Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (4), 37-46 (2016).
Series of Physical-Technical and Mathematical Sciences.

On fourth-order eigenvalue problems with indefinite weight

Ziyatkhan S. Aliyev · **Rada A. Huseynova**

Received: 06.10.2015 / Revised: 16.03.2016/ Accepted: 21.07.2016

Abstract. *In this paper we consider eigenvalue problems for the fourth order differential equations with sign-changing weight. We use the Ljusternik-Schnirelmann theory in* $\boldsymbol{C}^1$*-manifolds show that there exists two series of positive and negative eigenvalues of this problem. The first positive and negative eigenvalues are simple, and the corresponding eigenfunctions do not vanish in the interval* (0.1).

Keywords. eigenvalue, eigenfunction, fourth order eigenvalue problem, indefinite weight, Ljusternik-Schnirelmann

Mathematics Subject Classification (2010): 34B15, 34B18, 34C23, 34L05, 35P30, 58E05

1 Introduction

In this paper we consider the following spectral problem for the fourth order Sturm-Liouville equation

$$(\tau(x)y''(x))'' = \lambda r(x)y(x),\ x \in (0,1), \tag{1.1}$$

together with the Navier boundary conditions (used in the case of a beam supported at the ends)

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, the weight function $\tau(x)$ is positive and has absolutely continuous derivative on $[0,1]$, $r(x)$ is real-valued continuous sign-changing weight function on $[0,1]$.

The eigenvalue problem (1.1)-(1.2) were studied by Janczewsky [19] (see also [6]) for the case where $r(x) > 0,\ x \in [0,1]$. It is proved that the eigenvalues of (1.1)-(1.2) for $r(x) > 0,\ x \in [0,1]$, are all real and simple and form a sequence $0 < \mu_1 < \mu_2 < \ldots < \mu_k \mapsto +\infty$. Moreover, the k-th eigenfunction $v_k(x)$, corresponding to the k-th eigenvalue μ_k, has precisely $k-1$ simple zeros in $(0,1)$.

Z. S. Aliyev
E-mail: z_aliyev@mail.ru
Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan

R. A. Huseynova
E-mail: rada_huseynova@yahoo.com
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan

It is known that eigenvalue problems of the p-Laplacian and p-biharmonic operators have been studied extensively; we mention, for example, the works [2, 5, 7-18, 20, 21, 23, 24]. When dealing with eigenvalue problems of the p-Laplacian with indefinite weight was studied in [2, 5, 15, 21], of the p-biharmonic operator was studied in [10-12]. The oscillatory properties of eigenfunctions of p-Laplacian with indefinite weight well studied. But the oscillatory properties of eigenfunctions of p-biharmonic operator with indefinite weight we can say that has not yet been studied.

The purpose of this work is to study the spectrum of the boundary value problem (1.1)-(1.2) for the case where $r(x)$ is indefinite weight in $[0, 1]$ (i.e. for $\nu \in \{+, -\}$ we have **meas** $\{ I_r^\nu \} > 0$ where $I_r^\nu = \{ x \in \bar{I} : \nu\, r(x) > 0\}$). We prove that this spectral problem has two sequences of positive and negative eigenvalues tending to infinity and no other eigenvalues. Moreover, the least positive and negative eigenvalues are simple and their corresponding eigenfunctions have no zeros in the interval $(0, 1)$.

2 On critical points of smooth functionals on $\mathbf{C}^1$-manifold

Let **M** be a $\mathbf{C}^1$ Banach manifold (without boundary). Denote the tangent bundle of **M** by $T(\mathbf{M})$ and the tangent space of **M** at x by $T_x(M)$. Let $|| \cdot || : T(M) \to [0, +\infty$ be a continuous function such that

(i) for each $x \in \mathbf{M}$, the restriction of $|| \cdot ||$ to $T_x(\mathbf{M})$, denoted by $|| \cdot ||_x$ is an admissible norm on $T_x(M)$;

(ii) for each $x_0 \in \mathbf{M}$ and $k > 1$ there is a trivializing neighborhood U of x_0 such that

$$\frac{1}{k} \| \cdot \|_x \leq \| \cdot \|_{x_0} \leq k \| \cdot \|_x, \ x \in U.$$

The function $|| \cdot ||$ is called Finsler structure for $T(\mathbf{M})$. A regular manifold together with a Finsler structure for $T(\mathbf{M})$ is called Finsler manifold. Every paracompact $\mathbf{C}^1$ Banach manifold admits a Finsler structure [24].

Let **M** is Finsler manifold and $f \in \mathbf{C}^1(\mathbf{M}, \mathbb{R})$. Denote the differential of f at x by $df(x)$. Then $df(x)$ is an element of the cotangent space of **M** at x, $T_x(\mathbf{M})^*$. A point $x \in M$ is said to be critical point of f if $df(x) = 0$. The corresponding value $c = f(x)$ will be called a critical value.

If **M** is Finsler manifold, then cotangent bundle $T(\mathbf{M})^*$ has a dual structure given by

$$||w|| = \sup\{ \langle w, v\rangle \, : \, v \in T_x(\mathbf{M}), \, ||v||_x = 1\},$$

where $w \in T_x(\mathbf{M})^*$ and $\langle \cdot \, , \, \cdot \rangle$ is the duality pairing between $T_x(\mathbf{M})^*$ and $T_x(\mathbf{M})$. It follows that the mapping $||x \mapsto df(x)||$ will is defined and continuous for $f \in \mathbf{C}^1(\mathbf{M}, \mathbb{R})$. A function $f \in \mathbf{C}^1(\mathbf{M}, \mathbb{R})$ is said to satisfy *the Palais-Smale condition at the level* c, $c \in \mathbb{R}$ ($(\mathbf{PS})_c$ in short) if each sequence $\{x_n\}_{n=1}^{\infty} \subset M$ such that $f(x_n) \to c$ and $||df(x_n)|| \to 0$ as $n \to \infty$ has convergent subsequence.

Let **E** is a real Banach space and $\sum$ the collection of all symmetric subset of $\mathbf{E}\backslash\{0\}$ which are closed in **E** ($Y \subset \mathbf{E}$ is symmetric if $Y = -Y$). A nonempty set $Y \in \sum$ is said to be of genus k (denoted $\gamma(Y) = k$) if k is smallest integer with the property that there exists an odd continuous mapping from Y to $\mathbb{R}^k\backslash\{0\}$. If there is no such k, $\gamma(Y) = +\infty$, and if $Y = \emptyset$, $\gamma(Y) = 0$ (see [24]).

To solve eigenvalue problem (1.1)-(1.2), we employ the Ljusternik-Schnirelmann theory on $\mathbf{C}^1$-manifolds which provides a method for proving the existence of one or several critical points of a functional, with the aid of a concept of the genus. We will use the following results proved by Szulkin [24].

Theorem 2.1 [24; § 4, Corollary 4.1]. *Suppose that $\boldsymbol{M}$ is a closed symmetric $\boldsymbol{C}^1$-submanifold of $\boldsymbol{E}$ and $0 \notin \boldsymbol{M}$. Suppose also that $f \in \boldsymbol{C}^1(\boldsymbol{M}, \mathbb{R})$ is even and bounded below. Define*

$$c_j = \inf_{Y \subset \Gamma_j} \sup_{x \in Y} f(x),$$

*where $\Gamma_j = \{Y \subset \boldsymbol{M} \, : \, Y \in \sum, \, \gamma(Y) \geq j$ and Y is compact$\}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies **(PS)**$_c$ for all $c = c_j$, $j = 1, \, 2, \, ... \, k$, then f has at last k distinct pairs of critical points.*

A function $f : E \to \mathbb{R}$ is said to be Gâteaux differentiable if each $u \in E$ there exists a linear mapping $f'(u) \in E^*$ such that

$$\frac{d}{dt} f(u + tv) \bigg|_0 = \langle f'(u), v \rangle, \ \forall v \in E. \tag{2.1}$$

Let us remark that there is a different-weaker-definition of Gâteaux differentiability in which it is not required that the left-hand side of (2.1) by linear in$v$ (see e.g. [6]). A function $f \, : \, E \to \mathbb{R}$, where $\mathbf{M}$ is Finsler manifold, will be called Gâteaux differentiable if each $u \in \mathbf{M}$ and each chart $\phi : U \to T_x(\mathbf{M})$ at x, $f \circ \phi^{-1}$ is Gâteaux differentiable. The Gâteaux derivative df is $strong-to-weak^*$ $continuous$ if for each $x \in \mathbf{M}$, each sequence $u_n \to u$ as $n \to \infty$ and each chart $\phi : U \to T_x(\mathbf{M})$ at x

$$\left(f \circ \varphi^{-1}\right)' (\varphi(u_n)) \to \left(f \circ \varphi^{-1}\right)' (\varphi(u)) \ \ as \ n \to \infty$$

in the weak topology of $T_x(\mathbf{M})^*$

3 Existence of eigenvalues of problem (1.1)-(1.2)

Let $I = (0, 1)$ and $W^{k, \, p, \, \tau}(I)$ be the weighted Sobolev space that consist of all measurable real-valued functions u defined in I for which

$$||u||_{k, \, p, \, \tau} = \left\{ \sum_{m=0}^{k-1} \int_I |u^{(m)}(x)|^p dx + \int_I \tau(x) |u^{(k)}(x)|^p dx \right\}^{1/p} < \infty$$

and $W_0^{1, \, p}(I)$ is the closure of $C_0^\infty(I)$ in $W^{1, \, p}(I) = W_1^{1, \, p, \, 1}(I)$.

We denote $X = W_0^{1, \, 2}(I) \cup W^{2, \, 2, \, \tau}(I)$ with the norm

$$||u||_X \ = \ \left\{ \int_I p\,(x) |u''(x)|^2 dx \right\}^{\frac{1}{2}},$$

which by a weighted Friedrichs inequality is equivalent to the norm $||u||_{2, \, 2, \, \tau}$ of the space $W^{2, \, 2, \, \tau}(I)$. Further, we denote by X^* and $\langle \cdot \ , \ \cdot \rangle$ the dual space to X and the pairing between X and X^*. It is clear that X is a nonempty, well defined and closed subspace of $W^{2, \, 2, \, \tau}(I)$. However, it is easy to see that X is reflexive separable space with the induced norm of $W^{2, \, 2, \, \tau}(I)$ and uniformly convex. For simplicity we write $u_n \rightharpoonup u$ and $u_n \to u$ as $n \to \infty$ to denote the weak convergence and strong convergence of sequence $\{u_n\}_{n=1}^\infty \subset X$ in X, respectively (see [1]).

Firstly, we recall the definition of weak solution. A function $u \in X$ is said to be a weak solution of problem (1.1)-(1.2) if

$$\int_I p(x) \, u''(x) \, \varphi''(x) \, dx + \int_I q(x) \, u'(x) \, \varphi'(x) \, dx = \lambda \int_I r(x) \, u(x) \, \varphi(x) \, dx \tag{3.1}$$

for any $\phi \in X$.

For the regularity of weak solution, we have the following result.

Theorem 3.1. *Any weak solution* $u \in X$ *of problem* (1.1)-(1.2) *is also a classical solution of this problem, i.e.,* $u \in C^2(I)$, $pu'' \in C^2(I)$ *and* $u''(0) = u''(1) = 0$.

The proof of this theorem is similar to that of [12, Proposition 2.1].

Let us introduce the operators $L,\, H : X \to X^*$ by

$$\langle L(u), v \rangle = \int_I pu''v''dx,$$

$$\langle H(u), v \rangle = \int_I ruvdx$$

for all $u,\, v \in X$.

Remark 3.1. The operators $L,\, G$ are linear. Moreover, L continuously invertible and $||L(u)||_{X^*} = ||u||_X$ for any $u \in X$, where $||\cdot||_{X^*}$ is the dual norm associated with $||\cdot||_X$.

We define on X the following two functionals by

$$F(u) = \frac{1}{2}\int_I p\,|u''|^2 dx,$$

$$G(u) = \frac{1}{2}\int_I r|u|^2 dx.$$

Let $\mathcal{M} = \{u \in X \;:\; 2G(u) = 1\}$. It is easy to see that F and G are Gâteaux differentiable with

$$F' = L \;\text{ and }\; G' = H.$$

By Theorem 3.1 problem (1.1)-(1.2) for $\lambda > 0$ can be can be written in the equivalent form

$$F'(u) = \lambda G'(u),\;\; u \in \mathcal{M}, \tag{3.2}$$

or

$$L(u) = \lambda H(u),\;\; u \in \mathcal{M}. \tag{3.3}$$

It is known that (λ, u) solves (3.1) if and only if u is a critical point of F with respect to $\mathcal{M}$. Hence, for the proof the existence of eigenvalues of the problem (3.1), we will apply Theorem 2.1.

Lemma 3.1. $L : X \to X^*$ *is an hemicontinuous, bounded monotonous and coercive operator.*

Proof. It is obvious that the functional $F(u)$ is a convex. Then, by virtue of [22; Ch. 2, § 1.2, Proposition 1.1] it follows that $L : X \to X^*$ is an hemicontinuous and monotonous operator. The boundedness, continuity and coercivity of operator L are obvious. The proof of Lemma 2.1 is complete.

Remark 3.2. By [22; Ch. 2, § 2.2, Theorem 2.6] it follows from [22; Ch. 2, § 2.2, Proposition 2.2] that the operator $L : X \to X^*$ is strictly monotone.

Remark 3.3. It follows from Hölders inequality that

$$(L(u), v) = (L(v), u) = \int_I p(x)\, u''(x)\, v''(x)\, dx$$

$$\leq \left(\int_I p(x)\, u''^2(x)\; dx\right)^{\frac{1}{2}} \left(\int_I p(x)\, v''^2(x)\; dx\right)^{\frac{1}{2}} = ||u||_X ||v||_X.$$

which imply

$$(L(u) - L(v),\ u - v) = \|u\|_X^2 + \|v\|_X^2 - 2\,(Lu, v)$$
$$\geq \|u\|_X^2 + \|v\|_X^2 - 2\,\|u\|_X\,\|v\|_X = (\|u\|_X - \|v\|_X)^2\,. \tag{3.4}$$

Then the monotonicity (strongly monotonicity) of the operator L should also from (3.4).

Let T a mapping acting from X into X^*. T is said to belong to the class $(S+)$, if for any sequence $\{u_n\}_{n=1}^{\infty} \subset X$ with u_n converges weakly to $u \in X$ and $\limsup\limits_{n\to\infty} \langle Tu_n,\ u_n - u\rangle \leq 0$, it follows that u_n converges strongly to u in X. In this we write $T \in (S+)$ (see, for example, [10, 24]).

Lemma 3.2. *(i) The functionals $F, G : X \to \mathbb{R}$ are even, and are of class $\mathbf{C}^1$ on X; (ii) $\mathcal{M}$ is a closed C^1-manifold.*

Proof. (i) It is clear from the definitions that that F and G are even, and are of class $\mathbf{C}^1$ in X. (ii) Since $\mathcal{M} = G^{-1}\left(\frac{1}{2}\right)$, so $\mathcal{M}$ is closed. For any $u \in \mathcal{M}$ we have $G'(u) \neq 0$ (i.e. $G'(u)$ is onto for any $u \in \mathcal{M}$), hence G is a submersion. Then $\mathcal{M}$ is a C^1-manifold. The proof of Lemma 3.1 is complete.

The following lemma is the key to prove the existence of the eigenvalues of problem (1.1)-(1.2).

Lemma 3.3. *(i) $G' : X \to X^*$ is completely continuous; (ii) The functional F satisfies the Palais-Smale condition on $\mathcal{M}$, i.e., if each sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{M}$ such that $A(u_n)$ is bounded and*

$$\gamma_n = F'(u_n) - g_n G'(u_n) \to 0 \ as\ \ n \to \infty, \tag{3.5}$$

where $g_n = \frac{\langle F'(u_n), u_n\rangle}{\langle G'(u_n), u_n\rangle}$, then $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence in X.

Proof. (i) Step 1: Definition of G'. By the Sobolev embedding theorem (see [6]), it follows that $X \hookrightarrow C(\bar{I})$. Then for all $u, v \in X$ we obtain

$$\left| \int\limits_I r(x)u(x)v(x)dx \right| \leq\ |r|_\infty |u|_\infty |v|_\infty \leq \tilde{c}^2 |\,r\,|_\infty ||u||_X\, ||v||_X,$$

where $\tilde{c}$ is the constant of Sobolev's embedding and $|\cdot|_\infty$ is the max norm in $\bar{I}$. Hence G' is well defined.

Step 2: $G' : X \to X^*$ is completely continuous. Let $\{u_n\}_{n=1}^{\infty} \subset X$ be a sequence such that $u_n \rightharpoonup u$ as $n \to \infty$. We have to show that $G'(u_n) \to G'(u)$ in X, i.e.,

$$\sup_{v\in X,\ ||v||\leq 1} \left| \int\limits_I r\,(u_n - u)\,v\,dx \right| \to\ 0,\ \ \text{as } n \to \infty.$$

By $X \hookrightarrow C(\bar{I})$, we have

$$\sup_{v\in X,\ ||v||\leq 1} \left| \int\limits_I r\,(u_n - u)\,dx \right| \leq c\,|\,r\,|_\infty \sup_{\bar{I}} |u_n - u|\ .$$

It's obvious that

$$\sup_{\bar{I}} |u_n - u|\ \to 0 \text{ as } n \to \infty.$$

Consequently, G' is completely continuous in X.

(ii) Let $\{u_n\}_{n=1}^{\infty}$ is bounded in X. Then, without loss of generality, we can assume that u_n converges weakly in X for some function $u \in X$ and $||u_n||_X \to \alpha$ as $n \to \infty$. Consider two possible cases: $\alpha = 0$ and $\alpha \neq 0$.

In the case $\alpha = 0$ it follows that $u_n \to 0$ in X.

Let now $\alpha \neq 0$. Applying γ_n of (3.5) to u, we get

$$\sigma_n = \langle F'(u_n), u\rangle - \langle F'(u_n), u_n\rangle \langle G'(u_n), u\rangle \to 0 \text{ as } n \to \infty. \tag{3.6}$$

By (3.6) we have

$$\langle F'(u_n), u_n - u\rangle = \langle F'(u_n), u_n\rangle (1 - \langle G'(u_n), u\rangle) - \sigma_n,$$

which implies that

$$\limsup_{n\to\infty} \langle F'(u_n), u_n - u\rangle \leq c_0\alpha^2 \limsup_{n\to\infty} \langle G'(u_n), u\rangle \tag{3.7}$$

where $c_0 = \max\{ |p|_\infty, |q|_\infty\}$. Since $2G(u_n) = 1$ for all $n \in \mathbb{N}$, it follows that $2G(u) = 1$. Hence, by statement (i), we have $\langle G'(u), u\rangle = 1$. This yields that

$$1 - \langle G'(u_n), u\rangle = \langle B'(u), u\rangle - \langle G'(u_n), u\rangle \leq ||G'(u) - G'(u_n)||_{X^*}||u||_X. \tag{3.8}$$

By statement (i) it follows from (3.7) and (3.8) that

$$\limsup_{n\to\infty} \langle F'(u_n), u_n - u\rangle \leq 0. \tag{3.9}$$

We show that $F' \in (S+)$, which completes proof of the statement (ii). By (3.4) we have

$$\langle F'(u_n) - \langle F'(u), u_n - u\rangle \geq (||u_n||_X - ||u||_X)^2 \geq 0. \tag{3.10}$$

Since $u_n \rightharpoonup u$ as $n \to \infty$ it follows from (3.9) and (3.10) that

$$\lim_{n\to\infty} \langle F'(u_n) - F'(u), u_n - u\rangle = 0,$$

which implies that $||u_n||_X \to ||u||_X$ as $n \to \infty$. Therefore, since space X is uniformly convex, $u_n \rightharpoonup u$ implies that $u_n \to u$ as $n \to \infty$ in X. The proof of lemma is complete.

As above, we denote

$$\Gamma_n = \{K \subset \mathcal{M} : K \text{ is symmetric, compact and } \gamma(K) \geq n\}.$$

Lemma 3.4. *For all $k \in \mathbb{N}$ we have $\Gamma_k \neq \emptyset$.*

Proof. Since X is separable, there exist system $\{y_j\}_{j=1}^{\infty} \subset X$ linearly dense in X such that $\operatorname{supp} y_i \cap \operatorname{supp} y_j = \emptyset$ if $i \neq j$. By **meas**$\{ I_r^+\} > 0$ we can assume that $y_j \in \mathcal{M}$, i.e. $\int_I r(x)|y_j(x)|^2dx = 1$ for any $j \in \mathbb{N}$.

For $k \in \mathbb{N}$ we define $\Phi_k = \operatorname{span}\{y_1, y_2, \ldots, y_k\}$. It is clear that Φ_k is a vectorial subspace of X and $\dim \Phi_k = k$. Then for any $u \in \Phi_k$ there exists $(\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{R}^k$ such that $u = \sum_{i=1}^{k} \beta_i y_i$. Thus

$$F(u) = \sum_{i=1}^{k} |\beta_i|^2 F(y_i) = \frac{1}{2}\sum_{i=1}^{k} |\beta_i|^2.$$

It follows that the following map

$$u \mapsto (2F(u))^{1/2} = ||u||_{\Phi_K}$$

defines a norm on space Φ_k. Consequently, there exists a positive constant d such that

$$d||u||_X \leq (2F(u))^{1/2} \leq \frac{1}{d}||u||_X.$$

Let $S^{k-1} = \Phi_k \cap \{ u \in X : (2F(u))^{1/2} = 1\}$. The set S^{k-1} is the unit sphere of Φ_k. Hence S^{k-1} is closed, compact and symmetric. Then by [24, Prop. 2.3], $\Gamma(S_{k-1}) = k$ which implies that $S_{k-1} \in \Gamma_k$. Hence $\Gamma_k \neq \emptyset$.

Now we have our main result formulated as follows.

Theorem 3.2. *For any integer $k \in \mathbb{N}$ we have*

$$\lambda_k^+ = \inf_{K\subset \Gamma_k} \max_{u\in K} 2F(u)$$

is a critical value of F restricted on $\mathcal{M}$. More precisely, there exists $u_k^+ \in K_k \subset \Gamma_k$ such that

$$\lambda_k^+ = 2F(u_k) = \sup_{u\in K_k} 2F(u)$$

and (λ_k^+, u_k^+) is a solution of problem (1.1)-(1.2) *associated with the positive eigenvalue λ_k^+. Moreover, $\lambda_k^+ \to +\infty$ as $k \to \infty$.*

Proof. By Lemmas 3.1-3.4 and Theorem 2.1 we need only to prove that $\lambda_k^+ \to +\infty$ as $k \to \infty$. Let the system $\{v_j\}_{j=1}^\infty \subset X^*$ is conjugate to the system $\{y_j\}_{j=1}^\infty \subset X$, i.e. $\langle y_i, v_j\rangle = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. The v_j, $j \in \mathbb{N}$, are total for X^*.

For $k \in \mathbb{N}$, we denote $\Phi_k^\perp = \mathrm{span}\,\{y_{k+1}, y_{k+2}, ...\}$. It follows by [24; Proposition 2.3 (g)] that for any $A \in \Gamma_k$ we have $A \cap \Phi_{k-1}^\perp \neq \emptyset$. Then

$$t_k = \inf_{A\in\Gamma_k} \sup_{u\in A\cap \Phi_{k-1}^\perp} 2F(u) \to +\infty.$$

Indeed, otherwise, for sufficiently large k there exists $u_k \in \Phi_{k-1}^\perp$ with $||u_k||_{L_2} = 1$, where $||\cdot||_2$ denotes the $L_2(I)$-norm, and positive constant $\tilde{K}$ such that

$$t_k \leq 2F(y_k) \leq \tilde{K}.$$

Thus $||u_k||_X \leq \tilde{K}$, i.e. $\{u_k\}_{k=1}^\infty$ is bounded in X. For a subsequence still denoted by $\{u_k\}_{k=1}^\infty$, we have $u_k \rightharpoonup u$ in X and $u_k \to u$ in $L^2(I)$. By virtue of our choice of $\Phi_{k-1}^\perp$ we have $u_k \rightharpoonup 0$ in X. Hence $\langle y_k, v_j\rangle = 0$ for any $k \geq j$. This contradicts the assumption $||u_k||_{L_2} = 1$ for any sufficiently large $k \in \mathbb{N}$. Then it follows by inequality $\lambda_k \geq t_k$ that $\lambda_k^+ \to +\infty$ as $k \to \infty$. The proof of theorem is complete.

Remark 3.4. It is a trivial fact that

$$\lambda_1^+ = \inf \left\{ \int_I pu''^2 dx \,:\, u \in X \text{ and } \int_I ru^2 dx = 1 \right\}. \tag{3.11}$$

Corollary 3.1. *The following relation holds:*

$$0 < \lambda_1 \leq \lambda_2 \leq \; ... \; \lambda_k \mapsto +\infty.$$

Proof. For all $i \leq j$ $(i, j \in \mathbb{N})$ we have $\Gamma_i \supset \Gamma_j$. Hence from the definition of eigenvalues we get $\lambda_i \leq \lambda_j$. This completes the proof.

Corollary 3.2. *The problem* (1.1)-(1.2) *has a decreasing sequence of the negative eigenvalues $\{\lambda_k^-\}_{k=1}^\infty$, such that $\lim\limits_{k\to\infty} \lambda_k^- = -\infty$.*

Proof. It is clear that $I_r^- = -I_{-r}^+$. Then we have $\mathrm{meas}\{I_{-r}^+\} = \mathrm{meas}\{I_r^-\} > 0$. The problem (1.1)-(1.2) can be rewritten in the following equivalent form

$$\begin{gathered}(\tau(x)u''(x))'' = \hat{\lambda}\,\hat{r}(x)u(x),\ x \in (0,1),\\ u(0) = u(1) = u''(0) = u''(1) = 0,\end{gathered} \tag{3.12}$$

where $\hat{\lambda} = -\lambda$ and $\hat{r}(x) = -r(x)$, $x \in \bar{I}$. By Theorem 3.2 the problem (3.12) has an increasing sequence of the positive eigenvalues $\lambda_{k,*}^+$ such that $\lambda_{k,*}^+ \to +\infty$. If we put $\lambda_k^- = -\lambda_{k,*}^+$ then we obtain that $\{\lambda_k^-\}_{k=1}^\infty$ is decreasing sequence of the negative eigenvalues of problem (1.1)-(1.2) such that $\lim\limits_{k\to\infty} \lambda_k^- = -\infty$. The proof of corollary is complete.

4 Properties of principal eigenvalues of problem (1.1)-(1.2)

Now we investigate the existence of principal eigenvalues, i.e., eigenvalues corresponding to eigenfunctions which does not vanish in the interval I.

Theorem 4.1. *The eigenvalue λ_1^+ is simple and the corresponding eigenfunction u_1^+ do not vanish in the interval I.*

Proof. Let $J_\lambda(u) = F(u) - \lambda G(u)$. It is obvious that u_1^+ is an eigenfunction associated to λ_1^+ if and only if

$$J_{\lambda_1^+}(u_1) = F(u_1) - \lambda_1^+ G(u_1) = 0 = \inf_{u\in X\setminus\{0\}}\{F(u) - \lambda_1^+ G(u)\}.$$

Since $J_{\lambda_1^+}(u_1) = J_{\lambda_1^+}(|u_1|)$ it follows that all eigenfunctions corresponding to λ_1^+ do not change sign in I.

Let $r_1(x)$ is continuous function on $\bar{I}$ such that $r(x) + r_1(x) > 0$ for all $x \in \bar{I}$. Then (λ_1^+, u_1^+) is a solution of following eigenvalue problem

$$\begin{gathered}(\tau(x)u''(x))'' + h(x)u(x) = \lambda\tau(x)u(x),\ x \in (0,1),\\ u(0) = u(1) = u''(0) = u''(1) = 0,\end{gathered} \tag{4.1}$$

where $h(x) = \lambda_1^+ r_1(x)$ and $\tau(x) = r(x) + r_1(x)$, $x \in \bar{I}$. Note that $h(x), \tau(x) \in C[0,1]$ and $\tau(x) > 0$, $x \in \bar{I}$. By [4; Theorem 1] (see also [3]) the eigenvalues of the boundary value problem (4.1) are real, simple and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^\infty$, moreover the eigenfunction $v_k(x)$ corresponding to the eigenvalue μ_k has $k-1$ simple zeros in the interval I. Since $u_1^+(x)$ do not change sign in I it follows that $\lambda_1^+ = \mu_1$ and $u_1^+(x) = Cv_1(x)$ where $C = \mathrm{const} \neq 0$. Thus all eigenfunctions corresponding to λ_1^+ are positive or negative in I.

Let u and v be eigenfunctions corresponding to the eigenvalue λ_1^+. Then functions $M(t,x) = \max\{u(x), t\,v(x)\}$ and $M(t,x) = \min\{u(x), t\,v(x)\}$ belong to X and satisfy the following relation

$$J_{\lambda_1^+}(M(t,\cdot)) + J_{\lambda_1^+}(m(t,\cdot)) = J_{\lambda_1^+}(u) + J_{\lambda_1^+}(tv) = 0.$$

(see [23]). Consequently, by virtue of (3.11) we have

$$J_{\lambda_1^+}(M(t,\cdot)) = J_{\lambda_1^+}(m(t,\cdot)) = 0.$$

Thus $M(t,x)$, $x \in \bar{I}$, be the solution of problem (1.1)-(1.2) and by weight Sobolev embedding $X \hookrightarrow C^{1,\alpha}$ we have $M(t,\cdot) \in C^{1,\alpha}$ where $\alpha \in (0, \frac{1}{2})$. For any $x_0 \in (0,1)$ we put $t_0 = \frac{u(x_0)}{v(x_0)} > 0$. Since $M(t_0, x_0) = u(x_0) = t_0 v(x_0)$ it follows that

$$u(x_0 + \Delta) - u(x_0) \leq M(t_0, x_0 + \Delta) - M(t_0, x_0).$$

for all sufficiently small number Δ. Dividing this inequality by $\Delta > 0$ and $\Delta < 0$ and letting Δ tend to ± 0 we obtain $u'(x_0) = M'_x(t_0, x_0)$. Similarly we can find that $M'_x(t_0, x_0) = t_0 v'(x_0)$. It follows from the last two equalities that $\left(\frac{u}{v}\right)'(x_0) = 0$ which implies that $\frac{u(x)}{v(x)} \equiv const$ in I. Thus the eigenvalue λ_1^+ is simple. The proof of this lemma is complete.

Corollary 4.1. *The eigenvalue λ_1^- is simple and the corresponding eigenfunction v_1^- do not vanish in the interval I.*

Remark 4.1. λ_1^+ (λ_1^-) is a unique positive (negative) principal eigenvalue of the problem (1.1)-(1.2).

References

1. Adams, R.A.: Sobolev spaces, *Academic Press, New York* (1975).
2. Afrouzi, G.A., Brown, K.J.: *On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions*, Proc. Amer. Math. Soc., **127** (1), 125-130 (1999).
3. Aliyev, Z.S.: *Some global results for nonlinear fourth order eigenvalue problems*, Cent. Eur. J. Math., **12** (12), 1811-1828 (2014).
4. Aliev, Z.S., Agaev, E.A.: *Oscillation properties of the eigenfunctions of fourth-order completely regular Sturmian systems*, Doklady Mathematics, **90** (3), 657-659 (2014).
5. Anane, A., Chakrone, O., Monssa, M.: *Spectrum of one dimensional p-Laplacian with indefinite weight*, Electr. J. Qualitative Theory Differential Equations, **2002** (17), 11 p. (2002).
6. Banks, D.O., Kurowski, G.J.: *A Prufer transformation for the equation of the vibrating beam*, Trans. Amer. Math. Soc., **199**, 203-222 (1974).
7. Benedikt, J.: *On the discreteness of the spectra of the Dirichlet and Neumann p-biharmonic problem*, Abstr. Appl. Anal., **9**, 777-792 (2004).
8. Benedikt, J.: *On simplicity of spectra of p-biharmonic equations*, Nonlinear Anal. TMA., **58** (7-8), 835-853 (2004).
9. Drabek, P., Otani, M.: *Global bifurcation result for the p-biharmonic operator*, Electron. J. Diff. Eqns., **2001** (48), 1-19 (2001).
10. El Khalil, A., Kellati, S., Touzani, A.: On the spectrum of the p-biharmonic operator, Electronic J. Diff. Eqns., **2002**, 2002-Fez, 170191, *Conference on Partial Differential Equation*.
11. El Khalil, A., Kellati, S., Touzani, A.: *On the principal frequency curve of the p-biharmonic operator*, Arab Journal of Mathematical Sciences, **17**, 89-99 (2011).
12. Dai, G.: Spectrum of Navier p-biharmonic problem with sign-changing weight, arXiv:1207.7159v1 [math.CA].
13. Drabek, P.: *On the Global Bifurcation for a class of degenerate equations*, Annali di Matematica pura ed applicata, **159** (1), 1-16 (1991).
14. Hess, P., Kato, T.: *On some linear and nonlinear eigenvalue problems with indefinite weight function*, Comm. Partial Differential Equations, **5**, 999-1030 (1980).
15. Huang, Y.X.: *Eigenvalues of the p-Laplacian in R^N with indefinite weight*, Comment. Math.Univ.Carolin. **36** (3), 519-527 (1995).

16. Idogawa, T., Otani, M.: *The first eigenvalues of some abstact elliptic operator*, Funkcialaj Ekvacioj, **38**, 1-9 (1995).
17. Kim, I.S.: *Nonlinear eigenvalue problems for the p-Laplasian in weighted Sobolev space*, Proceedings of the 3-th Asian Conference on Nonlinear Analysis and Optimization, Matsue, Japan, 133-149 (2012).
18. Kratochvil, A., Necas, J.: *On discreteness of spectrum of nonlinear SturmLiouville problem of the fourth order*, Comment. Math. Univ. Carolinae, **12**(4), 639-653 (1971) (in Russian).
19. Janczewsky, S.: *Oscillation theorems for the differential boundary value problems of the fourth order*, Ann. Math., **29** (2), 521-542 (1928).
20. Le, A.: *Eigenvalue problems for the p-Laplacian*, Nonlinear Analysis, **64**, 1057-1099 (2006).
21. Le, A., Schmitt, K.: *Variational eigenvalues of degenerate eigenvalue problems for the weighted p-Laplacian*, Adv. Nonlinear Stud., **5** 573-585 (2005).
22. Lions, J.L.: Some Methods of Solving Non-Linear Boundary Value Problems, Dunod-Gauthier-Villars, Paris (1969).
23. Otani, M., Teshima, T.: *On the ifrst eigenvalue of some quasilinear elliptic equations*, Proc. Japan Acad. Ser. A, **64**, 8-10 (1988).
24. Szulkin, A.: *Ljusternik-Schnirelmann theory on **C1**-manifolds*, Ann. I. H. Poincare, Anal. non lineaire, **5**, 119-139 (1988).