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Discreteness spectrum and asymptotic distribution of eigenvalues of operator differential equations of higher order on semi-axis

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Abstract. In the present paper asymptotic distribution of eigenvalues of operator-differential equations of higher order on a semi-axis is studied. The uniform asymptotics of the Green function for large values of spectral parameter is obtained. Using the asymptotics of the Green function, the discretness of the spectrum is proved. Using the Keldysh – Tauberian theorem, an asymptotic formula is obtained for the distribution function of eigenvalues of the considered operator.

Keywords. operator-differential equation, spectr, eigenvalue, Green function, Hilbert space

Mathematics Subject Classification (2010): 35J25, 47A10, 58J40

1 Introduction

Let H be a seperable Hilbert space. In the space $H = L_2 [H; 0 \leq x < \infty]$ consider the differential expression

$$l(y) = (-1)^n \left(p(x) y^{(n)}\right)^{(n)} + \sum_{j=2}^{2n} Q_j(x) y^{(2n-j)}, \quad 0 \leq x < \infty \tag{1.1}$$

with boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0. \tag{1.2}$$

Here $0 \leq l, \leq l_2 \leq ... \leq l_n \leq 2n-1, y \in H_1$, and the derivatives are understood in the strong sense. Everywhere by $Q(x)$ we'll denote $Q_{2n}(x)$.

Let D' be a totality of all the functions of the form $\sum_{k=1}^{p} \varphi_k(x) f_{k'}$ where $\varphi_k(x)$ are finite, $2n$-times continuously differentiable scalar functions and $f_k \in D(Q)$. Determine the operator L' generated by expresson (1.1) and boundary conditions (1.2) with domain

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of definition D'. Subject to certain conditious, the operator L' is a positive and symmetric operator in H_1. We'll assume that the closure L of the operator L' is a self-adjoint and lower bounded operator in H_1.

At some assimptions relative to operator coefficients $P(x), Q(x), Q_j(x), j=\overline{2,2n-1}$ that we will show below the operator L generated by expression (1.1) and boundary conditions (1.2) has a pure discrete spectrum.

Let λ_1, λ_2,...,λ_n, be eigenvalues and $f_1(x)$, $f_2(x)$, ..., $f_n(x)$, be corresponding orthonormal eigenfunctions of the operator L. By $N(\lambda)$ we'll denote the number of eigenvalues of the operator L, smaller than the given λ, i.e.,

$$N(\lambda)=\sum_{\lambda_n<\lambda} 1$$

and is called the distribution function of eigenvalues of the given operator. Our principal task is studying the asymptotic behavior of the function $N(\lambda)$ as $\lambda\to\infty$.

Note, that the asymptotics of the function $N(\lambda)$ for the Stourm-Lioville operator equation first was obtained by A.G.Kostyuchenko and B.M.Levitan [7]. A similar problem for equation (1.1) at $n=1$ was solved by E.Abdukadirov [1], M.Bayramoglu [4] and G.I.Aslanov's [3] papers were devoted to the construction of the Green function for of a higher order equation and investigations of the asymptotics of the function $N(\lambda)$.

In the paper of A.A.Abudov and G.I.Aslanov [2] the asymptotics of the function $N(\lambda)$ was obtained for the operator L, generated by expression (1.1) when intermediate terms in expression (1.1) are absent. A similar problem for expression (1.1) whis boundary conditions $y(0)=y'(0)=...=y^{(n-1)}(0)=0$ was solved by G.I.Kasumova [5].

We'll assume that the coefficients of expression (1.1) satisfy the following conditions:

1. The operator function $p(x)$ is n- times differentiable everywhere in $(0,\infty)$ and for all $h\in H$

$$m(h,h)_H\le(p(x)h,h)_H\le M(h,h)_H\,,\ m,M>0.$$

2. The operators $Q(x)$ for almost all x are self-adjoint in H, moreover in H there exists a general for all x everywhere compact in H the set $D\{Q(x)\}=D(Q)$ on which $Q(x)$ are determined, self-adjoint and uniformly low bounded, i.e. there exists $d>0$ such that for x and for all $f\in D(Q)$

$$(Q(x)f,f)_H>d(f,f)\,.$$

3. There exists a constant number $C>0$, $0<a<\dfrac{2n+1}{2n}$ such that for all x and $|x-\xi|\le 1$ the inequality

$$\left\|[Q(\xi)-Q(x)]\,Q^{-a}(x)\right\|_H<C|x-\xi|$$

is true.

4. For $|x-\xi|>1$ $\left\|K(\xi)\exp\left(-\dfrac{Jm\varepsilon_1}{2}|x-\xi|\,\omega\right)\right\|_H<C,$

$$K(x)=P^{-\frac{1}{2}}(x)\,Q(x)\,P^{-\frac{1}{2}}(x)\,,\ \ \omega=\left\{K(x)+\mu P^{-1}(x)\right\}^{\frac{1}{2n}},\quad \mu>0.$$

$$Jm\varepsilon_1=\min_i\left\{Jm\varepsilon_i>0,\ \varepsilon_i^{2n}=-1\right\}.$$

5. For all $x \in (0, \infty)$

$$\left\| Q(x), P^{\pm\frac{1}{2}}(x), Q^{-1}(x) \right\| < C, \quad \left\| Q(\xi) P^{-\frac{1}{2}}(x) P^{\frac{1}{2}}(\xi) Q^{-1}(x) \right\| < C.$$

6. For all $x \in [0, \infty)$

$$\left\| Q_j(x) Q^{\frac{i-j}{2n}+\varepsilon}(x) \right\|_H < C \ , j = 3, 4, ..., 2n-1, \ \ \varepsilon > 0$$

7. Let's assume that $Q(x)$ for almost all x is inverse to a completely continuous operator. Then the operator $K(x)$ also will be inverse to a completely continuous operator. Let's denote by $\alpha_1(x), \alpha_2(x), ..., \alpha_n(x), ...$ the eigenvalues of the operator $K(x)$. We'll assume that the functions $\alpha_i(x)$ are measurable, and $\alpha_1(x) \geq 1$. Further, let's assume that almost for all $x \in (0, \infty)$ the series

$$\sum_{i=1}^{\infty} \alpha_i^{\frac{1-4n}{2n}}(x)$$

converges and its sum $F(x) \in L_1[0, \infty)$.

2 Green function and discreteness spectrum of the problem (1.1)-(1.2)

For studying the asymptotic behavior of the function $N(\lambda)$ as $\lambda \to \infty$, at first we study some properties of the Green function of the operator L. Construct the Green function of the operator L_0 generated by the differential expression

$$l(y) = (-1)^n \left(P(x) y^{(n)} \right)^{(n)} + Q(x) y + \mu y \tag{2.1}$$

and the boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0, \ \ 0 \leq l, \leq l_2 \leq ... \leq l_n \leq 2n-1.$$

As is known [5], the Green function $G_0(x, \eta, \mu)$ of the operator L_0 satisfies the following integral equation:

$$G_0(x, \eta, \mu) = G_1(x, \eta, \mu) - \int_0^{\infty} G_1(x, \xi, \mu) [Q(x) - Q(\xi)] G_0(\xi, \eta, \mu) +$$

$$+ \frac{1}{2ni} \int_0^{\infty} P^{-\frac{1}{2}}(x) \sum_{k=1}^{n} \varepsilon_k \exp(i\epsilon_k |x - \xi| \omega) P^{-\frac{1}{2}}(x) [p(\xi) - p(x)] G_0(\xi, \eta, \mu) d\xi +$$

$$+ (-1)^n \sum_{m=1}^{n} C_n^m \int_0^{\infty} G_{1\xi}^{(2n-m)}(x, \xi, \mu) P_{\xi}^{(m)}(\xi) G_0(\xi, \eta, \mu) d\xi, \tag{2.2}$$

where $G_1(x, \eta, \mu)$ is the Green function of the following problem

$$\begin{cases} (-1)^n \left(p(x) y^{(n)} \right)^{(n)} + Q(\xi) y + \mu y = \delta(x - \xi) \\ y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0. \end{cases} \tag{2.3}$$

Here "ξ" is a fixed point. The function $G_1(x,\eta,\mu)$ is determined in the following form

$$G_1(x,\eta,\mu)=\frac{1}{2\pi i}P^{-\frac{1}{2}}(x)\,\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k\exp\left(i\varepsilon_k\left|x-\eta\right|\omega\right)P^{-\frac{1}{2}}(x)+$$

$$+\frac{1}{2\pi i}P^{-\frac{1}{2}}(x)\,\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k\exp\left(i\varepsilon_k\left(x+\eta\right)\omega\right)P^{-\frac{1}{2}}(x). \tag{2.4}$$

From formula (2.4) it follows that

$$G_1(x,\eta,\mu)=\frac{1}{2\pi i}P^{-\frac{1}{2}}(x)\,\omega^{1-2n}\sum_{k=1}^{n}\varepsilon_k\exp\left(i\varepsilon_k\left|x-\eta\right|\omega\right)P^{-\frac{1}{2}}(x)\left[E+r(x,\eta,\mu)\right] \tag{2.5}$$

and as $\mu\to\infty$ we have $\|r(x,\eta,\mu)\|_H=o(1)$ uniformly with respect to (x,η).

As it noted above, the Green function $G_0(x,\eta,\mu)$ of the operator L_0 satisfies the inteqral equation (2.2). For investigating the solution of the integral equation (2.2), following the paper [8] we introdise the Banach spaces $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_4^{(s)}$ and X_5 $(p\geq 1, s\geq 0)$ whose elements are the operator functions $A(x,\eta)$ in the space H, and the norms are determined in the following way:

$$\|A(,\eta)\|_{X_1}^2=\int_0^\infty\left\{\int_0^\infty\|A(x,\eta)\|_H^2\,d\eta\right\}dx,$$

$$\|A(,\eta)\|_{X_2}^2=\int_0^\infty\left\{\int_0^\infty\|A(x,\eta)\|_2^2\,d\eta\right\}dx.$$

(Here $\|A(x,\eta)\|_2$ denote the Hilbert-Schmit norm (absolute norm) of the operator function $A(x,\eta)$ in H).

$$\|A(x,\eta)\|_{X_3^{(p)}}=\left[\sup_{0\leq x<\infty}\int_0^\infty\|A(x,p)\|_H^p\,d\eta\right]^{\frac{1}{p}},$$

$$\|A(x,\eta)\|_{X_2^{(s)}}=\int_0^\infty dx\left\{\int_0^\infty\|A(x,\eta)\,Q^s(\eta)\|_2^2\,d\eta\right\},$$

$$\|A(x,\eta)\|_{X_4^{(s)}}=\sup_{0\leq x<\infty}\int_0^\infty\|A(x,\eta)\,Q^s(\eta)\|_H\,d\eta,$$

$$\|A(x,\eta)\|_{X_5}=\sup_{0\leq x<\infty}\ \sup_{0\leq\eta<\infty}\|A(x,\eta)\|_H\,.$$

Determine the following inteqral operator:

$$NA(x,\eta)=\int_0^\infty G_1(x,\xi;\mu)\left[Q(\xi)-Q(x)\right]A(\xi,\eta)\,d\xi-$$

$$-\int_0^\infty \frac{1}{2ni} P^{-\frac{1}{2}}(x)\,\omega^{1-2n} \sum_{k=1}^{n} \varepsilon_k \exp\left(i\varepsilon_k |x-\xi|\,\omega\right) P^{-\frac{1}{2}}(x)\left[p(\xi)-p(x)\right] A(\xi,\eta)\,d\xi+$$

$$+\int_0^\infty (-1)^n \sum_{m=1}^{n} C_n^m G_{1\xi}^{(2n-m)}(x,\xi,\mu)\,P_\xi^{(m)}(\xi)\,A(\xi,\eta)\,d\xi. \tag{2.6}$$

It holds the following important lemma.

Lemma 2.1 *If the operator valued function $Q(x)$ satisfies conditions 1)-7), then for sufficiently large $\mu > 0$ the operator N is contractive in the spaces $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_4^{(s)}, X_5$.*

In all the considered Banach spaces, the equation (2.2) has a unique solution that may be obtained by means of the iterative method if the operator function $G_1(x,\eta,\mu)$ belongs to the appropriate space.

Using conditions 1)-7), we get that the function belongs to the spaces $X_3^{(2)}$ and X_2. Therefore for sufficiently large $\mu > 0$, the function $G_0(x,\eta,\mu)$ is also an element of the spaces $X_3^{(2)}$ and X_2.

It is proved that the solution of inteqral equation (2.2) is the Green function of the operator L_0, i.e. satisfies all the main properties of the Green function.

The Green function $G(x,\eta,\mu)$ of the operator L generated by the expression (1.1) and boundary conditions (1.2) is sought in the form

$$G(x,\eta,\mu) = G_0(x,\eta,\mu) - \int_0^\infty G_0(x,\xi,\mu)\,\rho(\xi,\eta)\,d\xi. \tag{2.7}$$

Using the main properties of the Green function $G_0(x,\eta,\mu)$ of the operator L_0, for determining $\rho(x,\eta)$ we get the following inteqral equation

$$\rho(x,\eta) + \sum_{j=2}^{2n} Q_j(x)\,\frac{\partial^{2n-j} G_0(x,\eta,\mu)}{\partial x^{2n-j}} + \sum_{j=2}^{2n} Q_j(x) \int_0^\infty \frac{\partial^{2n-j} G_0}{\partial x^{2n-j}}\,\rho(\xi,\eta)\,d\eta = 0. \tag{2.8}$$

If we denote $F(x,\eta,\mu) = -\sum\limits_{j=2}^{2n} Q_j(x)\,\dfrac{\partial^{2n-j} G_0}{\partial x^{2n-j}}$ equation (2.8) is written in the form:

$$\rho(x,\eta) = F(x,\eta,\mu) - \int_0^\infty F(x,\xi,\mu)\,\rho(\xi,\eta)\,d\xi. \tag{2.9}$$

Using the asymptotic functions $G_1(x,\eta,\mu)$ and $G_0(x,\eta,\mu)$, we can get the following estimation for the norm of the operator-function $F(x,\eta,\mu)$

$$\|F(x,\eta,\mu)\|_H = c\mu^{-\epsilon} e^{-Jm\omega_1 \sqrt[2n]{\mu}|x-\eta|}.$$

Hence

$$\sup_{0\le x<\infty} \int_0^\infty \|F(x,\eta,\mu)\|_H^2\,d\eta \le c\mu^{-2\varepsilon}.$$

From this estimation it follows that the function $F(x,\eta,\mu)$ is an element of the space $X_3^{(2)}$ and as $\mu \to \infty$ it converges to zero with respect to the norm of the space $X_3^{(2)}$. Hence it follows that the solution of equation (2.9) as $\mu \to \infty$ asymptotically behaves as the function $F(x,\eta,\mu)$. As a result, from the integral relation (2.7) we get the following asymptotic equality

$$G(x,\eta,\mu) = G_0(x,\eta,\mu)\left[E + \alpha(x,\eta,\mu)\right] \qquad (2.10),$$

where $\|\alpha(x,\eta,\mu)\|_H = o(1)$ as $\mu \to \infty$.

From equation (2.2) and estimation (2.5), (2.10) we finally get:

$$G(x,\eta,\mu) = g(x,\eta,\mu)\left[E + \beta(x,\eta,\mu)\right], \qquad (2.11)$$

where $\|\beta(x,\eta,\mu)\|_H = o(1)$ as $\mu \to \infty$ and

$$g(x,\eta,\mu) = \frac{1}{2ni} P^{-\frac{1}{2}}(x)\,\omega^{1-2n} \sum_{k=1}^{n} \varepsilon_k \exp\left(i\varepsilon_k |x-\xi|\,\omega\right) P^{-\frac{1}{2}}(x). \qquad (2.12)$$

Above we showed that for the function $g(x,\eta,\mu)$ the following estimation is fulfilled

$$\int_0^\infty \left\{ \int_0^\infty \|G(x,\eta,\mu)\|_2^2 \, d\eta \right\} dx < \infty.$$

Hence it follows that the integral operator with the Kernel $G(x,\eta,\mu)$ is an operator of Hilbert-Schmidt type. Since the function $G(x,\eta,\mu)$ is a kernel of the operator $R_\lambda = (L+\mu E)^{-1}$, we get that the operator L has a discrete spectrum $\lambda_1, \lambda_2, ..., \lambda_n, ...$ with a unique limit point at infinity.

3 Asymptotic distribution of eigenvalues of the problem (1.1)-(1.2)

The following theorem holds.

Theorem 3.1 *If conditions 1)-7) are fulfilled and the series* $\sum_{k=1}^{\infty} \int_0^\infty \int_0^\infty \frac{dxds}{(\beta_k(x,s)+\mu)^2}$ *converges, then as* $\mu \to \infty$ *the following relation holds*

$$\sum_{n=1}^{\infty} \frac{1}{(\lambda_n+\mu)^2} \sim \frac{1}{2\pi} \sum_{k=1}^{\infty} \int_0^\infty \int_0^\infty \frac{dxds}{(\beta_k(x,s)+\mu)^2}. \qquad (3.1)$$

Here $\beta_k(x,s)$ *are the eigenvales of the operator* $P(x)\,s^{2n} + Q(x)$ *in space* H.

Particularly from this theorem we obtain the convergences of the series

$$\sum_{n=1}^{\infty} \frac{1}{(\lambda_n+\mu)^2}.$$

Hence, it follows that as $\mu \to \infty$

$$\int_0^\infty \frac{dN(\lambda)}{(\lambda+\mu)^2} = \sum_{n=1}^{\infty} \frac{1}{(\lambda_n+\mu)^2} \sim \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^\infty \int_0^\infty \frac{dxds}{(\beta_n(x,s)+\mu)^2}. \qquad (3.2)$$

In order to get the asymptotic formula for the function (λ) by means of Titchmarsh's Tauberian theorem [9] from formula (3.2) we must put on the function $\beta_n(x,s)$ the following limitation:

8. there exist such positive constants c_1 and c_2 that

$$\frac{c_1}{t^2}\sum_{n=1}^{\infty}\iint\limits_{\beta_n(x,s)\le t} dxds \le \sum_{n=1}^{\infty}\iint\limits_{\beta_n(x,s)\ge t} \beta^{-2}(x,s)\,dxds \le \frac{c_2}{t^2}\sum_{n=1}^{\infty}\iint\limits_{\beta_n(x,s)\le t} dxds.$$

Assuming that this condition is fulfilled and applying the mentioned above Titchmarsh's-Tauberian theorem from formula (3.2) we fund the following asymptotic formula for $N(\lambda)$:

$$N(\lambda) \sim \frac{1}{2\pi}\sum_{n=1}^{\infty}\iint\limits_{\beta_n(x,s)<\lambda} dxds \text{ as } \lambda\to\infty. \tag{3.3}$$

Thus we get

Theorem 3.2 *Let conditions 1)-7) and tauberian condition 8) be fulfilled. Then for the function $N_s(\lambda)$ the following asymptotic formula (3.3) holds.*

References

1. Abdukadirov, E.: *On Green function of Sturm-Lioville equation with operator coefficients*. DAN SSSR, **195** (3), 519–522 (1970) (in Russian).
2. Abudov, A.A., Aslanov, G.I.: *Distribution of eigenvalues of operator-differential equations of order* $2n$. Izves. AN Azerb. SSR, ser. Fiz.techn. i matem. Nauk, (1), 9–14 (1980) (in Russian).
3. Aslanov, G.I.: *Asimptotics of number of eigenvalues of ordinary differential equations with operator coefficients*. DAN Azerb. SSR, **3** (3), 3-7 (1976) (in Russian).
4. Bayramoglu, M.: *Asymptotics of number of eigenvalues of ordinary differential equations with operator coefficients*. Func. Anal. I ego prilozh., Baku, "Elm", 122–149 (1971) (in Russian).
5. Kasumova, G.I.: *Asymptotic distribtion of eigenvalues of operator differential equations of higher order on the whole axis*. Proceedings of IMM of NAS of Azerbaijan, Baku, "Elm", **27**, 19–24 (2007).
6. Keldysh, M.V.: *On one tauberian theorem*. Trudi Matem. in-ta im. Steklova, **38**, 77–86 (1951).
7. Kostyuchenko, A.G., Levitan B.M.: *On asymptotic behavior of eigenvalues of the Sturm-Lioville operator problem*. Func. Anal. I ego prilozh., **1**, (1), 86–96 (1967) (in Russian).
8. Levitan, B.M.: *Investigation of Green function of Sturm-Lioville equation with operator coefficient*. Matem. sb., **76** (118), 239–270 (1968).
9. Titchmarsh, E.: Expansion in eigenfunctions of differential equation. II. *M.Izd-vo inost. Lit.*, (1961) (in Russian).