Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (1), 134-138 (2016).
Series of Physical-Technical and Mathematical Sciences.

On asymptotic behavior of the mean value of the family of the first exit time of random walk described by a nonlinear function of first order autoregression process $(AR(1))$

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Received: 21.10.2015 / Revised: 22.01.2016/ Accepted: 16.05.2016

Abstract. *In the paper we study asymptotic behavior of the mean value of the family of the first exit time of random walk described by a nonlinear function of first order autoregression process* $(AR(1))$.

Keywords. random walks, first order autoregression process, first exit time

Mathematics Subject Classification (2010): 60F05

1 Introduction

As is known (see [2-6]) first order autoregression processes $(AR(1))$ are determined as the solutions of the equation of the following form

$$X_n = \beta X_{n-1} + \xi_n, n \geq 1,$$

where $\{\xi_n\}$ is the sequence of independent identically distributed random values (innovation) determined on some probability space $(\Omega, \mathcal{F}, P)$, and β is some fixed number.

For simplicity we assume that the initial value of the process $X_0 = x = const$.

Note that the $AR(1)$ sequence allows to construct mathematical models in many applied fields of probability theory and mathematical statistics ([2-6]).

Assume

$$T_n = \sum_{k=1}^{n} X_k X_{k-1}, \quad S_n = \sum_{k=1}^{n} X_{k-1}^2$$

This work was supported by the science Development Foundation under the President of the Republic of Azerbaijan. Grant N EIF-2013-9 (15)-46/13/1.

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and

$$\beta_n = \frac{T_n}{S_n}, \ \ n \geq 1.$$

The law of large numbers and central limit theorem for the processes in T_n, S_n and β_n were studied in the papers [2] and [6]. These limit theorems enable to study some boundary value problems for random walks described by the processes X_n, T_n, S_n ([5], [7], [10]).

Let us consider the family of the first exit time:

$$\tau_a = \inf \{n \geq 1 : \theta_n > a\}, \qquad (1.1)$$

where $\theta_n = n\beta_n$ and $a \geq 0$.

Such families of the stoping times often arise in the problems of verification of statistical hypothesis that play the role of the number of necessary observations in the serial test ([2], [12]).

In the present paper we study asymptotic behavior of the mean value $E\tau_a$ as $a \to \infty$. Such problems for the processes X_n, T_n and S_n were studied in the papers [5], [7], [10]. Note that problems of finding asymptotic mean value of stoping times occupy a central place in theory of nonlinear renewal ([1], [12]).

2 Formulation and proof of the main result

Theorem 2.1 *Let $E\xi_1 = 0$, $D\xi_1 = 1$ and $0 < \beta < 1$.*

Then it holds

$$\frac{E\tau_a}{a} \to \frac{1}{\beta} \quad as \quad a \to \infty.$$

For proving the theorem we need the following lemmas.

Lemma 2.1 *Let $E\xi_1 = 0$, $D\xi_1 = 1$ and $|\beta| < 1$.*

Then the following relations are fulfilled

$$\frac{T_n}{n} \overset{a.s.}{\to} \frac{\beta}{1-\beta^2} \quad and \quad \frac{S_n}{n} \overset{a.s.}{\to} \frac{1}{1-\beta^2}$$

as $n \to \infty$.

The statement of lemma 2.1 was proved in [2].

Lemma 2.2 *Let the theorem conditions be fulfilled. Then it holds*

$$\frac{\tau_a}{a} \overset{a.s.}{\to} \frac{1}{\beta} \quad as \ \ a \to \infty.$$

Proof. At first we prove that

$$\tau_a \overset{a.s.}{\to} \infty \quad as \quad a \to \infty \qquad (2.1)$$

Indeed, the quantity τ_a as a function of the argument a increases. Therefore there exists a finite or infinite limit $\tau_\infty = \lim\limits_{a\to\infty} \tau_a$ i.e. we have

$$P\left(\tau_\infty = \lim_{a\to\infty} \tau_a \leq \infty\right) = 1.$$

Then from the property of continuity of probability measure it holds

$$P(\tau_\infty \leq n) = \lim_{a\to\infty} P(\tau_a \leq n) = \lim_{a\to\infty} P\left(\sup_{1\leq k\leq n} \theta_n > a\right) = 0$$

for all $n \geq 1$.

Hence it follows that $P(\tau_\infty = \infty) = 1$. From lemma 2.1 it follows

$$\frac{\theta_n}{n} \overset{a.s.}{\to} \beta \quad as \;\; n \to \infty. \tag{2.2}$$

Then according to theorem 2.1 of the paper [1, p. 10] from (2.1) and (2.2) it follows that

$$\frac{\theta\tau_a}{\tau_a} \overset{a.s.}{\to} \beta \quad as \;\; a \to \infty. \tag{2.3}$$

By the definition of the stopping moment (1.1) we have

$$\frac{\theta_{\tau_a - 1}}{\tau_a} \leq \frac{a}{\tau_a} < \frac{\theta_{\tau_a}}{\tau_a}. \tag{2.4}$$

The statement of lemma 2.2 follows from (2.2) and (2.4).

Lemma 2.3 *Subject to the theorem conditions, the family* $\frac{\tau_a}{a}$, $\;a \geq a_0$ *is uniformly integrable, where* $a_0 > 0$ *is some number.*

Proof. By the definition of uniform integrability it suffices to show that the following relation is fulfilled

$$\sup_a \int\limits_{\left\{\omega: \frac{\tau_a}{a} > c\right\}} \frac{\tau_a}{a} dP \to o \quad as \quad c \to \infty. \tag{2.5}$$

For $\varepsilon \in (0, \beta)$ we assume

$$N_a = \left[\frac{a}{\beta - \varepsilon}\right] + 1.$$

It is clear that for $n > N_a$ we have $a < n(\beta - \varepsilon)$. Therefore by the definition of the first exit time τ_a we can write

$$P(\tau_a > n) \leq P(\theta_n \leq a) \leq P(\theta_n < n(\beta - \varepsilon)) =$$
$$= P(\theta_n - n\beta < -n\varepsilon) = P(n(\beta_n - \beta) < -n\varepsilon) =$$
$$= P\left(\sqrt{n}(\beta_n - \beta) < -n\varepsilon\right).$$

As is seen,

$$\delta_n = P\left(\sqrt{n}(\beta_n - \beta) < -n\varepsilon\right), \;\; n \geq 1$$

is independent of δ_n and a for $n > N_a$ the following estimation is fulfilled:

$$P(\tau_a > N_a) \leq \delta_n. \tag{2.6}$$

It is clear that for proving (2.5) it suffices to show that

$$\int\limits_{\{\omega: \tau_a > N_a\}} \tau_a dP \to o \quad as \quad a \to \infty.$$

or

$$\sum_{n=N_a}^{\infty} P\left(\tau_a > n\right) = o\left(1\right) \quad as \quad n \to \infty$$

or

$$\sum_{n=N_a}^{\infty} \delta_n = o\left(1\right) \quad as \quad n \to \infty. \tag{2.7}$$

In the paper [6] it is proved a central limit theorem for the process β_n according to which in the conditions of the proved theorem it holds the limit relation

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\beta_n - \beta\right) \le x\right) = \Phi\left(\lambda x\right), \tag{2.8}$$

where

$$\lambda = \frac{1}{\sqrt{1-\beta^2}} \;\; and \;\; \Phi\left(x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^{2/2}} dy.$$

From (2.8) it follows that for sufficiently large a it holds

$$\sum_{n=N_a}^{\infty} \delta_n = O\left(\sum_{n=N_a}^{\infty} \Phi\left(-\lambda\varepsilon\sqrt{n}\right)\right). \tag{2.9}$$

By the equality $\Phi\left(-x\right) = 1 - \Phi\left(x\right)$, from the well known asymptotic equivalence (see [11, p.192])

$$1 - \Phi\left(x\right) \,\tilde{}\, \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \to \infty$$

it follows that

$$\sum_{n=N_a}^{\infty} \Phi\left(-\lambda\varepsilon\sqrt{n}\right) = o\left(1\right) \quad as \quad a \to \infty. \tag{2.10}$$

From (2.10) and (2.9) it follows (2.7).

For proving the theorem, it suffices to note that its statement follows from lemma 2.1 and 2.2 and the known theorem on limit passage under the sign of mathematical expectation (see for instance theorem 2.1 of the paper [1, p. 166]).

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