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Solvability of a boundary value problem for second order differential-operator equations with a spectral parameter in both the equation and boundary conditions

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Abstract. *In Hilbert space H we study noncoercive solvability of a boundary value problem for second order elliptic differential equations with a spectral parameter in the equation and boundary conditions in the case when in one of boundary conditions in addition to the spectral parameter there exists a linear bounded operator in the principal part of the given condition*

Keywords. differential-operator equations, elliptic equations, spectral parameter, izomorfizm, noncoercive estimation.

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1 Introduction

Boundary value problems for second order differential operator equations in the case when one and the same spectral parameter enters into the equation and boundary conditions were studied in different aspects in the papers of V.I. Gorbachuk and M.A. Rybak [14], M. A. Rybak [16], M. A .Denche [12], B. A. Aliev [2,3,4], B. A. Aliev and Ya. Yakubov [5], A. Aibeche , A. Favini and Mezoued [1], M. Bayramoglu and N.M.Aslanova [11], B. A. Aliev and N. K. Kurbanova [6], B. A. Aliev , N. K. Kurbanova and Ya.Yakubov [11], Ya.Yakubov [7] and others.

In the paper [8] B.A.Aliev and Ya.Yakubov, in UMD Banach space E the following boundary value problem was studied for a second order elliptic differential-operator equation with a spectral parameter in the case when one of the boundary conditions contains a linear bounded operator:

$$L(\lambda, D)\, u := \lambda u(x) - u''(x) + Au(x) + (A_1 u)(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

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$$
\begin{gathered}
L_1 u := \alpha u'(1) + B u(0) + \sum_{j=1}^{N_1} \gamma_{1j} u'(x_{1j}) + T_1 u = f_1, \\
L_2 u := \beta u'(0) + \sum_{j=1}^{N_2} \gamma_{2j} u'(x_{2j}) + T_2 u = f_2,
\end{gathered} \tag{1.2}
$$

where λ is a complex parameter, A is an inversible R- sectorial operator in E: B is a linear unbounded operator in H subjected to the operator $A^{1/2}$ in the certain sense; A_1 is a linear bounded operator in $L_p((0,1);E)$; T_k, $k=1,2$ are linear operators from $L_p((0,1);E)$ in E: $\alpha, \beta, \gamma_{kj}$ are any fixed complex numbers, $\alpha \neq 0, \beta \neq 0$; $x_{kj} \in (0,1)$; N_k some integers; $D := \dfrac{d}{dx}$.

In the paper [8] for problem (1.1), (1.2) at rather large $|\lambda|$ from some angle containing a positive semi- axis, a theorem on an isomorphism between the solutions and the right hand side of problem (1.1), (1.2) in space $L_p((0,1);E)$, $p \in (1,\infty)$, was proved , and it was established that for boundary value problem (1.1), (1.2) it holds coercive solvability with respect to u.

Solvability of boundary value problems in Hilbert space H for fourth order differential-operator equations (without a spectral parameter) with unbounded operators in one of boundary conditions was studied in the papers of A. R. Aliev and E. S. Rzayev [9], E.S.Al-Aidarous, A.R.Aliev, E. S. Rzayev and H. A. Zedan [10].

In the present paper, in separable Hilbert space H we study solvability of a boundary value problem of equation (1.1) with $A_1 = 0$, with the following boundary conditions

$$
\begin{gathered}
L_1(\lambda) u := \alpha u'(1) + \lambda B u(0) + \sum_{j=1}^{N_1} \gamma_{1_j} u'(x_{1j}) = f_1 \\
L_2 u := \beta u'(0) + \sum_{j=1}^{N_2} \gamma_{2_j} u'(x_{2j}) = f_2.
\end{gathered} \tag{1.3}
$$

As is seen, unlike boundary conditions (1.2), in boundary conditions (1.3) these is a spectral parameter in the principal part of the boundary condition. Existence of a spectral parameter in boundary conditions (1.3) qualitatively influences on solvability of boundary value problem (1.1), (1.3). The matter is that when we look for the solution of problem (1.1), (1.3) belonging to $W_p^2((0,1);H(A),H)$ the element f_2 can not be taken from interpolational space $(H(A),H)_{1/2+1/2p,p}$ that is dictated by the theorem on traces, it is taken from narrower interpolational space, more exactly, from $(H(A),H)_{1/2,p}$, though the element f_1 is taken from $(H(A),H)_{1/2+1/2p,p}$. As a spectral parameter exists in boundary conditions (1.3), we can not take the operator B unbounded as it was taken from boundary conditions (1.2), we take it only bounded.

As the element f_2 can't be found in interpolational space, $(H(A),H)_{1/2+1/2p,p}$ by studying the solvability of boundary value problem (1.1), (1.3), it is impossible to take the vector-valued function $f(x)$ from the space $L_p((0,1);H)$, $p \in (1;+\infty)$, as it was done in the paper [8]. We have to take it from narrower space, more exactly from $L_p((0,1);H(A^{1/2}))$. As a result, in the present paper, for problem (1.1),(1.3) a theorem on isomorpisim dosen't hold.

For the solution of problem (1.1), (1.3) belonging to $W_p^2((0,1);H(A),H)$ for rather large $|\lambda|$ from the angle $|\arg \mu| \leq \varphi < \pi$ noncorrosive estimation with respect to u was obtained, where $\varphi \in [0,\pi)$ is any fixed number, i.e. for boundary value problem (1.1), (1.3) the noncoercive estimation with respect to u was obtained.

Note that similar cases happen also by studying solvability of boundary value problems for equation (1.1) with the following Birkhoff-Tamarkin irregular boundary conditions:

$$\begin{aligned} L_{10}u &:= \alpha u'(0) + \beta u'(1) + \gamma u(0) + \delta u(1) = f_1, \\ L_{20}u &:= \alpha u(0) - \beta u(1) = f_2. \end{aligned} \tag{1.4}$$

In S.Yakubov and Ya.Yakubov's monograph [18] boundary value problem (1.1), (1.4) is studied in Hilbert space H. It is shown that by studying solvability of boundary value problems of the form (1.1), (1.4) in the space $L_p((0,1);H)$ it is not succeeded to take the elements f_k, $k=1,2$ from the natural interpolational space, they are taken from narrower interpolational space. There with, the right hand side of equation (1.1) i.e. $f(x)$, is taken not from $L_p((0,1);H)$ as in [8], but from the space $L_p((0,1);H(A^{1/2})$. In the monograph [18] it is proved that if in (1.4) $f_k \in (H(A),H)_{1-k/2+1/2p,p}$ and $f(x) \in L_p((0,1);H)$ are taken, then for rather large from the angle $|\arg \mu| \le \varphi$ for the solution of problem (1.1), (1.4) belonging to $W_p^2((0,1);H(A),H)$, a weaker estimation is obtained.

Note that solvability for boundary value problems of type (1.1), (1.4) in UMD Banach space were studied in A. Favini and Ya. Yakubov's paper [13].

Note that even if in boundary conditions (1.3) $B = I$ (a unit operator) is taken, then for boundary value problem (1.1), (1.3) the obtained result is also new and is stated first in this paper.

Let us introduce definitions and notion that are used in the present paper.

Let E_1 and E_2 be Banach spaces. The set (u,v) of all vectors of the form $u \in E_1$, where $v \in E_2$ with ordinary coordinate linear operations and with the norm

$$\|(u,v)\|_{E_1 \dot{+} E_2} := \|u\|_{E_1} + \|v\|_{E_2}$$

is a Banach space and is called the direct sum of Banach spaces E_1 and E_2 .

Let E_1 and E_2 be two Banach spaces. Denote by $B(E_1,E)$ a Banach space of all linear bounded operators acting from E_1 to E_2 with ordinary operator norm. In special case we assume $B(E) := B(E,E)$.

Definition 1.1 *Linear closed operator A in Hilbert space H will be called strongly positive if the domain of definition $D(A)$ is dense in H for some $\varphi \in [0,\pi)$ for all points from the angle $|\arg \mu| \le \varphi$ (including $\mu = 0$) these exist the operators $(A+\mu I)^{-1}$ and for these μ it holds the estimation*

$$\left\|(A+\mu I)^{-1}\right\|_{B(H)} \le C\,(1+|\mu|)^{-1},$$

where I is a unit operator in H, $C = const > 0$. For $\varphi = 0$, the operator A is said to be positive .

The simplest example of positive operators are selfadjoint positive-definite operators acting in a Hilbert space. Note that the strong positivity of the operator A yields the strong positivity of the operator $A^\alpha, \alpha \in (0,1)$.

Let A be a strongly positive operator in H. As the inverse operator A^{-1} is bounded in H, then

$$H\left(A^n\right) := \left\{u : u \in D\left(A^n\right),\ \|u\|_{H(A^n)} = \|A^n u\|_H\right\},\ \ n \in \mathbb{N},$$

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator A^n. If the operator A is strongly positive in H, it is known that the operator $-A$ is a generator of the semigroup e^{-tA} analytic for $t>0$, and this semigroup exponentially decreases, i.e., there exist two numbers $C>0,\ \sigma_0>0$ such that $\left\|e^{-tA}\right\| \le Ce^{-\sigma_0 t}$, $0 \le t < +\infty$. By [15,theorem 1.5.5], the operator $-A^{1/2}$ generates an analytic semigroup for $t>0$, decreasing at infinity.

Definition 1.2 *[17, theorem 1.14.5] Let A is a strongly positive operator in H, Then international spaces $(H(A^n), H)_{\theta,p}$ of Hilbert spaces $H(A^n)$ and H, are defined by the equality*

$$(H(A^n), H)_{\theta,p} := \{u : u \in H, \ \|u\|_{(H(A^n),H)_{\theta,p}} :=$$

$$= \int_0^{+\infty} t^{-1+n\theta p} \left\|A^n e^{-tA} u\right\|_H^p dt < \infty \right\}, \ \theta \in (0,1), \ p > 1, \ n \in \mathbb{N}.$$

We denote , $(H(A^n), H)_{0,p} := H(A^n)$, $(H(A^n), H)_{1,p} := H$.

Denote by $L_p((0,1); H)$ $(1 < p < \infty)$ a Banach space (for $p = 2$ a Hilbert space) of vector-functions $x \to u(x) : [0,1] \to H$ strongly measurable and summable in $p-th$ with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p dx\right)^{1/p} < \infty$$

and by $W_p^{2n}((0,1); H(A^n), H) := \left\{u : A^n u, u^{(2n)} \in L_p((0,1); H)\right\}$ denote a Banach space of vector-functions with the norm

$$\|u\|_{W_p^{2n}((0,1);H(A^n),H)} := \|A^n u\|_{L_p((0,1);H)} + \left\|u^{(2n)}\right\|_{L_p((0,1);H)}.$$

It is known that [17, theorem 1.8.2] if $u \in W_p^{2n}((0,1); H(A^n), H)$ then,

$$u^{(j)}(\cdot) \in (H(A^n), H)_{\frac{j+\frac{1}{p}}{2n},p}, \quad j = 0, \ldots, 2n-1.$$

2 Homogeneous Equations

As first, consider the following boundary value problem in separable Hilbert space H

$$L(\lambda, D) u := \lambda u(x) - u''(x) + Au(x) = 0, \ x \in (0,1), \tag{2.1}$$

$$\begin{gathered} L_1(\lambda) u := \alpha u'(1) + \lambda B u(0) + \sum_{j=1}^{N_1} \gamma_{1j} u'(x_{1j}) = f_1, \\ L_2 u := \beta u'(0) + \sum_{j=1}^{N_2} \gamma_{2j} u'(x_{2j}) = f_2. \end{gathered} \tag{2.2}$$

Theorem 2.1 *Let the following conditions be fulfilled:*
1. A is a strongly positive operator in H;
2.The linear operator B is bounded from H into H and from $H(A)$ into $H(A)$;
3. $\alpha, \beta, \gamma_{kj}$ are some complex numbers, and $\alpha \neq 0$, $\beta \neq 0$; $x_{kj} \in (0,1)$.

Then for, $f_1 \in (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$, $f_2 \in (H(A), H)_{\frac{1}{2p},p}$ and for sufficiently large $|\lambda|$ from the angle $|\arg \lambda| \leq \varphi < \pi$, problem (2.1), (2.2) has a unique solution $u \in W_p^2((0,1); H(A), H)$, such that $u(0) \in D(B)$, and the following noncoercive estimate holds for this solution:

$$\begin{gathered} |\lambda| \|u\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \\ \leq C \sum_{k=1}^{2} \left(\|f_k\|_{(H(A),H)_{1-\frac{k}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{k}{2}-\frac{1}{2p}} \|f_k\|_H\right). \end{gathered} \tag{2.3}$$

Proof. By [18, lemma 5.4.2/6] for $|\arg\lambda| \le \varphi < \pi$, there exists the analytic for $x > 0$ and strongly continuous for semigroup $e^{-x(A+\lambda I)^{1/2}}$. By [18,lemma 5.3.2/1], for the function $u(x)$ be the solution of equation (2.1) belonging to $W_p^2((0,1);H(A),H),\ p\in(1,\infty)$ it is necessary and sufficient that for $|\arg\lambda| \le \varphi < \pi$,

$$u\left(x\right)=e^{-x(A+\lambda I)^{1/2}}g_1+e^{-(1-x)(A+\lambda I)^{1/2}}g_2, \tag{2.4}$$

where $g_1, g_2 \in \left(H\left(A\right), H\right)_{\frac{1}{2p},p}$. By [17,theorem 1.8.2] (see also [18, theorem 1.7.7/1]) $u\left(0\right)\in$
$\left(H\left(A\right), H\right)_{\frac{1}{2p},p}$. From condition 2 and interpolation theorem [17, theorem 1.3.3/(a)] it follows that the operator B is bounded from $\left(H\left(A\right), H\right)_{\theta,p}$ into $\left(H\left(A\right), H\right)_{\theta,p}$ for any $\theta\in(0,1)$. So, $u\left(0\right)\in D\left(B\right)$.

Require the function (2.4) to satisfy the boundary conditions (2.2). Then we get the following system for the elements $g_1$and g_2.

$$\left[-\alpha\left(A+\lambda I\right)^{1/2}e^{-(A+\lambda I)^{1/2}}+\sum_{j=1}^{N_1}\gamma_{1j}e^{-x_{1j}(A+\lambda I)^{1/2}}+\lambda B\right]g_1$$
$$+\left[\alpha\left(A+\lambda I\right)^{1/2}+\lambda Be^{-(A+\lambda I)^{1/2}}\right.$$
$$\left.+\sum_{j=1}^{N_1}\gamma_{1j}\left(A+\lambda I\right)^{1/2}e^{-(1-x_{1j})(A+\lambda I)^{1/2}}\right]g_2=f_1,$$
$$\left[-\beta\left(A+\lambda I\right)^{1/2}-\sum_{j=1}^{N_2}\gamma_{2j}\left(A+\lambda I\right)^{1/2}e^{-x_{2j}(A+\lambda I)^{1/2}}\right]g_1$$
$$+\left[\beta\left(A+\lambda I\right)^{1/2}e^{-(A+\lambda I)^{1/2}}\right.$$
$$\left.+\sum_{j=1}^{N_2}\gamma_{2j}\left(A+\lambda I\right)^{1/2}e^{-(1-x_{2j})(A+\lambda I)^{1/2}}\right]g_2=f_2. \tag{2.5}$$

We write system (2.5) in space $\mathbb{H}:=\left(H\left(A\right), H\right)_{\frac{1}{2}+\frac{1}{2p},p}\dot{+}\left(H\left(A\right), H\right)_{\frac{1}{2p},p}$ in the from of the operator equation

$$\left(A\left(\lambda\right)+R\left(\lambda\right)\right)\begin{pmatrix}g_1\\ g_2\end{pmatrix}=\begin{pmatrix}f_1\\ f_2\end{pmatrix}, \tag{2.6}$$

where $A\left(\lambda\right)$ and $R\left(\lambda\right)$ are operator matrices of dimension 2×2:

$$A\left(\lambda\right)=\begin{pmatrix}\lambda B & \alpha\left(A+\lambda I\right)^{1/2}\\ -\beta\left(A+\lambda I\right)^{1/2} & 0\end{pmatrix},$$
$$D\left(A\left(\lambda\right)\right):=\left(H\left(A\right), H\right)_{\frac{1}{2p},p}\dot{+}\left(H\left(A\right), H\right)_{\frac{1}{2p},p}$$

and

$$R\left(\lambda\right):=\begin{pmatrix}K_{11}\left(\lambda\right) & K_{12}\left(\lambda\right)\\ K_{21}\left(\lambda\right) & K_{22}\left(\lambda\right)\end{pmatrix},\quad D\left(R\left(\lambda\right)\right):=\mathbb{H},$$

where

$$K_{11}(\lambda) := -\alpha (A+\lambda I)^{1/2} e^{-(A+\lambda I)^{1/2}} - \sum_{j=1}^{N_1} \gamma_{1j} (A+\lambda I)^{1/2} e^{-x_{1j}(A+\lambda I)^{1/2}},$$

$$K_{12}(\lambda) := \lambda B e^{-(A+\lambda I)^{1/2}} + \sum_{j=1}^{N_1} \gamma_{1j} (A+\lambda I)^{1/2} e^{-(1-x_{1j})(A+\lambda I)^{1/2}}$$

$$K_{21}(\lambda) := -\sum_{j=1}^{N_2} \gamma_{2j} (A+\lambda I)^{1/2} e^{-x_{2j}(A+\lambda I)^{1/2}},$$

$$K_{22}(\lambda) := \beta (A+\lambda I)^{1/2} e^{-(A+\lambda I)^{1/2}} + \sum_{j=1}^{N_2} \gamma_{2j} (A+\lambda I)^{1/2} e^{-(1-x_{2j})(A+\lambda I)^{1/2}}.$$

Show that the operator $A(\lambda)$ in space H for λ from the angle $|\arg\lambda| \leq \varphi < \pi$, has bounded inverse $A(\lambda)^{-1}$, acting from $\mathbb{H}$ into $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ and it holds the estimation

$$\left\| A(\lambda)^{-1} \right\|_{B\left(\mathbb{H},(H(A),H)_{\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2p},p}\right)} \leq C, \tag{2.7}$$

where $C > 0-$ is a constant independent of λ . As formally $A(\lambda)^{-1}$ has the form

$$A^{-1}(\lambda) = \begin{pmatrix} 0 & -\frac{1}{\beta}(A+\lambda I)^{-1/2} \\ \frac{1}{\alpha}(A+\lambda I)^{-1/2} & \frac{1}{\alpha\beta}\lambda (A+\lambda I)^{\mathbf{-1/2}} B (A+\lambda I)^{\mathbf{-1/2}} \end{pmatrix},$$

then for that it sufficiencies to show that:

a) the operator $(A+\lambda I)^{-\frac{1}{2}}$, for $|\arg\lambda| \leq \varphi < \pi$, is bounded from $(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\| (A+\lambda I)^{-1/2} \right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)} \leq C, \tag{2.8}$$

where $C > 0$ is a constant independent on λ;

b) the operator $(A+\lambda I)^{-\frac{1}{2}}$, for $|\arg\lambda| \leq \varphi < \pi$, is bounded from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\| (A+\lambda I)^{-\frac{1}{2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \leq C(1+|\lambda|)^{-\frac{1}{2}}, \tag{2.9}$$

where $C > 0$ is a constant independent on λ;

c) operator $\lambda (A+\lambda I)^{-\frac{1}{2}} B (A+\lambda I)^{-1/2}$, for $|\arg\lambda| \leq \varphi < \pi$, bounded from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$, and it holds the estimate

$$\left\| \lambda (A+\lambda I)^{-\frac{1}{2}} B (A+\lambda I)^{-\frac{1}{2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \leq C, \tag{2.10}$$

where $C > 0$ is a constant independent of λ.

Item a) was proved in the paper [8]. Prove b), by [18, lemma 5.4.2/6], for $|\arg\lambda| \le \varphi < \pi$, the operator $(A+\lambda I)^{-\frac{1}{2}}$ is bounded from H into H and it holds the estimate

$$\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le C\,(1+|\lambda|)^{-\frac{1}{2}}\,. \tag{2.11}$$

Then it is obvious that

$$\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H(A))} = \left\|A\,(A+\lambda I)^{-\frac{1}{2}}\,A^{-1}\right\|_{B(H)}$$
$$= \left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le C\,(1+|\lambda|)^{-\frac{1}{2}}\,. \tag{2.12}$$

From (2.11) and (2.12), by the interpolation theorem [17, theorem 1.3.3/(a)], it follows that the operator $(A+\lambda I)^{-\frac{1}{2}}$, for $|\arg\lambda| \le \varphi < \pi$, bounded from $(H(A),H)_{\theta,p}$ into $(H(A),H)_{\theta,p}$, for any $\theta\in(0,1)$, and it holds the estimate

$$\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B\left((H(A),H)_{\theta,p}\right)} \le C\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|^{1-\theta}_{B(H)}$$
$$\cdot\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|^{\theta}_{B((H(A)))} \le C\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le C\,(1+|\lambda|)^{-\frac{1}{2}}\,. \tag{2.13}$$

We take in (2.13) $\theta = \frac{1}{2p}$. Then we get (2.9), i.e. b) is proved. Prove c), by (2.11) and condition 2, for $|\arg\lambda| \le \varphi < \pi$ we have

$$\left\|\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,B\,(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le |\lambda|\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)}$$
$$\times\|B\|_{B(H)}\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le C\,|\lambda|\,(1+|\lambda|)^{-1} \le C, \tag{2.14}$$

$$\left\|\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,B\,(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H(A))} = \left\|A\left[\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,B\,(A+\lambda I)^{-\frac{1}{2}}\right]A^{-1}\right\|$$
$$= \left\|\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,ABA^{-1}\,(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le |\lambda|\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)}\left\|ABA^{-1}\right\|$$
$$\times\left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \le C\,|\lambda|\,(1+|\lambda|)^{-1} \le C. \tag{2.15}$$

By the above interpolational theorem, from estimations (2.14) and (2.15) it follows that the operator $\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,B\,(A+\lambda I)^{-\frac{1}{2}}$, for $|\arg\lambda| \le \varphi < \pi$, bounded from $(H(A),H)_{\theta,p}$ into $(H(A),H)_{\theta,p}$ for any $\theta\in(0,1)$ in particular, for $\theta = \frac{1}{2p}$ it holds estimation (2.10), i.e. c) is proved. Estimations (2.8), (2.9) and (2.10) yield (2.7). Then from equation (2.6) we have

$$\left(I + A\,(\lambda)^{-1}\,R\,(\lambda)\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A\,(\lambda)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{2.16}$$

From representations $A\,(\lambda)^{-1}$ and $R\,(\lambda)$ it is seen that the product $A\,(\lambda)^{-1}\,R\,(\lambda)$ is an operator -matrix whose members are linear combinations of the operators

$$e^{-(A+\lambda I)^{1/2}},\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,Be^{-(A+\lambda I)^{1/2}},e^{-x_{kj}(A+\lambda I)^{1/2}},e^{-\left(1-x_{kj}\right)(A+\lambda I)^{1/2}},$$
$$\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,Be^{-x_{2j}(A+\lambda I)^{1/2}},\lambda\,(A+\lambda I)^{-\frac{1}{2}}\,Be^{-(1-x_{2j})(A+\lambda I)^{1/2}}.$$

We can show that all operators in the operator-matrix $A(\lambda)^{-1} R(\lambda)$, for $|\arg\lambda| \le \varphi < \pi$, bounded from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$. For example, we show this for the operator $\lambda(A+\lambda I)^{-\frac{1}{2}} \times Be^{-(A+\lambda I)^{1/2}}$. By [18, lemma 5.4.2/6] and condition 2 for $|\arg\lambda| \le \varphi < \pi$, we have

$$\left\|\lambda(A+\lambda I)^{-\frac{1}{2}} Be^{-(A+\lambda I)^{1/2}}\right\|_{B(H)} \le |\lambda| \left\|(A+\lambda I)^{-\frac{1}{2}}\right\|_{B(H)} \|B\|_{B(H)}$$

$$\times \left\|e^{-(A+\lambda I)^{1/2}}\right\|_{B(H)} \le C|\lambda|(1+|\lambda|)^{-1/2} e^{-\omega|\lambda|^{1/2}}$$

$$\le C|\lambda|^{1/2} e^{-\omega|\lambda|^{1/2}},\ \exists C>0, \omega>0; \tag{2.17}$$

$$\left\|\lambda(A+\lambda I)^{-\frac{1}{2}} Be^{-(A+\lambda I)^{1/2}}\right\|_{B(H(A))}$$

$$= |\lambda| \left\|A\left[(A+\lambda I)^{-1/2} Be^{-(A+\lambda I)^{1/2}}\right]A^{-1}\right\|_{B(H)}$$

$$\le |\lambda| \left\|(A+\lambda I)^{-1/2}\right\|_{B(H)} \left\|ABA^{-1}\right\|_{B(H)}$$

$$\times \left\|e^{-(A+\lambda I)^{1/2}}\right\|_{B(H)} \le C|\lambda|^{\frac{1}{2}} e^{-\omega|\lambda|^{1/2}}, \exists C>0, \omega>0. \tag{2.18}$$

By interpolational theorem [17, theorem 1.3.3/(a)], from estimations (2.17), (2.18) it follows that for $|\arg\lambda| \le \varphi < \pi$, the operator $\lambda(A+\lambda I)^{-1/2} Be^{-(A+\lambda I)^{1/2}}$ is bounded from $(H(A), H)_{\theta,p}$ into $(H(A), H)_{\theta,p}$, for any $\theta \in (0,1)$, including from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$ (for $\theta = \frac{1}{2p}$) and it holds the estimate

$$\left\|\lambda(A+\lambda I)^{-1/2} Be^{-(A+\lambda I)^{1/2}}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)}$$

$$\le C|\lambda|^{\frac{1}{2}} e^{-\omega|\lambda|^{1/2}},\ \exists C>0, \omega>0. \tag{2.19}$$

In a similar way it is proved that remaining members of the operator-matrix $A(\lambda)^{-1} R(\lambda)$ bounded from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$ and for the norms of these members, for $|\arg\lambda| \le \varphi < \pi$, the estimations of type (2.19) hold. So, for rather large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, the operator $A(\lambda)^{-1} R(\lambda)$ is bounded from $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\|A(\lambda)^{-1} R(\lambda)\right\|_{B\left((H(A),H)_{\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2p},p}\right)}$$

$$\le C|\lambda|^{\frac{1}{2}} e^{-\omega|\lambda|^{\frac{1}{2}}} < 1,\ \exists C>0, \omega>0. \tag{2.20}$$

Hence, by the Neumann identity, for rather large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$,

$$\left(I + A(\lambda)^{-1} R(\lambda)\right)^{-1} = I + \sum_{k=1}^{\infty} (-1)^k \left(A(\lambda)^{-1} R(\lambda)\right)^k, \tag{2.21}$$

where the series in the right hand side of (2.21) converges in the norm of the space of bounded operators in $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$. By (2.20) and (2.21) from (2.16) for $|\arg\lambda| \le \varphi < \pi$ and $|\lambda| \to \infty$, we have

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(I + A(\lambda)^{-1} R(\lambda)\right)^{-1} A(\lambda)^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Consequently, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, the elements g_1 and g_2 can be represented in the from:

$$g_k = (C_{k1}(\lambda) + R_{k1}(\lambda)) f_1 + (C_{k2}(\lambda) + R_{k2}(\lambda)) f_2, \quad k = 1, 2, \tag{2.22}$$

where

$$C_{11}(\lambda) = 0, \quad C_{12}(\lambda) = -\tfrac{1}{\beta}(A + \lambda I)^{-\frac{1}{2}}, \ \ C_{21}(\lambda) = \tfrac{1}{\alpha}(A + \lambda I)^{-\frac{1}{2}},$$
$$C_{22}(\lambda) = \tfrac{1}{\alpha\beta}\lambda (A + \lambda I)^{-\frac{1}{2}} B (A + \lambda I)^{-\frac{1}{2}}, \qquad R_{kj}(\lambda), \ (k, j = 1, 2) -$$

are some bounded operators from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$, for $|\arg\lambda| \le \varphi$ and $|\lambda| \to \infty$. Moreover, using the estimaties (2.7) and (2.20), one can show that, for $|\arg\lambda| \le \varphi < \pi$ and $|\lambda| \to \infty$,

$$\|R_{kj}(\lambda)\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \le C|\lambda|^{\frac{1}{2}} e^{-\omega|\lambda|^{\frac{1}{2}}}, \ \exists C, \ \omega > 0. \tag{2.23}$$

From the representation $A(\lambda)^{-1}$ and $A(\lambda)^{-1} R(\lambda)$ it also follows that, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, for the operators $R_{kj}(\lambda)$ the estimate:

$$\|R_{kj}(\lambda)\|_{B(H)} \le C|\lambda|^{\frac{1}{2}} e^{-\omega|\lambda|^{\frac{1}{2}}}, \quad \exists C, \omega > 0. \tag{2.24}$$

Substituting (2.22) in (2.4), we get

$$u(x) = \sum_{k=1}^{2} \Big\{ e^{-x(A+\lambda I)^{1/2}} (C_{1k}(\lambda) + R_{1k}(\lambda))$$
$$+ e^{-(1-x)(A+\lambda I)^{1/2}} (C_{2k}(\lambda) + R_{2k}(\lambda)) \} f_k.$$

Then, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, we have

$$|\lambda| \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)}$$
$$\le C \sum_{k=1}^{2} \Bigg\{ |\lambda| \Bigg[\left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} C_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p}$$
$$+ \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} R_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p}$$
$$+ \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} C_{2k}(\lambda) f_k \right\|_H^p dx \right)^{1/p}$$

$$+\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/p}} R_{2k}(\lambda) f_k\right\|_H^p dx\right)^{1/p}\Bigg]$$

$$+\left(1+\left\|A(A+\lambda I)^{-1}\right\|\right)$$

$$\times\Bigg[\left(\int_0^1 \left\|(A+\lambda I) e^{-x(A+\lambda I)^{1/2}} C_{1k}(\lambda) f_k\right\|_H^p dx\right)^{1/p}$$

$$+\left(\int_0^1 \left\|(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} C_{2k}(\lambda) f_k\right\|_H^p dx\right)^{1/p}$$

$$+\left(\int_0^1 \left\|(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} R_{2k}(\lambda) f_k\right\|_H^p dx\right)^{1/p}\Bigg]\Bigg\}. \qquad (2.25)$$

By [18, theorem 5.4.2/1] and estimations (2.10) and (2.14) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi$, for the first term of the right hand side of inequality (2.25) (for $k=2$) we have:

$$|\lambda|\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/2}} C_{22}(\lambda) f_2\right\|_H^p dx\right)^{1/p}$$

$$=|\lambda|\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/2}} \frac{1}{\alpha\beta}\lambda (A+\lambda I)^{-1/2} B (A+\lambda I)^{-1/2} f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C|\lambda|\cdot\left\|(A+\lambda I)^{-1}\right\|_{B(H)}$$

$$\times\left(\int_0^1 \left\|(A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} \lambda (A+\lambda I)^{-1/2} B (A+\lambda I)^{-1/2} f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C\Bigg(\left\|\lambda (A+\lambda I)^{-1/2} B (A+\lambda I)^{-1/2} f_2\right\|_{(H(A),H)_{\frac{1}{2p},p}}$$

$$+|\lambda|^{1-\frac{1}{2p}}\left\|\lambda (A+\lambda I)^{-1/2} B (A+\lambda I)^{-1/2} f_2\right\|_H\Bigg)$$

$$\le C\left(\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_2\|_H\right).$$

By [18, theorem 5.4.2/1] for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi$, for the first term of the right hand side of inequality (2.25) (for $k=2$) we have

$$|\lambda|\left(\int_0^1 \left\|e^{-x(A+\lambda I)^{1/2}} C_{12}(\lambda) f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C|\lambda|\left(\int_0^1 \left\|e^{-x(A+\lambda I)^{1/2}} (A+\lambda I)^{-1/2} f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C|\lambda|\left\|(A+\lambda I)^{-1}\right\|_{B(H)}\left(\int_0^1 \left\|(A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C\left(\int_0^1 \left\|(A+\lambda I)^{1/2}e^{-x(A+\lambda I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C\left\|(A+\lambda I)^{-1/2}\right\|_{B(H)}\left(\int_0^1 \left\|(A+\lambda I)e^{-x(A+\lambda I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}$$

$$\le C(1+|\lambda|)^{-1/2}\left(\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|f_2\|_H\right)$$

$$\le C\left(\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\|f_2\|_H\right). \quad (2.26)$$

Estimate the integral with $C_{21}(\lambda)f_1$ in the third term of the right hand side of inequality (2.25). Again, according to [18, theorem 5.4.2/1] and for the same λ we have

$$|\lambda|\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/2}}C_{21}(\lambda)f_1\right\|_H^p dx\right)^{1/p}$$

$$=|\lambda|\left(\int_0^1 \left\|e^{-(1-x)(A+\lambda I)^{1/2}}\frac{1}{\alpha}(A+\lambda I)^{-1/2}f_1\right\|_H^p dx\right)^{1/p}$$

$$\le C|\lambda|\left\|(A+\lambda I)^{-1}\right\|_{B(H)}\cdot\left(\int_0^1 \left\|(A+\lambda I)^{1/2}e^{-(1-x)(A+\lambda I)^{1/2}}f_1\right\|_H^p dx\right)^{1/p}$$

$$\le C\left(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+|\lambda|^{\frac{1}{2}-\frac{1}{2p}}\|f_1\|_H\right).$$

The remaining summands of the right hand side are estimated in the same way. Theorem 2.1 is proved.

3 Nonhomogeneous Equations

Now we consider a boundary value problem for an nonhomogeneous equation with a parameter in H, i.e. the problem

$$L(\lambda,D)u:=\lambda u(x)-u''(x)+Au(x)=f(x),\quad x\in(0,1), \quad (3.1)$$

$$L_1(\lambda)u:=\alpha u'(1)+\lambda Bu(0)+\sum_{j=1}^{N_1}\gamma_{1j}u'(x_{1j})=f_1,$$
$$L_2u:=\beta u'(0)+\sum_{j=1}^{N_2}\gamma_{2j}u'(x_{2j})=f_2. \quad (3.2)$$

Theorem 3.1 *Let the conditions of theorem 2.1 be fulfilled:*

Then, for $f\in L_p\left((0,1);H\left(A^{1/2}\right)\right)$, $f_1\in(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$, $f_2\in(H(A),H)_{\frac{1}{2p},p}$ *and for sufficiently large* $|\lambda|$ *from the angle* $|\arg\lambda|\le\varphi<\pi$, *problem (3.1), (3.2) has a unique solution from* $W_p^2((0,1);H(A),H)$ *and for this solution it holds the following noncoercive estimation*

$$|\lambda|\|u\|_{L_p((0,1);H)}+\left\|u''\right\|_{L_p((0,1);H)}+\|Au\|_{L_p((0,1);H)}$$
$$\le C\Big[|\lambda|\|f\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)}$$
$$+\sum_{k=1}^{2}\left(\|f_k\|_{(H(A),H)_{1-\frac{k}{2}+\frac{1}{2p},p}}+|\lambda|^{\frac{k}{2}-\frac{1}{2p}}\|f_k\|_H\right)\Big]. \quad (3.3)$$

Proof. The uniqueness follows from theorem 2.1. The solution of problem (3.1),(3.2), belonging to
$W_p^2((0,1);H(A),H)$ in the form of the sum $u(x)=u_1(x)+u_2(x)$, where $u_1(x)$ if $[0,1]$ is the contraction on [0,1] of the solution of the equation

$$L(\lambda,D)\tilde{u}_1(x)=\tilde{f}(x),\quad x\in\mathbb{R}=(-\infty;+\infty),\tag{3.4}$$

where $\tilde{f}(x):=f(x)$, if $x\in[0,1]$, and $\tilde{f}(x)=0$, if $x\notin[0,1]$, and $u_2(x)$ is the solution of the problem

$$\begin{array}{c} L(\lambda,D)u_2=0, x\in(0,1), \\ L_1(\lambda)u_2=f_1-L_1(\lambda)u_1, \\ L_2u_2=f_2-L_2u_1. \end{array}\tag{3.5}$$

As is shown in the proof [13, theorem 2] if $f\in L_p\left(R;H\left(A^{1/2}\right)\right)$, then for $|\arg\lambda|\le\varphi<\pi$, the solution of equation (3.4) belongs to the space $W_p^2\left(R;H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)$ and for the solution is holds the estimation

$$\|\tilde{u}_1\|_{W_p^2\left(R;H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)}\le C\left\|\tilde{f}\right\|_{L_p\left(R;H\left(A^{1/2}\right)\right)}\tag{3.6}$$

uniformly with respect to λ. Hence it follows that $u_1\in W_p^2\left((0,1);H\left(A^{3/2}\right),\ H\left(A^{1/2}\right)\right)\subset$

$W_p^2((0,1);H(A),H)$. Then from (3.4) and (3.6) it follows that

$$\begin{aligned}|\lambda|\,\|u_1\|_{L_p((0,1);H)}+\left\|u_1''\right\|_{L_p((0,1);H)}+\|Au_1\|_{L_p((0;1);H)}\\ \le C\,\|f\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)}.\end{aligned}\tag{3.7}$$

By [17, theorem 1.8.2] (see also [18, theorem 1.7.7/1]), for any fixed $x_0\in[0,1]$,

$$u_1'(x_0)\in\left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1}{2}+\frac{1}{2p},p},$$

$$u_1(x_0)\in\left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1}{2p},p}.$$

By [17, lemma 1.7.3/1, 1.7.3/6 and 1.7.3/5], for $k=0,1$, we have

$$\left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1+kp}{2p},p}=\left(H\left(A^{1/2}\right),H\left(A^{3/2}\right)\right)_{1-\frac{1+kp}{2p},p}$$

$$=\left(H,H\left(A^{3/2}\right)\right)_{1-\frac{1+kp}{3p},p}=\left(H,H\left(A^2\right)\right)_{\frac{3}{4}-\frac{1+kp}{4p},p}=\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1+kp}{4p},p}.$$

Consequently,

$$u_1'(x_0)\in\left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p},\quad u_1(x_0)\in\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}.$$

By [17, theorem 1.3.3/(b) and theorem 1.15.2/(f)], we have

$$\left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p}=\left(H,H\left(A^2\right)\right)_{\frac{1}{2}\left(1-\frac{1}{2p}\right),p}$$

$$=(H,H(A))_{1-\frac{1}{2p},p}=(H(A),H)_{\frac{1}{2p},p}.$$

On the other hand, at the beginning of the proof of theorem 2.1 it was shown that the operator B is bounded from $(H(A), H)_{\theta,p}$ into $(H(A), H)_{\theta,p}$ for any $\theta \in (0,1)$. Hence

$$L_1(\lambda) u_1 \in (H(A), H)_{\frac{1}{2p},p} \subset (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}. \tag{3.8}$$

Similarly

$$L_2 u_1 \in (H(A), H)_{\frac{1}{2p},p}. \tag{3.9}$$

Furthermore

$$\begin{aligned} &\left(H\left(A^2\right), H\right)_{\frac{1}{4}+\frac{1}{4p},p} \subset \left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p} \\ &= (H(A), H)_{\frac{1}{2p},p} \subset (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}. \end{aligned} \tag{3.10}$$

Then by theorem 2.1, for sufficiently large $|\lambda|$ from the angle $|\arg| \le \varphi < \pi$, for the solution of problem (3.5) we have

$$\begin{aligned} &|\lambda| \|u_2\|_{L_p((0,1);H)} + \left\|u_2''\right\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)} \\ &\le C\left[\|f_1 - L_1(\lambda) u_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_1 - L_1(\lambda) u_1\|_H + \right. \\ &\left. \|f_2 - L_2 u_1\|_{(H(A),H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_2 - L_2 u_1\|_H\right] \\ &\le C\left(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + \|f_2\|_{(H(A),H)_{\frac{1}{2p}}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_1\|_H\right. \\ &+ |\lambda|^{1-\frac{1}{2p}} \|f_2\|_H + \left\|u_1'(1)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda| \|Bu_1(0)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} \\ &+ \sum_{j=1}^{N_1} \left\|u_1'(x_{1j})\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + \left\|u_1'(0)\right\|_{(H(A),H)_{\frac{1}{2p},p}} \\ &\left. + \sum_{j=1}^{N_2} \left\|u_1'(x_{2j})\right\|_{(H(A),H)_{\frac{1}{2p},p}}\right). \end{aligned} \tag{3.11}$$

By [17, theorem 1.8.2] (see. also [18, theorem 1.7.7/1]) and inequality (3.7) we have $(\forall x_0 \in [0,1])$

$$\begin{aligned} &\left\|u_1'(x_0)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} \le C \|u_1\|_{W_p^2((0,1);H(A),H)} \\ &\le C \|f\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)}. \end{aligned} \tag{3.12}$$

Estimate the expression $|\lambda| \cdot \|Bu_1(0)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}$. Above we have already noted that the operator B is bounded from $(H(A), H)_{\theta,p}$ into $(H(A), H)_{\theta,p}$, for any $\theta \in (0,1)$. Then by [17, theorem 1.8.2] (see. also [18, theorem 1.7.7/1]) and estimation (3.10) and (3.7) we have

$$\begin{aligned} &|\lambda| \|Bu_1(0)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} \le |\lambda| \|B\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}\right)} \\ &\times \|u_1(0)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} \le C|\lambda| \cdot \|u_1(0)\|_{(H(A),H)_{\frac{1}{2p},p}} \end{aligned}$$

$$\leq C\left|\lambda\right|\left\|u_1\right\|_{W_p^2((0,1);H(A),H)} \leq C\left|\lambda\right|\left\|f\right\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)}. \quad (3.13)$$

Now estimate the norms $\left\|u_1'(x_0)\right\|_{(H(A),H)_{\frac{1}{2p},p}}$, for any fixed $x_0 \in [0,1]$. Using [17, theorem 1.15.2 and 1.15.4/2], we can show

$$\left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p} = \left(H\left(A^{\frac{3}{2}}\right),H\left(A^{\frac{1}{2}}\right)\right)_{\frac{1}{2}+\frac{1}{2p},p}.$$

Then by [17, theorem 1.8.2] (see. also [18, theorem 1.7.7/1]) and (3.6),we have

$$\left\|u_1'(x_0)\right\|_{(H(A),H)_{\frac{1}{2p},p}} = \left\|u_1'(x_0)\right\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}}$$

$$= \left\|u_1'(x_0)\right\|_{\left(H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)_{\frac{1}{2}+\frac{1}{2p},p}}$$

$$\leq C\left\|u_1\right\|_{W_p^2\left((0,1);H\left(A^{3/2}\right),H\left(A^{1/2}\right)\right)} \leq C\left\|f\right\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)}. \quad (3.14)$$

Taking into account estimations in (3.12)-(3.14) in (3.11) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$, we have

$$|\lambda|\left\|u_2\right\|_{L_p((0,1);H)} + \left\|u_2''\right\|_{L_p((0,1);H)} + \left\|Au_2\right\|_{L_p((0,1);H)}$$

$$\leq C\left[|\lambda|\left\|f\right\|_{L_p\left((0,1);H\left(A^{1/2}\right)\right)} + \right.$$

$$\left. + \sum_{k=1}^{2}\left(\left\|f_k\right\|_{(H(A),H)_{1-\frac{k}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{k}{2}-\frac{1}{2p}}\left\|f_k\right\|_H\right)\right]. \quad (3.15)$$

From (3.7) and (3.15) it follows (3.3). Theorem 3.1 is proved.

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