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Global existence and nonexistence of solution for Cauchy problem for a class of fourth order semi-linear pseudo-hyperbolic equations with structural damping

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Abstract. A sufficient condition for the existence of global solutions is established, and the rate of decay of solutions to the Cauchy problem for a semi-linear pseudo-hyperbolic equation with structural damping is found. The nonexistence of global solutions is also studied. An analogue of the critical Fujita exponents is obtained for the considered problem.

Keywords. Cauchy problem, global existence, pseudo-hyperbolic equation, structural damping, Fujita exponent.

Mathematics Subject Classification (2010): Primary 35L30, 35L75, 35L82, 35L76, Secondary 35B40, 35B45

1 Introduction

New approaches and mathematical models for describing various physical processes have been developed during the last decades. New mathematical models provide more adequate and simpler descriptions of problems and make it possible to pass from ideal models to actual situations and to predict new physical effects. The possibilities of analyzing problems can be expanded by applying the mathematical formalism of fractional derivatives. In this context, the theory of nonlinear equations with fractional derivatives has been actively developed (see [7,14,15,19,20,22]). A task of primary importance for nonlinear equations is to analyze the existence or nonexistence of global solutions [18]. This task is also of particular interest for nonlinear hyperbolic and pseudo-hyperbolic equations with structural damping, i.e., fractional order damping with respect to spatial variables (see, for example, [9,11]). The Cauchy problem for the semilinear wave equation with structural damping

$$u_{tt} - \Delta u + (-\Delta)^{\alpha} u_t = f(u),$$
$$u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x),$$

where $\sigma \in$ (0, 1/2] and

$$|f(u_1) - f(u_2)| \leq C(|u_1| + |u_2|)^{p-1} \, |u_1 - u_2|,$$

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has been considered by investigate in works M. D'Abbicco, M. Reissig [11] and M. D'Abbicco, M.R. Ebert [10, 11] for some $p > 1$. In [9–11], some global existence results for small data solutions to (1.1) have been proved for various values of σ and p. In the work of Ikehata R, Natsume M. [13] obtain the decay estimates for the total energy and L_2-norm solutions of the Cauchy problem for a linear wave equation with structural dissipation.

We should also mention the recent work [1] which considered the Cauchy problem for pseudo-hyperbolic equations and systems with Riemann-Liouville type fractional dissipation.

In this paper, we consider the Cauchy problem for the pseudo hyperbolic equation with structural damping

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + (-\Delta)^{\alpha} u_t = f(u), \ \ x \in R^n, \ \ t > 0, \tag{1.1}$$

$$u(0, x) = \varphi(x), \ u_t(0, x) = \psi(x), \ \ x \in R^n, \tag{1.2}$$

where $(-\Delta)^{\alpha}$ is determined by the Fourier transformation, $0 \leq \alpha < 1$ and $|f(v)| \approx |v|^p$, $v \in R$, for some $p > 1$. We will investigate the question of global solvability.

The Cauchy problem for semi-linear pseudo-hyperbolic equations has been studied by various authors (see, e.g., [3, 5, 21, 25, 26]). More specifically, Xu Runzhang and Liu Yacheng [21] considered the Cauchy problem for a class of nonlinear equations with double dispersion

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u = \Delta f(u),$$

where $|f(u)| \approx |u|^p$, $u \in R$. With the help of the potential well method, without using local solutions, they proved theorems on the existence of global solutions. They also studied the nonexistence of global solutions. For a broad class of convex functions and a one-dimensional double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = (f(u))_{xx},$$

more accurate conditions for the existence of global solutions were obtained in [25]. In [3, 5], an analogue of the Fujita criterion [12, 17] for the existence and nonexistence of global solutions was obtained for a broad class of semi-linear pseudo-hyperbolic equations

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u_t + u_t = f(u).$$

We would like to notice that, the linear part of the problem (1.1), (1.2) was also investigated in [8], where the decay rate of energetic norm was obtained. The aim of present paper is to obtain exact conditions on the growth of nonlinearities which ensure the existence of the global solutions.

In this work, we give a detailed review of results on the global existence and nonexistence for the solutions of the problem (1.1), (1.2) announced in [2].

The work is organized as follows. In the next section, we state our problem and our main results. In Section 3, some auxiliary lemmas are presented. In Section 4, we study the linear problem and establish some decay estimates. In Section 5, we obtain some a priori estimates which aid us in proving the existence theorem for global solutions. Section 6 deals with the proof of the local solvability theorem. In Section 7, we obtain the analog of the Fujita's criterion on the absence of global solutions (see [12]). And, in Section 8, we make a remark on the smoothness of solutions.

2 Preliminaries and Statement of Results

In the sequel, $L_p = L_p(\mathbb{R}^n),\ 1 \leq p \leq \infty$, will denote the usual Lebesgue space with the norm $\|\cdot\|_{L_p(R^n)}$. $L_{p,loc}$ is a class of functions for which the p-th power of the absolute value is locally summable. Given a nonnegative integer s, $W_2^s(\mathbb{R}^n)$ will stand for the Sobolev space of L_2 functions equipped with the norm $\|\cdot\|_{W_2^s(\mathbb{R}^n)}$:

$$\|v\|_{W_2^s(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{v}(\xi)|^2 \, d\xi \right\}^{1/2},$$

where $\hat{v}(\xi)$ is the Fourier transform of the function $v(x)$ and, $C^k(I; W_2^s(\mathbb{R}^n))$ will denote the space of *k*-times continuously differentiable functions on the interval *I* with the values in the Sobolev space $W_2^s(\mathbb{R}^n)$.

Throughout this paper, the constants C and c are positive generic constants which can be different in different places.

Before stating our results, we make the following assumptions:

(L1). Suppose the

$$|f(u_1) - f(u_2)| \leq a(u_1, u_2)\, |u_1 - u_2|,$$

where $a(u_1, u_2) \in C\left(\mathbb{R}^2, \mathbb{R}_+\right),\ a(u_1, u_2) \geq 0$.

(L2). $a(u_1, u_2) \leq c\left(|u_1|^{p-1} + |u_2|^{p-1}\right)$, with

$$p \in [1, \infty) \quad \text{if} \quad n = 4; \tag{2.1}$$

$$p \in \left[1, \frac{n+2}{n-4}\right) \quad \text{if} \quad n \geq 4. \tag{2.2}$$

Using the results of [4], we have the following local well-posedness for the Cauchy problem (1.1)-(1.2).

Theorem 2.1 *Let* $0 \leq \alpha < 1$*, and let the conditions (L1) and (L2) be satisfied. Then, for any* $\varphi \in W_2^2(\mathbb{R}^n),\ \psi \in W_2^1(\mathbb{R}^n)$*, there exists* $T_0 > 0$ *such that problem* (1.1), (1.2) *has a unique solution*

$$u \in C\left([0,\, T_0];\, W_2^2(\mathbb{R}^n)\,\right) \cap C^1\left([0,\, T_0];\ W_2^1(\mathbb{R}^n)\right).$$

Moreover, if T' *is the length of the maximum interval of existence of the solution* $u \in C\left([0,\, T');\ W_2^2(\mathbb{R}^n)\right) \cap C^1\left([0,\, T'); W_2^1(\mathbb{R}^n)\right)$*, then either*

(a) $T' = T$*,*

or

(b) $\lim\limits_{t \to T'-0} E(t) = +\infty$*, where* $E(t) = \|u(t, \cdot)\|_{W_2^2(\mathbb{R}^n)} + \|u'(t, \cdot)\|_{W_2^1(\mathbb{R}^n)}$.

Let

$$\cup_\delta(m, k) = \Big\{ (\varphi, \psi) : \varphi \in W_2^m(\mathbb{R}^n) \cap L_1(\mathbb{R}^n), \psi \in W_2^k(\mathbb{R}^n) \cap L_1(\mathbb{R}^n),$$

$$\|\varphi\|_{W_2^m(\mathbb{R}^n)} + \|\varphi\|_{L_1(\mathbb{R}^n)} + \|\psi\|_{W_2^k(\mathbb{R}^n)} + \|\psi\|_{L_1(\mathbb{R}^n)} < \delta \Big\}.$$

The main goal of this paper is to prove the following theorem on global solvability.

Theorem 2.2 *Let $1 \leq n \leq 7$, and let the conditions (L1) and (L2) be satisfied, where*

$$p \in \left(1 + \frac{4}{n-2\alpha}, +\infty\right), \ 0 \leq \alpha < \frac{n}{4} \text{ if } n \leq 4 \tag{2.3}$$

$$p \in \left(2, \frac{n+2}{n-4}\right), \ 0 \leq \alpha < 1 \text{ if } 5 \leq n \leq 7. \tag{2.4}$$

Then there exists $\delta_0 > 0$ such that for any $(\varphi, \psi) \in \bigcup_{\delta_0}(\frac{n}{4} + 3, 2)$ the problem (1.1)-(1.2) *has a unique solution $u(\cdot) \in C\left([0,\infty); W_2^2(\mathbb{R}^n)\right) \cap C^1\left([0,\infty); W_2^1(\mathbb{R}^n)\right)$ which satisfies the decay property*

$$\|u(t,\cdot)\|_{L_2(\mathbb{R}^n)} \leq c(\delta_0)(1+t)^{-\sigma_0}, \ t \in [0,\infty); \tag{2.5}$$

$$\left\|\nabla^2 u(t,\cdot)\right\|_{L_2(\mathbb{R}^n)} \leq c(\delta_0)(1+t)^{-\sigma_1}, \ t \in [0,\infty); \tag{2.6}$$

$$\|u_t(t,\cdot)\|_{L_2(\mathbb{R}^n)} \leq c(\delta_0)(1+t)^{-\eta_0}, \ t \in [0,\infty]; \tag{2.7}$$

$$\|\nabla u_t(t,\cdot)\|_{L_2(\mathbb{R}^n)} \leq c(\delta_0)(1+t)^{-\eta_1}, \ t \in [0,\infty), \tag{2.8}$$

where $\sigma_0 = \frac{n-4\alpha}{4(2-\alpha)}$, $\quad \sigma_1 = \sigma_0 + \frac{1}{2-\alpha}$,

$$\eta_0 = \min\left\{\frac{(p-1)n - 2p\alpha}{2(2-\alpha)}, \frac{n+8-4\alpha}{4(2-\alpha)}, \frac{1}{2(1-\alpha)}\right\},$$

$$\eta_1 = \min\left\{\frac{(p-1)n - 2p\alpha}{2(2-\alpha)}, \frac{n+10-4\alpha}{4(2-\alpha)}, \frac{1}{1-\alpha}\right\}, \ \text{and } c(\cdot) \in C(R_+, R_+)$$

does not depend on $t > 0$.

Next, let's consider the counterpart of the condition (2.3). In other words, under the assumption

$$1 < p \leq 1 + \frac{4}{n-2\alpha}, \ \ 0 \leq \alpha < 1,$$

we will try to derive the blow-up property of the Cauchy problem (1.1), (1.2).

With this purpose, we consider the following inequality in the domain $R_+ \times \mathbb{R}^n$:

$$Lu = u_{tt} - \Delta u_{tt} + \Delta^2 u + (-\Delta)^\alpha u_t \geq \ |u|^p, \tag{2.9}$$

with the initial conditions

$$u(0,x) = \varphi(x), \ \ u_t(0,x) = \psi(x), \ \ x \in \mathbb{R}^n. \tag{2.10}$$

By the "weak solution" of the inequality (2.8) with the initial conditions (2.9), where $\varphi, \Delta\varphi, \psi, \Delta\psi \in L_1(\mathbb{R}^n)$, we mean a function $u(t,x) \in L_{p,loc}(R_+ \times \mathbb{R}^n)$ which satisfies the following inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^n} [\varphi(x) + \Delta\varphi(x)]\, \xi_t(0,x)\, dx - \int_{\mathbb{R}^n} [\psi(x) + \Delta\psi(x)]\, \varsigma(0,x)\, dx \\
&+ \int_{\mathbb{R}^n} F^{-1}[|\xi|^{2\alpha} F[\varphi]](x) \varsigma(0,x)\, dx - \int_0^\infty \int_{\mathbb{R}^n} u(t,x) F^{-1}[|\cdot|^{2\alpha} F[\varsigma_t(t,x)]] dx dt \\
&+ \int_0^\infty \int_{\mathbb{R}^n} u(t,x) \left[\varsigma_{tt}(t,x) - \Delta\varsigma_{tt}(t,x) + \Delta^2 \varsigma(t,x)\, dx dt\right] \\
&\geq \int_0^\infty \int_{\mathbb{R}^n} |u|^p\, \varsigma(t,x)\, dx dt,
\end{aligned} \tag{2.11}$$

where F is the Fourier transformation, $\varsigma(t,x) \in C_0^4(R_+ \times \mathbb{R}^n)$; and $\varsigma(t,x) \geq 0$ is an arbitrary function.

Theorem 2.3 *Let*

$$n > 2\alpha \geq 0, \;\; 1 < p \leq 1 + \frac{4}{n - 2\alpha} \tag{2.12}$$

and

$$\int_{\mathbb{R}^n} [\psi (x) + \Delta\psi (x)] \, dx - \int_{\mathbb{R}^n} (-\Delta)^{\alpha} \varphi(x) dx \geq 0. \tag{2.13}$$

Then the problem (2.9), (2.10) *has no nontrivial weak solution.*

3 Some Auxiliary Decay Estimates

The solutions of the problem (1.1), (1.2) can be represented in the form

$$u (t, x) = u_0 (t, x) * \varphi (x) + u_1 (t, x) * \psi (x)$$

$$+ \int_0^t u_1 (t - \tau, x) * (I - \Delta)^{-1} f (u (\tau, x)) \; d\tau, \tag{3.1}$$

where $*$ denotes the convolution respect to the variable x:

$$u_0 (t, x) = F^{-1} [\hat{u}_0] (t, x) \, , \;\; u_1 (t, x) = F^{-1} [\hat{u}_1] (t, x) \, .$$

F^{-1} above is the inverse Fourier transform, while the functions $\hat{u}_0 (t, \xi)$ and $\hat{u}_1 (t, \xi)$ are the solutions of the problems

$$L_\xi \hat{u}_0 (t, \xi) = 0, \quad \hat{u}_0 (0, \xi) = 1, \;\; \hat{u}_{0_t} (0, \xi) = 0 \tag{3.2}$$

and

$$L_\xi \hat{u}_1 = 0, \quad \hat{u}_1 (0, \xi) = 0, \;\; \hat{u}_{1_t} (0, \xi) = 1, \tag{3.3}$$

respectively, where

$$L_\xi \hat{u}_i = \hat{u}_{i_{tt}} (t, \xi) + \frac{|\xi|^4}{1 + |\xi|^2} \hat{u}_i (t, \xi) + \frac{|\xi|^{2\alpha}}{1 + |\xi|^2} \hat{u}_i (t, \xi) \, , \;\; i = 0, 1.$$

Note that the discriminant

$$D = \frac{1}{1 + |\xi|^2} D_\xi, \;\; D_\xi = |\xi|^{4\alpha} - 4 \; |\xi|^4 \left(1 + |\xi|^2\right)$$

of the characteristic equation

$$\left(1 + |\xi|^2\right) \lambda^2 + |\xi|^{2\alpha} \lambda + |\xi|^4 = 0$$

is positive on a certain interval $(0, t_0)$ and negative on $(t_0, +\infty)$.

It is easy to see that

$$\hat{u}_0(t, \xi) = \begin{cases} \frac{\sqrt{D_\xi} - |\xi|^{2\alpha}}{2\sqrt{D_\xi}} e^{\lambda_1(\xi)t} + \frac{|\xi|^{2\alpha} + \sqrt{D_\xi}}{2\sqrt{D_\xi}} e^{\lambda_2(\xi)t}, & |\xi| < t_0, \\ e^{-\frac{|\xi|^{2\alpha}}{2(1+|\xi|^2)}t} \left[cos \frac{\sqrt{-D_\xi}}{(1+|\xi|^2)} t \; + \frac{|\xi|^{2\alpha}}{\sqrt{-D_\xi}} sin \frac{\sqrt{-D_\xi}}{(1+|\xi|^2)} t \right], & |\xi| > t_0 \end{cases}$$

and

$$\hat{u}_1(t,\xi)=\begin{cases}\frac{2(1+|\xi|^2)}{\sqrt{D_\xi}}[e^{\lambda_2(\xi)t}-e^{\lambda_1(\xi)t}], & |\xi|<t_0,\\ \frac{2(1+|\xi|^2)}{\sqrt{-D_\xi}}e^{-\frac{|\xi|^{2\alpha}}{1+|\xi|^2}t}sin\frac{\sqrt{-D_\xi}}{(1+|\xi|^2)}t, & |\xi|>t_0\end{cases}$$

are the solutions of (3.2) and (3.3), respectively, where

$$\lambda_1(\xi)=\frac{-|\xi|^{2\alpha}}{2(1+|\xi|^2)}-\frac{\sqrt{D_\xi}}{2(1+|\xi|^2)},\quad \lambda_2(\xi)=\frac{-|\xi|^{2\alpha}}{2(1+|\xi|^2)}+\frac{\sqrt{D_\xi}}{2(1+|\xi|^2)}$$

and

$$D_\xi=|\xi|^{4\alpha}-4|\xi|^4(1+|\xi|^2).$$

Define the numbers $T_i=\min\left\{\frac{i}{4},1\right\}$, $i=0,1,...$. In the sequel, we will use the notations:
$D^\beta_{x_i}=\frac{\partial^\beta}{\partial x_i}$, $i=1,..,n$, $D_t=\frac{\partial}{\partial t}$.

Lemma 3.1 *Let* $\beta\ge 0$, $\lambda\ge\beta-1$, $\alpha\in[0,T_n)$, $1\le n\le 4+4\lambda-6\beta$, $\psi(\cdot)\in L_1(\mathbb{R}^n)\bigcap W_2^\lambda(\mathbb{R}^n)$. *Then*

$$\left\|D^\beta_{x_i}(u_1(t,\cdot)*\psi(\cdot))\right\|_{L_2(R^n)}\le c\,(1+t)^{-\gamma_{2,\beta}}\left[\|\psi\|_{L_1(\mathbb{R}^n)}+\left\|(I-\Delta)^{\frac{\lambda}{2}}\psi\right\|_{L_2(R^n)}\right],$$

where $i=1,...,n;\ \ \gamma_{2\beta}=\frac{n}{4(2-\alpha)}+\frac{\beta-2\alpha}{2(2-\alpha)}$.

Lemma 3.2 *Let* $0\le\beta\le k$, $\alpha\in[0,T_n)$, $1\le n\le 4\kappa-6\beta$, $\varphi(\cdot)\in L_1(\mathbb{R}^n)\cap W_2^\kappa(\mathbb{R}^n)$. *Then*

$$\left\|D^\beta_{x_i}(u_0(t,\cdot)*\varphi)\right\|_{L_2(R^n)}\le c\,(1+t)^{-\gamma_{2\beta}}\left[\|\varphi\|_{L_1(\mathbb{R}^n)}+\|\varphi\|_{W_2^\kappa(\mathbb{R}^n)}\right],i=1,...,n.$$

Lemma 3.3 *Let* $\chi\in\{0,1\}$, $\alpha\in[0,T_n)$, $1\le n\le 7$, $\psi(.)\in L_1(\mathbb{R}^n)\cap W_2^2(\mathbb{R}^n)$. *Then*

$$\left\|D^\chi_{x_i}D_t(u_1(t,\cdot)*\psi(\cdot))\right\|_{L_2(R^n)}\le c\,(1+t)^{-\eta_{1\chi}}\left[\|\psi\|_{L_1(\mathbb{R}^n)}+\|(I-\Delta)\,\psi\|_{L_2(R^n)}\right],$$

where $i=1,...,n$; $\eta_{1\chi}=\min\left\{\frac{n+8-8\alpha+2\chi}{4(2-\alpha)},\frac{n+2\chi}{4\alpha},\frac{2-\chi}{2(1-\alpha)}\right\}$.

Lemma 3.4 *Let* $\chi\in\{0,1\}$, $\alpha\in[0,T_n)$, $1\le n\le 7$, $\varphi(\cdot)\in L_1(\mathbb{R}^n)\cap W_2^\kappa(\mathbb{R}^n)$, $\kappa=3+\frac{n}{4}$. *Then*

$$\left\|D^\chi_{x_i}D_t(u_0(t,\cdot)*\varphi)\right\|_{L_2(R^n)}\le c\,(1+t)^{-\,\eta_{2\chi}}\left[\|\varphi\|_{L_1(\mathbb{R}^n)}+\|\varphi\|_{W_2^\kappa(\mathbb{R}^n)}\right],$$

$$\eta_{2\chi}=\min\left\{\frac{\kappa-\chi-1}{2(1-\alpha)},\frac{n+8+2\chi-4\alpha}{4(2-\alpha)}\right\}.$$

4 Proofs of Lemmas

To prove Lemmas 3.1-3.4, we will use the following inequalities:
If $a > -1;\ \ b > 0;\ \ c > 0$ and $\delta > 0$, then

$$\int_0^\delta y^a e^{-ct^b} dy \le M(1+t)^{-\frac{a+1}{b}}, t \ge 0, \tag{4.1}$$

where $M \ge 0$.
If $a > 0,\ \ b > 0,\ \ \ \delta > 0$ and $c > 0$, then

$$\sup_{0\le y\le\delta} y^a e^{-cty^b} \le c(1+t)^{\frac{a}{b}}, \qquad t \ge 0. \tag{4.2}$$

Proof of Lemma 3.1. From Plancherel equality it follows that

$$\left\|D^\beta(u_1 * \psi)\right\|^2_{L_2(R_n)} = \left\||\xi|^\beta\left(\hat{u}_1\cdot\hat{\psi}\right)\right\|^2_{L_2(R_n)}$$

$$= \left\||\xi|^\beta\left(\hat{u}_1\cdot\hat{\psi}\right)\right\|^2_{L_2(|\xi|\le t_0-\delta)} = \left\||\xi|^\beta\left(\hat{u}_1\cdot\hat{\psi}\right)\right\|^2_{L_2(|\xi|\le t_0-\delta)}$$

$$+\left\||\xi|^\beta\left(\hat{u}_1\cdot\hat{\psi}\right)\right\|^2_{L_2(t_0-\delta<|\xi|<t_0)} + \left\||\xi|^\beta\left(\hat{u}_1\cdot\hat{\psi}\right)\right\|^2_{L_2(t_0<|\xi|\le t_0+\delta)}$$

$$+\left\||\xi|^\beta\left(\hat{u}_1\cdot\hat{\psi}\right)\right\|^2_{L_2(|\xi|>\ t_0+\delta)} = I_1 + I_2 + I_3 + I_4. \tag{4.3}$$

Let's estimate every term separately. Taking into account the representation for the solution of the problem (3.3), we have the following estimate:

$$I_1 \le 8\int_{|\xi|\le t_0-\delta} |\xi|^{2\beta}\frac{1+|\xi|^2}{D_\xi}\left(e^{2\lambda_1(\xi)t} + e^{2\lambda_2(\xi)t}\right)\left|\hat{\psi}(\xi)\right|^2 d\xi = I_{11} + I_{12}.$$

From

$$\sqrt{D_\xi} \approx |\xi|^{2\alpha}, \lambda_1(\xi) \le -c\,|\xi|^{2\alpha},\ \ |\xi| \le t_0 - \delta$$

and the inequality (4.2), we have

$$I_{11} \le c_1\int_{|\xi|\le t_0-\delta} |\xi|^{2\beta-4\alpha} e^{-c_1|\xi|^{2\alpha}t}\left|\hat{\psi}(\xi)\right|^2 d\xi$$

$$\le c_1\sup_{\xi\in\mathbb{R}^n}\left|\hat{\psi}(\xi)\right|^2\int_{|\xi|\le t_0-\delta}|\xi|^{2\beta-4\alpha} e^{-c_1|\xi|^{2\alpha}t}d\xi$$

$$\le c\int_0^{t_0-\delta} s^{\beta-2\alpha+\frac{n}{2}-1}e^{-c_1 s^\alpha}ds\cdot\|\psi\|^2_{L_1(\mathbb{R}^n)} \le c(1+t)^{-2\gamma_{1\beta}}\,\|\psi\|^2_{L_1(\mathbb{R}^n)}, \tag{4.4}$$

where $\gamma_{1\beta} = \frac{\beta-2\alpha}{2\alpha} + \frac{n}{4\alpha}$.
As $\lambda_2(\xi) = \frac{-|\xi|^4(1+|\xi|^2)}{2|\xi|^{2\alpha}(1+|\xi|^2)[1+\sqrt{1-4|\xi|^{4(1-\alpha)}(1+|\xi|^2)}} \le -c_1\,|\xi|^{2(2-\alpha)}$, $|\xi| \le t_0 - \delta$, from (4.1) we have

$$I_{12} \le c_1\int_{|\xi|\le t_0-\delta}|\xi|^{2\beta-4\alpha} e^{-c_1|\xi|^{4-2\alpha}t}d\xi\cdot\sup_{\xi\in R^n}\left|\,\hat{\psi}(\xi)\right|^2$$

$$\le c(1+t)^{-2\gamma_{2\beta}}\ \|\psi\|^2_{L_1(\mathbb{R}^n)}, \tag{4.5}$$

where $\gamma_{2\beta} = \frac{n}{4(2-\alpha)} + \frac{\beta-2\alpha}{2(2-\alpha)}$.

Denoting $g(\xi) = \frac{\sqrt{D_\xi}}{2\left(1+|\xi|^2\right)}$, we have

$$I_2 \le \int_{t_0-\delta<|\xi|\le t_0} |\xi|^{2\beta} e^{-\frac{|\xi|^{4\alpha}}{4(1+|\xi|^2)}} g^{-2}(\xi) \left| e^{g(\xi)t} - e^{-g(\xi)t} \right|^2 \left|\hat{\psi}(\xi)\right|^2 d\xi$$

$$\le \int_{t_0-\delta<|\xi|\le t_0} |\xi|^{2\beta} e^{-\frac{|\xi|^{4\alpha}}{4(1+|\xi|^2)}t} \left[\left| \frac{e^{g(\xi)t}-1}{g(\xi)t} \right| + \left| \frac{e^{-g(\xi)t}-1}{-g(\xi)t} \right| \right]^2 \cdot t \cdot \left|\hat{\psi}(\xi)\right|^2 d\xi$$

$$\le ce^{-2\omega t} \left\|\psi\right\|^2, \quad \omega > 0. \tag{4.6}$$

Similarly, denoting $g_1(\xi) = \frac{\sqrt{-D_\xi}}{2\left(1+|\xi|^2\right)}$, we have

$$I_3 \le \int_{t_0-\delta<|\xi|\le t_0} |\xi|^{2\beta} e^{-\frac{|\xi|^{4\alpha}}{4(1+|\xi|^2)}t} g_1^{-2}(\xi) \left|\sin\left(g_1(\xi)t\right)\right|^2 \left|\hat{\psi}(\xi)\right|^2 d\xi \le ce^{-2\omega t} \left\|\psi\right\|^2_{L_2(R^n)} . \tag{4.7}$$

Now let's estimate I_4 for $0 \le \alpha < 1$:

$$I_4 = \int_{|\xi|\ge t_0+\delta} |\xi|^{2\beta} g_1^{-2}(\xi) e^{-g_1(\xi)t} \sin^2\left(g_1(\xi)t\right). \left|\hat{\psi}(\xi)\right|^2 d\xi$$

$$\le c \int_{|\xi|\ge t_0+\delta} |\xi|^{2\beta-2-2\lambda} e^{-\frac{2|\xi|^{2\alpha}}{1+|\xi|^2}t} \left| (1+|\xi|^2)^{\frac{\lambda}{2}} \hat{\psi}(\xi) \right|^2 d\xi$$

$$\le c \sup_{|\xi|\ge t_0+\delta} |\xi|^{2\beta-2-2\lambda} e^{-\frac{2|\xi|^{2\alpha}}{1+|\xi|^2}t} \cdot \int_{|\xi|\ge t_0+\delta} \left| (1+|\xi|^2)^{\frac{\lambda}{2}} \hat{\psi}(\xi) \right|^2 d\xi.$$

Taking into account the inequalities (4.2), we obtain

$$I_4 \le c(1+t)^{-2\gamma_{3\beta\lambda}} \left\| (I-\Delta)^{\frac{\lambda}{2}} \psi \right\|^2_{L_2(\mathbb{R}^n)}, \tag{4.8}$$

where $\gamma_{3\beta\lambda} = \frac{1+\lambda-\beta}{2(1-\alpha)}$.

Let $\beta \ge 0,\ \ \lambda \ge \beta - 1,\ \alpha \in [0, T_n),\ \ 1 \le n \le 4 + 4\lambda - 6\beta$. Then it is clear that

$$\min\{\ \gamma_{1\beta}, \gamma_{2\beta}, \gamma_{3\beta\,\lambda}\} = \gamma_{2\beta} > 0. \tag{4.9}$$

The assertion of Lemma 3.1 follows from (4.3)-(4.9), and the inequality

$$e^{-\omega t} \le c\,(1+t)^{-\gamma_{2,\beta}} \tag{4.10}$$

is obviously true.

Proof of Lemma 3.2. By Plancherel equality, we get

$$\left\| D^\beta (u_0 * \varphi) \right\|^2_{L_2(R^n)} = \left\| |\xi|^\beta \left(\hat{u}_0 \cdot \hat{\varphi}\right) \right\|^2_{L_2(R^n)} = \left\| |\xi|^\beta \left(\hat{u}_0 \cdot \hat{\varphi}\right) \right\|^2_{L_2(|\xi|\le t_0-\delta)}$$

$$+ \left\| |\xi|^\beta \left(\hat{u}_0 \cdot \hat{\varphi}\right) \right\|^2_{L_2(t_0-\delta<|\xi|<t_0)} + \left\| |\xi|^\beta \left(\hat{u}_0 \cdot \hat{\varphi}\right) \right\|^2_{L_2(t_0<|\xi|\le t_0+\delta)}$$

$$+ \left\| |\xi|^\beta \left(\hat{u}_0 \cdot \hat{\varphi}\right) \right\|^2_{L_2(|\xi|>\ t_0+\delta)} = J_1 + J_2 + J_3 + J_4. \tag{4.11}$$

It's clear that $D > 0$ if $|\xi| \le t_0$. Therefore

$$J_1 \le c \int_{|\xi|\le t_0-\delta} |\xi|^{2\beta} \frac{(\sqrt{D_\xi} - |\xi|^{2\alpha})^2}{4D_\xi} e^{2\lambda_1(\xi)t} |\hat{\varphi}(\xi)|^2 d\xi$$

$$+ \int_{|\xi|\le t_0-\delta} |\xi|^{2\beta} \frac{(\sqrt{D_\xi} + |\xi|^{2\alpha})^2}{4D_\xi} e^{2\lambda_2(\xi)t} |\hat{\varphi}(\xi)|^2 d\xi = J_{11} + J_{12}. \tag{4.12}$$

Obviously,

$$\frac{(\sqrt{D_\xi} - |\xi|^{2\alpha})^2}{4D_\xi} \approx |\xi|^{8(1-\alpha)} \quad \text{as} \quad |\xi| \to 0.$$

Then from (4.12), by virtue of (4.1), we have

$$J_{11} \le \int_{|\xi|\le t_0-\delta} |\xi|^{2\beta+8(1-\alpha)} e^{-|\xi|^{2\alpha}t} |\hat{\varphi}(\xi)|^2 d\xi$$

$$\le c \sup_{\xi\in\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 \int_{|\xi|\le t_0-\delta} |\xi|^{2\beta+8(1-\alpha)} e^{-c_1|\xi|^{2\alpha}t} d\xi$$

$$\le c\,(1+t)^{-2r_{1\beta}} \, \|\varphi\|^2_{L_1(\mathbb{R}^n)}, \tag{4.13}$$

where $r_{1\beta} = \frac{2\beta+8(1-\alpha)+n}{4\alpha}$.

On the other hand,

$$\frac{(\sqrt{D_\xi} + |\xi|^{2\alpha})^2}{4D_\xi} \approx 1 \quad \text{and} \quad \lambda_2(\xi) \approx -|\xi|^{2(2-\alpha)} \quad \text{as} \quad |\xi| \to 0.$$

It follows that if $0 \le \alpha < 1$, then

$$J_{12} \le \int_{|\xi|\le t_0-\delta} |\xi|^{2\beta} e^{-t|\xi|^{2(2-\alpha)}} |\hat{\varphi}(\xi)|^2 d\xi$$

$$\le c \sup_{\xi\in\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 \int_{|\xi|\le t_0-\delta} |\xi|^{2\beta} e^{-c_1|\xi|^{2(2-\alpha)}t} d\xi$$

$$\le c \|\varphi\|^2_{L_1(\mathbb{R}^n)} \cdot \int_0^{t_0+\delta} s^{\beta+\frac{n}{2}-1} e^{-c_1 t\, s^{2-\alpha}} ds \le c(1+t)^{-r_{2\beta}} \|\varphi\|^2_{L_1(\mathbb{R}^n)}, \tag{4.14}$$

where $r_{2\beta} = \frac{2\beta+n}{4(2-\alpha)}$.

On the other hand, $\lambda_1(\xi) \le -\frac{|\xi|^{2\alpha}}{2(1+|\xi|^2)} \le -\omega$, $\lambda_2(\xi) \le -\frac{4|\xi|^4}{\sqrt{D}+|\xi|^{2\alpha}} \le -\omega$ and

$$\left| \frac{\exp(g(\xi)t) - \exp(-g(\xi)t)}{g(\xi)t} \right| \le c \text{ if } t_0 - \delta < |\xi| < t_0.$$

Then

$$J_2 \le c \int_{t_0-\delta<|\xi|\le t_0} |\xi|^{2\beta} e^{2\lambda_1(\xi)t} |\hat{\varphi}(\xi)|^2 d\xi + \int_{t_0-\delta<|\xi|\le t_0} |\xi|^{2\beta} e^{2\lambda_2(\xi)t} |\hat{\varphi}(\xi)|^2 d\xi$$

$$+ \int_{|\xi|\le t_0-\delta} e^{-\frac{|\xi|^{2\alpha}t}{1+|\xi|^2}} |\xi|^{2\beta} \left| \frac{\exp(g(\xi)t) - \exp(-g(\xi)t)}{g(\xi)t} \right|^2 |\hat{\varphi}(\xi)|^2 d\xi \le ce^{-2\omega t} \|\varphi\|^2. \tag{4.15}$$

Similarly we have

$$J_3 \leq ce^{-2\omega t} \left\| \varphi \right\|^2_{L_2(R^n)} . \tag{4.16}$$

Using the obvious inequalities

$$\sup_{s \geq c} s^{-a} e^{-ts^{-b}} \leq \left(\frac{a}{b} \right) t^{-\frac{a}{b}} e^{-\frac{a}{b}}, a > 0,\ b > 0,\ c > 0,$$

which follow from (4.2), we obtain

$$J_4 \leq c \int_{|\xi| \geq t_0 + \delta} |\xi|^{2\beta} e^{-t|\xi|^{2(\alpha-1)}} \left| \hat{\varphi}(\xi) \right|^2 d\xi$$

$$\leq c \int_{|\xi| \geq t_0 - \delta} |\xi|^{2\beta - 2\kappa} e^{-t|\xi|^{2(\alpha-1)}} \left| (1 + |\xi|^2)^{\frac{\kappa}{2}} \hat{\varphi}(\xi) \right|^2 d\xi$$

$$\leq c \sup_{s \geq (t_0 + \delta)^2} s^{\beta - \kappa} e^{-\frac{t}{s^{1-\alpha}}} \left\| \varphi \right\|^2_{W_2^k(\mathbb{R}^n)} \leq c(1+t)^{-2r_{3\beta k}} \left\| \varphi \right\|^2_{W_2^k(\mathbb{R}^n)} , \tag{4.17}$$

where $r_{3\beta k} = \frac{\lambda - \beta}{2(1-\alpha)}$.

Similarly, we have

$$J_5 \leq c \int_{|\xi| \geq t_0 - \delta} |\xi|^{2\beta - 6 - 2k + 4\alpha} e^{-|\xi|^{2(\alpha-1)} t} \left| (1 + |\xi|^2)^{\frac{k}{2}} \hat{\varphi}(\xi) \right|^2 d\xi$$

$$\leq c \sup_{s \geq (t_0 - \delta)^2} \left[s^{\beta - 3 - k + 2\alpha} e^{-\frac{ct}{s^{1-\alpha}}} \right] \cdot \left\| \varphi \right\|^2_{W_2^\kappa(\mathbb{R}^n)} \leq c(1+t)^{-2r_{4\beta k}} \left\| \varphi \right\|^2_{W_2^\kappa(\mathbb{R}^n)} ,$$

where $r_{4\beta k} = \frac{3 + k - \beta - 2\alpha}{2(1-\alpha)}$.

Let $0 \leq \beta \leq \kappa$, $\alpha \in [0, T_n)$, $1 \leq n \leq 4\kappa - 6\beta$. Then it is clear that

$$\min \left\{ r_{1\beta}, r_{2\beta}, r_{3\beta k}, r_{4\beta k} \right\} \geq \gamma_{2\beta} > 0, \tag{4.18}$$

where T_n is defined as in Lemma 3.1.

The assertion of Lemma 3.2 follows from (4.11)-(4.18).

Lemmas 3.3 and 3.4 can be proved in the same way.

5 Proof of Theorem 2.2 on Global Solvability

If $\beta = 0,\ \lambda = 2,\ \kappa = 3 + \frac{n}{4}$, then, taking into account Lemmas 3.1 and 3.2, from (3.1) we obtain the following estimate:

$$\left\| u\left(t, \cdot\right) \right\|_{L_2(\mathbb{R}^n)} \leq c\left(1 + t\right)^{-\gamma_{20}} d$$

$$+ c \int_0^t \left(1 + t - \tau\right)^{-\gamma_{20}} \left[\left\| f\left(u\left(\tau, \cdot\right)\right) \right\|_{L_1(\mathbb{R}^n)} + \left\| f\left(u\left(\tau, \cdot\right)\right) \right\|_{L_2(\mathbb{R}^n)} \right] d\tau, \tag{5.1}$$

where

$$d = \left\| \varphi \right\|_{W_2^k(\mathbb{R}^n)} + \left\| \varphi \right\|_{L_1(\mathbb{R}^n)} + \left\| \psi \right\|_{W_2^2(\mathbb{R}^n)} + \left\| \psi \right\|_{L_1(\mathbb{R}^n)} .$$

If $\beta = 2,\ \lambda = 2,\ \kappa = 3 + \frac{n}{4}$, then, taking into account Lemmas 3.1 and 3.2, from (3.2) we obtain the following estimate:

$$\left\| \nabla^2 u\left(t, \cdot\right) \right\|_{L_2(\mathbb{R}^n)} \leq c\left(1 + t\right)^{-\gamma_{22}} d$$

$$+c\int_0^t (1+t-\tau)^{-\gamma_{22}} \left[\|f(u(\tau,\cdot))\|_{L_1(\mathbb{R}^n)} + \|f(u(\tau,\cdot))\|_{L_2(\mathbb{R}^n)}\right] d\tau. \tag{5.2}$$

On the other hand, in view of conditions (L1) and (L2),

$$\|f(u)\|_{L_1(\mathbb{R}^n)} \le c\|u\|^p_{L_p(\mathbb{R}^n)}, \|f(u)\|_{L_2(\mathbb{R}^n)} \le c\|u\|^p_{L_{2p}(\mathbb{R}^n)}.$$

Further, using Gagliardo-Nirenberg type multiplicative inequality (see [6]) we have

$$\|f(u(t,\cdot))\|_{L_1(\mathbb{R}^n)} \le c\|u\|^{p(1-\theta_1)}_{L_2(\mathbb{R}^n)} \cdot \left(\left\|\nabla^2 u(t,\cdot)\right\|_{L_2(\mathbb{R}^n)}\right)^{p\theta_1}, \tag{5.3}$$

$$\|f(u(t,\cdot))\|_{L_2(\mathbb{R}^n)} \le c\|u\|^{p(1-\theta_2)}_{L_2(\mathbb{R}^n)} \cdot \left(\left\|\nabla^2 u(t,\cdot)\right\|_{L_2(\mathbb{R}^n)}\right)^{p\theta_2}, \tag{5.4}$$

where

$$\theta_1 = \frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right) \in [0,1], \theta_2 = \frac{n}{4}\left(1-\frac{1}{p}\right) \in [0,1]. \tag{5.5}$$

Denote

$$G_1(t) = (1+t)^{\gamma_{20}}\|u(t,\cdot)\|_{L_2(\mathbb{R}^n)}, G_2(t) = (1+t)^{\gamma_{22}}\left\|\nabla^2 u(t,\cdot)\right\|_{L_2(\mathbb{R}^n)}. \tag{5.6}$$

Then from (4.3)-(5.4) we obtain

$$G_1(t) \le cd + c(1+t)^{\gamma_{20}}\int_0^t (1+t-\tau)^{-\gamma_{20}}$$

$$\times\left[(1+\tau)^{-h_1} G_1^{p(1-\theta_1)}(\tau) G_2^{p\theta_1}(\tau) + (1+\tau)^{-h_2} G_1^{p(1-\theta_2)}(\tau) G_2^{p\theta_2}(\tau)\right] d\tau \tag{5.7}$$

and

$$G_2(t) \le cd + c(1+t)^{\gamma_{22}}\int_0^t (1+t-\tau)^{-\gamma_{22}}$$

$$\times\left[(1+\tau)^{-h_1} G_1^{p(1-\theta_1)}(\tau) G_2^{p\theta_1}(\tau) + (1+\tau)^{-h_2} G_1^{p(1-\theta_2)}(\tau) G_2^{p\theta_2}(\tau)\right] d\tau, \tag{5.8}$$

where

$$h_1 = p(1-\theta_1)\gamma_{20} + p\theta_1(\gamma_{20} + \frac{1}{2-\alpha}), \tag{5.9}$$

$$h_2 = p(1-\theta_2)\gamma_{20} + p\theta_2(\gamma_{20} + \frac{1}{2-\alpha}). \tag{5.10}$$

From (5.5), (5.9) and (5.10) it follows that

$$\begin{aligned} h_1 &= p\gamma_{20} + p\theta_1.\tfrac{1}{2-\alpha} = p\tfrac{n-4\alpha}{4(2-\alpha)} + \tfrac{pn}{2(2-\alpha)}(\tfrac{1}{2}-\tfrac{1}{p}) \\ &= \tfrac{p(n-2\alpha)}{2(2-\alpha)} - \tfrac{n}{2(2-\alpha)} \end{aligned} \tag{5.11}$$

and

$$h_2 = h_1 + \frac{n}{4(2-\alpha)}. \tag{5.12}$$

Denoting $Y(t) = \sup\limits_{0\le\tau\le t}(G_1(\tau)+G_2(\tau))$, from (5.6)-(5.10) we obtain

$$Y(t) \le cd + \left\{(1+t)^{\gamma_{20}}\int_0^t (1+t-\tau)^{\gamma_{20}}\left[(1+\tau)^{-h_1} + (1+\tau)^{-h_2}\right] d\tau\right.$$

$$+\left(1+t\right)^{\gamma_{22}}\int_0^t\left(1+t-\tau\right)^{-\gamma_{22}}\left[\left(1+\tau\right)^{-h_1}+\left(1+\tau\right)^{-h_2}\right]d\tau\Bigg\}Y^p\left(t\right). \tag{5.13}$$

In view of the condition (2.3), (2.4) it follows from (5.11), (5.12) that $h_2 > h_1 > 1$. In this case we have (see [23])

$$\left(1+t\right)^{\gamma_{20}}\int_0^t\left(1+t-\tau\right)^{\gamma_{20}}\left[\left(1+\tau\right)^{-h_1}+\left(1+\tau\right)^{-h_2}\right]d\tau\le c, \tag{5.14}$$

$$\left(1+t\right)^{\gamma_{22}}\int_0^t\left(1+t-\tau\right)^{\gamma_{22}}\left[\left(1+\tau\right)^{-h_1}+\left(1+\tau\right)^{-h_2}\right]d\tau\le c. \tag{5.15}$$

From (5.14)-(5.15) it follows that

$$Y\left(t\right)\le cd+c_1Y^p\left(t\right),\quad t\in\left[0,T'\right).$$

The latter implies the validity of the following inequality for sufficiently small values of d:

$$Y\left(t\right)\le M,\quad t\in\left[0,T'\right),\quad c_1Y^p-Y+cd=0 \tag{5.16}$$

where M is the first positive root of the equation

$$c_1Y^p-Y+cd=0.$$

So we have proved the validity of the following a priori estimates:

$$\left\|u\left(t,\cdot\right)\right\|_{L_2(\mathbb{R}^n)}\le c\left(1+t\right)^{-\gamma_{20}},\ t\in\left[0,T'\right), \tag{5.17}$$

$$\left\|\nabla^2u\left(t,\cdot\right)\right\|_{L_2(\mathbb{R}^n)}\le c\left(1+t\right)^{-\gamma_{22}},\ t\in\left[0,T'\right). \tag{5.18}$$

On the other hand, from (3.2) it follows that

$$u_t\left(t,x\right)=u_{0t}\left(t,x\right)*\varphi\left(x\right)+u_{1t}\left(t,x\right)*\psi\left(x\right)$$
$$+\int_0^t u_{1t}\left(t-\tau,x\right)*\left(I-\Delta\right)^{-1}f\left(u\left(\tau,x\right)\right)\,d\tau\,.$$

Using Fourier transform, Plancherel theorem and Hausdorff-Young inequality [24], we obtain

$$\left\|\nabla^{\chi}D_tu\left(t,\cdot\right)\right\|_{L_2(\mathbb{R}^n)}\le\left\|\left|\cdot\right|^{\chi}D_t\hat{u}_0\left(t,\cdot\right)\cdot\hat{\varphi}(\cdot)\right\|_{L_2(\mathbb{R}^n)}+\left\|\left|\cdot\right|^{\chi}D_t\hat{u}_1\left(t,\cdot\right)\cdot\hat{\psi}(\cdot)\right\|_{L_2(\mathbb{R}^n)}$$
$$+\int_0^t\left\|\left|\cdot\right|^{\chi}D_t\hat{u}_1\left(t-\tau,\cdot\right)F\left[f\left(u\right)\right]\left(\tau,\cdot\right)\right\|_{L_2(\mathbb{R}^n)}d\tau,\ \ \chi=0,1. \tag{5.19}$$

Now, considering Lemmas 3.3 and 3.4 in 4 from (5.19), we obtain that

$$\left\|\nabla^{\chi}D_tu\left(t,\cdot\right)\right\|_{L_2(\mathbb{R}^n)}\le C\left(1+t\right)^{-\eta_{2\chi}t}\left[\left\|\varphi(\cdot)\right\|_{L_2(\mathbb{R}^n)}+\left\|\varphi(\cdot)\right\|_{W_2^{\kappa}(\mathbb{R}^n)}\right]$$
$$+C\left(1+t\right)^{-\eta_{1\chi}t}\left[\left\|\psi(\cdot)\right\|_{L_2(\mathbb{R}^n)}+\left\|\psi(\cdot)\right\|_{W_2^2(\mathbb{R}^n)}\right]$$
$$+\int_0^t\left(1+t-\tau\right)^{-\eta_{1\chi}t}\left[\left\|f(u(\tau,\cdot)\right\|_{L_1(\mathbb{R}^n)}+\left\|f(u(\tau,\cdot)\right\|_{L_2(\mathbb{R}^n)}\right]\,d\tau. \tag{5.20}$$

By virtue of (4.1)-(4.2), (5.3), (5.4), (5.11), (5.12), (5.17), (5.18), Lemmas 3.3 and 3.4, we have

$$\left\|u_t\left(t,\cdot\right)\right\|_{L_2(\mathbb{R}^n)}\le c\left(\delta_0\right)\left(1+t\right)^{-\eta_0},\ t\in\left[0,T'\right), \tag{5.21}$$

$$\|\nabla u_t(t,\cdot)\|_{L_2(\mathbb{R}^n)} \le c(\delta_0)(1+t)^{-\eta_1}, \ \ t\in[0,T'), \tag{5.22}$$

where

$$\eta_0 = \min\left\{\frac{(p-1)n-2p\alpha}{2(2-\alpha)}, \frac{n+8-4\alpha}{4(2-\alpha)}, \frac{1}{2(1-\alpha)}\right\},$$

$$\eta_1 = \min\left\{\frac{(p-1)n-2p\alpha}{2(2-\alpha)}, \frac{n-10-4\alpha}{4(2-\alpha)}, \frac{1}{1-\alpha}\right\}.$$

In view of (5.17), (5.18), (5.21) and (5.22), we have the following a priori estimate:

$$\|u(t,\cdot)\|_{W_2^2(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{W_2^1(\mathbb{R}^n)} \le c(d), \ t\in[0,T').$$

Hence the $T' = +\infty$, i.e. the Cauchy problem has a global solution for sufficiently small values of d.

6 Proof of Theorem 2.1 on Local Solvability

By substitution $v_1 = u$, $v_2 = u_t$, the problem (1.1)-(1.2) is reduced to the equivalent problem

$$\left.\begin{array}{l} w_t = Aw + F(w) \\ w(0) = w_0 \end{array}\right\} \tag{6.1}$$

in the Hilbert space $H = W_2^2(\mathbb{R}^n)\times W_2^1(\mathbb{R}^n)$, where

$$w = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \ \ w_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \ \ F(w) = \begin{pmatrix} 0 \\ Gf(u) \end{pmatrix},$$

$$Aw = \left(v_2, -\Delta^2 G\, v_1 - (-\Delta)^\alpha G v_2\right),$$

$$D(A) = W_2^3(\mathbb{R}^n)\times W_2^2(\mathbb{R}^n), \, G = (I-\Delta)^{-1}.$$

In H, we introduce the scalar product

$$\left\langle w^1, w^2\right\rangle = \int_{\mathbb{R}^n} \Delta v_1^1 \cdot \overline{\Delta v_1^2} dx + \int_{\mathbb{R}^n} v_1^1\cdot \overline{v_1^2} dx + \int_{\mathbb{R}^n} \nabla v_2^1 \cdot \overline{\nabla v_2^2} dx + \int_{\mathbb{R}^n} v_2^1\cdot\overline{v_2^2} dx.$$

Using the definitions of A and G as well as the inner product in H, we obtain that $Re\langle Aw, w\rangle \le c\|w\|^2, \ c>0$.

On the other hand, with the help of Fourier transform it is easy to show that the operator $A+I$ is invertible. So in view of [16] the following statement is true.

Lemma 6.1 *The linear operator A generates a strongly continuous semigroup in the space H.*

Lemma 6.2 *Under the conditions (L1) and (L2), the nonlinear operator F satisfies a local Lipschitz condition, i.e., for any $w^1, w^2 \in$ H*

$$\left\|F(w^1) - F(w^2)\right\|_{L_2(R^n)} \le c\left(\left\|w^1\right\|_{L_2(R^n)}, \left\|w^2\right\|_{L_2(R^n)}\right) \ \left\|w^1 - w^2\right\|_{L_2(R^n)}, \tag{6.2}$$

where $c(\cdot)\in C\left(R_+^2, R_+\right)$.

Proof. Suppose $w^1 = \begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix}, w^2 = \begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} \in \mathrm{H} = W_2^2(\mathbb{R}^n) \times W_2^1(\mathbb{R}^n)$. Then

$$\left\|F\left(w^1\right) - F\left(w^2\right)\right\|_{L_2(R^n)}^2 = \int_{\mathbb{R}^n} \left|(I-\Delta)^{-\frac{1}{2}}\left[f(v_1^1) - f(v_1^2)\right]\right|^2 dx.$$

Let $n < 4$. Using the limitations $(I-\Delta)^{-1}$ in H, and the conditions (L1)-(L2), we obtain that

$$\left\|F\left(w^1\right) - F\left(w^2\right)\right\|_{L_2(R^n)}^2 = \int_{\mathbb{R}^n} \left|a\left(v_1^1, v_1^2\right)\left[v_1^1 - v_1^2\right]\right|^2 dx$$

$$\leq \sup_{x \in R} a^2(v_1^1(x), v_1^2(x)) \cdot \int_{\mathbb{R}^n} \left|v_1^1 - v_1^2\right|^2 dx.$$

Hence, using the embedding $W_2^2(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, $n < 4$ we obtain the inequality (6.2).

In case of $n = 4$, applying Hölder's inequality and using the embedding $W_2^2(R^4) \subset L_\rho(R^4)$, for any $2 \leq \rho < \infty$ we obtain the inequality (6.2).

Now consider the case of $n > 4$. Then, applying Hölder's inequality, we have:

$$\int_{\mathbb{R}^n} \frac{1}{\left(1+|\xi|^2\right)}\left[\hat{f}(v_1^1) - \hat{f}(v_1^2)\right]^2 d\xi$$

$$\leq \left(\int_{\mathbb{R}^n} \frac{d\xi}{\left(1+|\xi|^2\right)^{\rho'}}\right)^{1/\rho} \left(\int_{\mathbb{R}^n} \left[\hat{f}(v_1^1) - \hat{f}(v_1^2)\right]^{2\rho} d\xi\right)^{1/\rho}.$$

On the other hand, when $n > 4$ and $\rho' > \frac{n}{2}$,

$$\int_{\mathbb{R}^n} \frac{d\xi}{\left(1+|\xi|^2\right)^{\rho'}} < +\infty.$$

We can choose $\rho = \frac{n}{n-2+\varepsilon}$, $\rho' = \frac{n}{2-\varepsilon}$, where $\varepsilon > 0$ is a sufficiently small number. It is easy to see that if $n > 4$, then $1 \leq m \leq 2$, where $m = \frac{2\rho}{2\rho-1} = \frac{2n}{n+2-\varepsilon}$. Then, using Hausdorff-Young inequality (see [24]), we have

$$I = \left(\int_{\mathbb{R}^n} \left[\hat{f}(v_1^1) - \hat{f}(v_1^2)\right]^{2\rho} d\xi\right)^{1/\rho} \leq C\left\|f(v_1^1) - f(v_1^2)\right\|_{L_{\frac{2n}{n+2-\varepsilon}}(\mathbb{R}^n)}^2 .$$

Using the conditions (Ll) and applying Hölder's inequality with exponents

$$\rho_1 = \frac{n+2-\varepsilon}{6-\varepsilon} \text{ and } \rho_2 = \frac{n+2-\varepsilon}{n-4},$$

we obtain

$$I = \left(\int_{\mathbb{R}^n} \left(\left|v_1^1\right|^{p-1} + \left|v_1^2\right|^{p-1} d\xi\right)^{\frac{2n}{n+2-\varepsilon}} \left|v_1^1 - v_1^2\right|^{\frac{2n}{n+2-\varepsilon}} d\xi\right)^{\frac{n+2-\varepsilon}{n}}$$

$$\leq \left(\int_{\mathbb{R}^n} \left(\left|v_1^1\right|^{p-1} + \left|v_1^2\right|^{p-1}\right)^{\frac{2n}{6-\varepsilon}} d\xi\right)^{\frac{6-\varepsilon}{n}} \cdot \left(\left|v_1^1 - v_1^2\right|^{\frac{2n}{n-4}} d\xi\right)^{\frac{n-4}{n}}$$

$$\leq C\left(\left\|v_1^1\right\|_{W_2^2(\mathbb{R}^n)}, \left\|v_1^2\right\|_{W_2^2(\mathbb{R}^n)}\right) \cdot \left\|v_1^1 - v_1^2\right\|_{W_2^2(\mathbb{R}^n)}^2 .$$

In view of Lemmas 6.5 and 6.6, all the conditions of the theorem on existence and uniqueness for local solutions of our problem are fulfilled [4].

7 Proof of Theorem 3.3 on Absence of Global Solutions

Consider the following test function:

$$\varsigma(t,x) = \varsigma_1(t)\,\varsigma_2(x),$$

where $\varsigma_1(t) = h_1\left(\frac{t^\chi}{d^2}\right)$, $\varsigma_2(x) = h_2\left(\frac{|x|^\mu}{d^2}\right)$, $\mu > 1$, $\chi > 1$, $d > 0$ some parameters, $h_i \in C_0^\infty(R_+)$ and $h_i(r) = \begin{cases} 1, & 0 \le r \le 1, \\ 0, & r \ge 2, \end{cases}$ $i = 1, 2$ (see [18]).

The definition of $\varsigma(t,x)$ implies

$$\frac{\partial \varsigma(0,x)}{\partial t} = 0. \tag{7.1}$$

Then, from (2.11) and (7.1) we get

$$H(\varphi,\psi) + c\int_0^\infty \int_{\mathbb{R}^n} |u|^p\,\varsigma(t,x)\,dxdt \le \int_0^\infty \int_{\mathbb{R}^n} u(t,x)\,[\varsigma_{tt}(t,x)$$
$$-\Delta\varsigma_{tt}(t,x) + \Delta^2\varsigma(t,x)]dxdt - \int_0^\infty \int_{\mathbb{R}^n} u(t,x)\,(-\Delta)^\alpha\,\varsigma_t(t,x)\,dxdt, \tag{7.2}$$

where

$$(-\Delta)^\alpha\,\varsigma_t(t,x) = F^{-1}\left[|\xi|^{2\alpha}\,F\left[\varsigma_t(t,\cdot)\right]\right],$$
$$H(\varphi,\psi) = \int_{|x|<2d^2} [\psi(x) + \Delta\psi(x) + F^{-1}[|\cdot|^{2\alpha}\,F[\varphi]](x)]h_2\left(\frac{|x|^\mu}{d}\right)dx.$$

Using Hölder's and Young's inequalities, we have

$$\int_0^\infty \int_{\mathbb{R}^n} u(t,x)\cdot\varsigma_{tt}(t,x)dxdt - \varepsilon\int_0^\infty \int_{\mathbb{R}^n} |u|^p\,\varsigma(t,x)dxdt$$
$$\le c(\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} |\varsigma_{tt}(t,x)|^{p'}\,\varsigma^{1-p'}(t,x)dxdt = I_1;$$
$$\int_0^\infty \int_{\mathbb{R}^n} u(t,x)\;\varsigma_{tt}(t,x)dxdt - \varepsilon\int_0^\infty \int_{\mathbb{R}^n} |u|^p\,\varsigma(t,x)dxdt$$
$$\le c(\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} |\Delta\varsigma_{tt}(t,x)|^{p'}\,\varsigma^{1-p'}(t,x)dxdt = I_2;$$
$$\int_0^\infty \int_{\mathbb{R}^n} u(t,x)\cdot\Delta^2\varsigma(t,x)dxdt - \varepsilon\int_0^\infty \int_{\mathbb{R}^n} |u|^p\,\varsigma(t,x)dxdt$$
$$\le c(\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} \left|\Delta^2\varsigma(t,x)\right|^{p'}\,\varsigma^{1-p'}(t,x)dxdt = I_3;$$
$$\int_0^\infty \int_{\mathbb{R}^n} u(t,x)\,(-\Delta)^\alpha\,\varsigma_t(t,x)dxdt - \varepsilon\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p\,\varsigma(t,x)dxdt$$
$$\le c(\varepsilon)\int_0^\infty \int_{\mathbb{R}^n} \varsigma^{1-p'}(t,x)[(-\Delta)^\alpha\,\varsigma_t(t,x)]^{p'}dxdt = I_4,$$

where $0 < \varepsilon < \frac{c}{4}$, $c(\varepsilon) = \frac{1}{p'(p\varepsilon)^{\frac{p'}{p}}}$.

Then, from (7.2) we obtain the estimate

$$H\left(\varphi,\psi\right)+\left(c-4\varepsilon\right)\int_{0\leq t^{\chi}\leq 2d^{2}}\int_{0\leq|x|^{\mu}\leq 2d^{2}}|u(x,t)|^{p}\,dxdt\leq c(\varepsilon)\left(I_{1}+I_{2}+I_{3}+I_{4}\right). \tag{7.3}$$

Let'us make a substitution in the right-hand side of (7.3): $t=d^{\frac{2}{\chi}}\tau$, $x_k=d^{\frac{2}{\mu}}x'_k$, $k=1,2,...,n$. Then we get

$$I_1=d^{\frac{2}{\chi}+\frac{2n}{\mu}-\frac{4p'}{\chi}}I_{11}\cdot I_{12};I_2=d^{\frac{2}{\chi}+\frac{2n}{\mu}-\frac{4p'}{\mu}-\frac{4p'}{\chi}}I_{21}\cdot I_{22};I_3=d^{\frac{2}{\chi}+\frac{2n}{\mu}-\frac{8p'}{\mu}}I_{31}\cdot I_{32}, \tag{7.4}$$

where

$$I_{11}=I_{21}=\int_0^{2^{1/\chi}}h_1^{1-p'}(\tau^{\chi})\left|\frac{\partial^2}{\partial\tau^2}\left(h_1\left(\tau^{\chi}\right)\right)\right|^{p'}d\tau;$$

$$I_{12}=\int_{|x'|^{\mu}\leq 2}h_2(\left|x'\right|^{\mu})dx';$$

$$I_{22}=\int_{|x'|^{\mu}\leq 2}h_2^{1-p'}(\left|x'\right|^{\mu})\left|\Delta_{x'}[h_2\left(\left|x'\right|^{\mu}\right)]\right|^{p'}dx';$$

$$I_{31}=\int_0^{2^{1/\chi}}h_1(\tau^{\chi})d\tau;$$

$$I_{32}=\int_{|x'|\leq 2}h_2^{1-p'}|x'|^{\mu}\left|\left(\Delta^2_{x'}h_2\left|x'|^{\mu}\right)\right)\right|^{p'}dx'.$$

Now let us transform:

$$I_4=\int_0^{\infty}h_1^{1-p'}\left(\frac{t^{\chi}}{d^2}\right)\left|\frac{\partial}{\partial t}\left[h_1\left(\frac{t^{\chi}}{d^2}\right)_t\right]\right|^{p'}dt\cdot\int_{\mathbb{R}^n}h_2^{1-p'}\left(\frac{|x|^{\mu}}{d^2}\right)\left|(-\Delta)^{\alpha}\left(h_2\frac{|x|^{\mu}}{d^2}\right)\right|^{p'}dx.$$

If $0\leq\alpha\leq 1$, then using the representation of fractal derivatives in the Fourier transformation and the change of variables:

$$x_k=d^{\frac{2}{\mu}}x'_k,\ \ \xi_k=d^{-\frac{2}{\mu}}\xi'_k,\ \ y_k=d^{\frac{2}{\mu}}y'_k,\ \ k=1,2,...,n \tag{7.5}$$

we have the following equality

$$\left(-\Delta_x\right)^{\alpha}h_2\left(\frac{|x|^{\mu}}{d^2}\right)=\int_{R^n}e^{i\xi x}|\xi|^{2\alpha}\left(\int_{R^n}e^{-i\xi y}h_2\left(\frac{|y|^{\mu}}{d^2}\right)dy\right)d\xi$$

$$=d^{\frac{4\alpha}{\mu}}\int_{R^n}e^{i\xi'x'}|\xi'|^{2\alpha}\left(\int_{R^n}e^{-i\xi'x'}h_2\left(|y'|^{\mu}\right)dy'\right)d\xi'=d^{-\frac{4\alpha}{\mu}}(-\Delta_x)^{\alpha}h_2\left(|x'|^{\mu}\right).$$

Thus, using variable substation (7.5) and $t=d^{\frac{2}{\chi}}\tau$, we get

$$I_4=d^{\frac{2}{\chi}+\frac{2n}{\mu}-\frac{2p'}{\chi}-\frac{4\alpha p'}{\mu}}\cdot I_{41}\cdot I_{42}, \tag{7.6}$$

where

$$I_{41}=\int_0^{2^{1/\chi}}h_1^{1-p'}\left(\tau^{\chi}\right)h_{1_\tau}\left(\tau^{\chi}\right)d\tau,$$

$$I_{42}=\int_{|x'|^{\mu}\leq 2}h_2^{1-p'}\left(|x'|^{\mu}\right)|\left(-\Delta_x\right)^{\alpha}h_2\left(|x'|^{\mu}\right)|^{p'}dx'.$$

Now we will choose $h_1(\cdot)$ and $h_2(\cdot)$ such that the integrals I_{ik} are finite, where $i = 1, ..., 4$ and $k = 1, 2$.

Then from (7.3), (7.4) and (7.6) we obtain that

$$I_1 \le Cd^{\sigma_1};\ \ I_2 \le Cd^{\sigma_2};$$
$$I_3 \le Cd^{\sigma_3};\ \ I_4 \le Cd^{\sigma_4},$$

where

$$\sigma_1 = \frac{2}{\chi} + \frac{2n}{\mu} - \frac{4p'}{\chi}; \sigma_2 = \frac{2}{\chi} + \frac{2n}{\mu} - \frac{4p'}{\chi} - \frac{4p'}{\mu};$$
$$\sigma_3 = \frac{2}{\chi} + \frac{2n}{\mu} - \frac{8p'}{\mu}; \sigma_4 = \frac{2}{\chi} + \frac{2n}{\mu} - \frac{2p'}{\chi} - \frac{4\alpha p'}{\mu}.$$

By choosing $\mu = 2(2-\alpha)\chi,\ \ \chi > 1$ and taking into account the conditions (2.12), we obtain:

$$\sigma = \sigma_3 = \sigma_4 = \frac{1}{\chi(2-\alpha)}[n - 2\alpha - \frac{4}{p-1}] \le 0;$$
$$\sigma_2 = \sigma - \frac{2p'(3-2\alpha)}{\chi(2-\alpha)} < 0;\ \ \sigma_1 = \sigma - \frac{4p'}{\chi(2-\alpha)}(1-\alpha) < 0.$$

If $\sigma < 0$, then for $d \to \infty$ we get

$$\int_{\mathbb{R}^n} \left[\psi\left(x\right) + \Delta\psi\left(x\right) + \left(-\Delta\right)^{\alpha}\varphi(x)\right]\ dx + (c - 4\varepsilon)\int_0^{\infty}\int_{\mathbb{R}^n} |u|^p\, dxdt \le 0. \tag{7.7}$$

Hence, by (2.13), it follows that

$$u\left(t,x\right) = 0. \tag{7.8}$$

If $\sigma = 0$, then from (7.7) we obtain that

$$\int_0^{\infty}\int_{\mathbb{R}^n} |u|^p\, dxdt < \infty. \tag{7.9}$$

Further, applying Hölder's inequality, from (7.2) we obtain:

$$H\left(\varphi,\psi\right) + c\int\limits_{0\le t^{\chi}\le 2d^2}\ \int\limits_{|x|^{\mu}\le 2d^2} \left|u\left(t,x\right)\right|^p dxdt \le \left[\left(\int\limits_{0\le t^{\chi}\le 2d^2}\ \int\limits_{d^2\le |x|^{\mu}\le 2d^2} \left|u\left(t,x\right)\right|^p dxdt\right)^{1/p}\right.$$
$$\left. + \left(\int\limits_{d^2\le t^{\chi}\le 2d^2}\ \int\limits_{0\le |x|^{\mu}\le 2d^2} \left|u\left(t,x\right)\right|^p dxdt\right)^{1/p}\right] \cdot J, \tag{7.10}$$

where

$$J = \left(\int\limits_{0\le t^{\chi}\le 2d^2}\ \int\limits_{|x|^{\mu}\le 2d^2} \left|\varsigma_{tt}\left(t,x\right)\right|^{p'} dxdt\right)^{1/p'} + \left(\int\limits_{0\le t^{\chi}\le 2d^2}\ \int\limits_{|x|^{\mu}\le 2d^2} \left|\Delta\varsigma_{tt}\left(t,x\right)\right|^{p'} dxdt\right)^{1/p'}$$
$$+ \left(\int\limits_{0\le t^{\chi}\le 2d^2}\ \int\limits_{|x|^{\mu}\le 2d^2} \left|\Delta^2\varsigma\left(t,x\right)\right|^{p'} dxdt\right)^{1/p'}$$

$$+\left(\int\limits_{0\le t^{\chi}\le 2d^2}\int\limits_{|x|^{\mu}\le 2d^2}\left|(-\Delta)^{\alpha}\varsigma_t(t,x)\right|^{p'}dxdt\right)^{1/p'}\le cd^{\sigma} \tag{7.11}$$

It follows from (7.9) that

$$\lim_{d\to\infty}\int_{d^2<t^{\chi}\le 2d^2}\int_{|x|^{\mu}\le 2d^2}|u(t,x)|^p\,dxdt=0.$$

$$\lim_{d\to 0}\int_{0\le t^{\chi}\le 2d^2}\int_{d^2\le|x|^{\mu}\le 2d^2}|u(x,t)|^p\,dxdt=0.$$

Considering this in (7.10) and using (2.13) and (7.11), we obtain that the relation (7.8) is true for $\sigma=0$, too.

8 Remark on the Smoothness of Solutions

It is seen from the proving process of Theorem 2.1 that if $\varphi(x)\in W_2^3(\mathbb{R}^n)$, $\psi(x)\in W_2^2(\mathbb{R}^n)$, then the local solution $u(t,x)$ belongs to a smoother class of functions. Namely,

$$u(\cdot)\in C\left(\left[0,T'\right);\ W_2^2(\mathbb{R}^n)\right)\cap C^1\left(\left[0,T'\right);W_2^1(\mathbb{R}^n)\right),$$

where T' is the length of the maximal interval of existence of the local solution. Besides, if the a priori estimate

$$\|u(t,\cdot)\|_{W_2^3(\mathbb{R}^n)}+\left\|u'(t,\cdot)\right\|_{W_2^2(\mathbb{R}^n)}\le C,\ \ t\in\left[0,T'\right), \tag{8.1}$$

is true for this solution, then $T'=+\infty$.

If $(\varphi,\psi)\in\bigcup_{\delta_0}(\frac{n}{4}+3,2)$, then, using the representation (2.13), a priori estimates (5.21), (5.22) and the method of reasoning of the proofs of Lemmas 3.1 and 3.2, we get the validity of the a priori estimate (7.11) for sufficiently small values of $\delta_0>0$. So in this case we have

$$u(\cdot)\in C\left([0,\infty);\ W_2^3(\mathbb{R}^n)\right)\cap C^1\left([0,\infty);W_2^2(\mathbb{R}^n)\right).$$

The results obtained for the asymptotic profile of the solution of the corresponding linear part allowed us to get the stated task.

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