}}}{(\delta_1 - c_1 d_{1,R})^2} \left|l^1_3\right| \preceq 1.$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_1) = \int\limits_{l^1_1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.30}$$

$$= \int\limits_{l^1_1} \frac{|\zeta - z_1| \, |d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}} |\zeta - z|^2} \preceq \frac{d_{1,R}}{(\delta_1 - c_1 d_{1,R})^2} \int\limits_{l^1_1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}}}$$

$$\preceq \frac{d_{1,R}}{(\delta_1 - c_1 d_{1,R})^2} \int\limits_{d_{1,R}}^{c_1 d_{1,R}} \frac{|d\zeta|}{s^{\frac{1}{\lambda_1}}} \preceq \frac{1}{d_{1,R}^{\frac{1}{\lambda_1}-2}} \preceq n^{1+\varepsilon}.$$

3.2) a) If $1 \leq \lambda_1 \leq 2$, then

$$J^1_{n,2}(l^1_2) = \int\limits_{l^1_2} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.31}$$

$$= \int\limits_{l^1_2 \cap \{\zeta: |\zeta - z_1| \geq |\zeta - z|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} + \int\limits_{l^1_2 \cap \{\zeta: |\zeta - z_1| < |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l^1_2 \cap \{\zeta: |\zeta - z_1| \geq |\zeta - z|\}} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} + \int\limits_{d(z_1, L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq \int\limits_{l^1_2 \cap \{\zeta: |\zeta - z_1| \geq |\zeta - z|\}} \frac{|\zeta - z_1| \, |d\zeta|}{|\zeta - z|^{1+\frac{1}{\lambda_1}}} + n^{1+\varepsilon}.$$

Denote by $I_{n,3}(z_1, z)$ the last integral. We estimate this integral analogously to integral $I_{n,2}(z_1, z)$. Consequently, in this case for $J^1_{n,2}(l^1_3)$ we will obtain:

$$J^1_{n,2}(l^1_3) \preceq n^{1+\varepsilon}, \ \forall \varepsilon > 0. \tag{4.32}$$

b) If $0 < \lambda_1 < 1$, then

$$J^1_{n,2}(l^1_2) = \int\limits_{l^1_2} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.33}$$

$$= \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| \geq |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{l_2^1 \cap \{\zeta : |\zeta - z_1| < |\zeta - z|\}} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}}$$

$$\leq \int\limits_{L^1} \frac{|d\zeta|}{|\zeta - z|^{\frac{1}{\lambda_1}+1}} + \int\limits_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}+1}} \preceq \int\limits_{d(z,\,L)}^{c_2} \frac{ds}{s^{\frac{1}{\lambda_1}+1}} + \int\limits_{d(z_1,\,L_R)}^{c_3} \frac{ds}{s^{\frac{1}{\lambda_1}+1}}$$

$$\preceq d^{-\frac{1}{\lambda_1}}(z,\ L) + d^{-\frac{1}{\lambda_1}}(z_1,\ L_R) \preceq n^{1+\varepsilon}.$$

3.3) a) If $1 \leq \lambda_1 \leq 2$, then

$$J_{n,2}^1(l_3^1) = \int\limits_{l_3^1} \frac{|\zeta - z_1|^{1-\frac{1}{\lambda_1}} |d\zeta|}{|\zeta - z|^2} \tag{4.34}$$

$$\preceq (diamL)^{1-\frac{1}{\lambda_1}} \int\limits_{l_3^1} \frac{|d\zeta|}{|\zeta - z|^2} \preceq \int\limits_{d(z,\,L)}^{c_4} \frac{ds}{s^2} \preceq \frac{1}{d(z,\ L)} \preceq n^{1+\varepsilon}.$$

b) If $0 < \lambda_1 < 1$, then

$$J_{n,2}^1(l_3^1) := \int\limits_{l_3^1} \frac{|d\zeta|}{|\zeta - z_1|^{\frac{1}{\lambda_1}-1} |\zeta - z|^2} \tag{4.35}$$

$$\preceq \frac{1}{\delta_1^{1-\frac{1}{\lambda_1}}} \int\limits_{d(z,\,L)}^{c_4} \frac{ds}{s^2} \preceq \frac{1}{d(z,\ L)} \preceq n^{1+\varepsilon}.$$

Combining estimations (4.7)-(4.35), we get:

$$\prod_{j=1}^{m} |z - z_j|^{\mu_j} |P_n(z)| \preceq n^{\frac{1}{p}+\varepsilon} \cdot \|P_n\|_{L_p},\ z \in L_R. \tag{4.36}$$

The estimation (4.36) satisfieds on L_R. We show that it is also carried out on L.

For any $\mu > 0$ and $z \in G_R$, let us set:

$$H_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{1/\mu},$$

where $B_{m,R}(z)$ is a Blashke function with respect to zeros $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$ in G_R, defined as in (4.1). The function $H_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. Then, applying maximal modulus prinsible to $[H_n(z)]^{1/\mu}(z - z_1)$, we have:

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{1/\mu}(z - z_1)\right|$$

$$\leq \max_{\zeta \in \overline{G_R}} \left|\left[\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right]^{1/\mu}(\zeta - z_1)\right| \leq \max_{\zeta \in L_R} |P_n(\zeta)|^{1/\mu} |\zeta - z_1|$$

$$\preceq \left(n^{\frac{1}{\alpha p}} \cdot \|P_n\|_{L_p}\right)^{1/\mu},\ z \in L,$$

and therefore, we find:

$$|(z-z_1)^{\mu}P_n(z)| \preceq n^{\frac{1}{\alpha p}} \cdot \|P_n\|_{L_p}, \ z \in L.$$

Since the system of points $\{z_j\}_{j=1}^m$ are isolated and according to assumption (4.11), we get:

$$\max_{z\in L}\left(\prod_{j=1}^{m}[|z-z_j|]^{\mu_j}|P_n(z)|\right) \preceq n^{\frac{1}{\alpha p}} \cdot \|P_n\|_{L_p}, \ p>0,$$

and we complete the proof of estimation (2.6).

42 Proof of Remark 2.2.

Proof. a) Lets $L := \{z : |z| = 1\}$, $h^*(z) \equiv 1$ and $P_n^*(z) = \sum_{j=0}^{n}(j+1)z^j$. Then, $L \in \widetilde{Q}_1$;

$$|P_n^*(z)| \le \sum_{j=0}^{n}\left|(j+1)z^j\right| = \frac{(n+1)(n+2)}{2}, |z|=1.$$

On the other hand,

$$|P_n^*(1)| = \frac{(n+1)(n+2)}{2}.$$

Therefore,

$$\|P_n^*\|_{L_\infty} = \frac{(n+1)(n+2)}{2}; \ \|P_n^*\|_{L_2(1,L)} = \sqrt{\frac{(n+1)(n+2)(2n+3)}{3}\pi}.$$

Then,

$$\|P_n^*\|_{L_\infty} = \sqrt{\frac{3(n+1)(n+2)}{4\pi(2n+3)}}\,\|P_n^*\|_{L_2(1,L)} \ge \sqrt{\frac{3}{8\pi}}\sqrt{n}\,\|P_n^*\|_{L_2(1,L)}.$$

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