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An inverse scattering problem for system of Dirac equations on the whole axis with conditions of discontinuity at some point

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Abstract. *The Dirac operator on the whole axis with conditions of discontinuity at some point is considered. Basic equations of the inverse problem are introduced. The uniqueness theorem of the solution of the inverse problem is proved.*

Keywords. inverse scattering problems, system of Dirac equations, discontinuity conditions, scattering function, Jost solution

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1 Introduction

Let us consider a system of Dirac equations

$$By' + \Omega(x)y = \lambda y, \ -\infty < x < +\infty \tag{1.1}$$

with conditions of discontinuity at some point $a \in (-\infty, +\infty)$

$$y(a-0) = My(a+0). \tag{1.2}$$

Here the following notations are used

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ M = \begin{pmatrix} \alpha & 0 \\ 2\beta & \alpha^{-1} \end{pmatrix}, \tag{1.3}$$

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where α and β are real numbers, $\alpha \neq 0$, $p(x)$, $q(x)$ are real-valued functions, satisfying the condition

$$\int_{-\infty}^{\infty} \|\Omega(x)\| \, dx < \infty, \tag{1.4}$$

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where $\|\cdot\|$ is an operator norm the Euclidean space C^2.

The aim of this work is to study the direct and inverse scattering problems for (1.1) with conditions of discontinuity (1.2). In the absence of the conditions of discontinuity, that is when $M = I_2$ is the unit matrix $(\alpha = 1, \beta = 0)$ in condition (1.2), some aspects of the direct and inverse problems for the system of Dirac equations were studied in [3-8], and with conditions of discontinuity in papers [2,9,10].

2 The Jost type solutions

Matrix functions $E^{\pm}(x,\lambda)$, satisfying the equation (1.1), the condition (1.2) and the condition at infinity

$$\lim_{x\to\pm\infty} E^{\pm}(x,\lambda)e^{\lambda Bx} = I_2 \tag{2.1}$$

we will call the Jost type matrix solutions.

Here and in the hereafter, the numbering of the formulas where there is $\pm$ or $\mp$ actually refers to two different formulas, one of which determined by the upper signs only and the other lower signs only.

It is easy to show that if $\Omega(x) \equiv 0$, then the matrix functions

$$E_0^{\pm}(x,\lambda) = \begin{cases} e^{-\lambda Bx} & \pm\, x > \pm a, \\ (M^{\pm 1})^{(-)}e^{-\lambda Bx} + (M^{\pm 1})^{(+)}e^{-\lambda B(2a-x)}, & \pm x < \pm a, \end{cases}$$

where $D^{(\pm)} = \frac{1}{2}\left(D \pm BDB\right)$ are the Jost type matrix solutions.

Theorem 2.1 *Under the condition (1.4) the equation (1.1) with the conditions of discontinuity (1.2) has the matrix solutions $E^{\pm}(x,\lambda)$ representable in the form*

$$E^{\pm}(x,\lambda) = E_0^{\pm}(x,\lambda) \pm \int_x^{\pm\infty} K^{\pm}(x,t)e^{-\lambda Bt}dt, \ \textit{for all real } \lambda \tag{2.2}$$

and kernels $K^{\pm}(x,t)$ satisfy the inequalities

$$\pm\int_x^{\pm\infty} \left\|K^{\pm}(x,t)\right\| dt \le e^{C\sigma^{\pm}(x)} - 1,$$

where C is some positive constant, and

$$\sigma^{\pm}(x) = \pm\int_x^{\pm\infty} \|\Omega(t)\|\, dt.$$

In addition, the following equalities

$$\lim_{t\to\pm 0}\int_a^{\pm\infty} \left|BK^{\pm}(x,x+t) - K^{\pm}(x,x+t)B \mp \Omega(x)\right| dx = 0,$$

$$\lim_{t\to\pm 0}\int_{\mp\infty}^{a} \left|BK^{\pm}(x,x+t) - K^{\pm}(x,x+t)B \mp \Omega(x)(M^{\mp 1})^{(-)}\right| dx = 0,$$

hold.

Proof. For matrix solution $E^+(x,\lambda)$, the theorem is proved in the work [10]. For solution $E^-(x,\lambda)$, the proof is similar.

Vector functions $e^\pm(x,\lambda) \stackrel{def}{=} E^\pm(x,\lambda)\binom{1}{\mp i}$ we will call the Jost type solutions of the equation (1.1). It is obvious that

$$e^\pm(x,\lambda) = e_0^\pm(x,\lambda) \pm \int\limits_x^{\pm\infty} K^\pm(x,t)\begin{pmatrix}1\\ \mp i\end{pmatrix} e^{\pm i\lambda t}dt, \tag{2.3}$$

where

$$e_0^\pm(x,\lambda) = E_0^\pm(x,\lambda)\begin{pmatrix}1\\ \mp i\end{pmatrix}$$

$$= \begin{cases} \binom{1}{\mp i}e^{\pm i\lambda t}, & \pm x > \pm a,\\ (M^{\pm 1})^{(-)}\binom{1}{\mp i}e^{\pm i\lambda x} + (M^{\pm 1})^{(+)}\binom{1}{\mp i}e^{\pm i\lambda(2a-x)}, & \pm x < \pm a, \end{cases}$$

3 The direct scattering problem

From the reality of potential $\Omega(x)$ and numbers α, β, it follows that for real λ $\overline{e^+(x,\lambda)}$ and $\overline{e^-(x,\lambda)}$ also are vector solutions of the problem (1.1), (1.2) as $e^+(x,\lambda)$ and $e^-(x,\lambda)$. As Wronskian of two solutions of the problem (1.1), (1.2) does not depend on x then we have

$$W[e^+(x,\lambda),\ \overline{e^+(x,\lambda)}] = e^+(x,\lambda)B\overline{e^+(x,\lambda)} = (1,-i)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 2i,$$

$$W[e^-(x,\lambda),\ \overline{e^-(x,\lambda)}] = -2i. \tag{3.1}$$

Hence, for $\lambda \in \mathbb{R} = (-\infty, +\infty)$ pairs $e^+(x,\lambda)$, $\overline{e^+(x,\lambda)}$ and $e^-(x,\lambda)$, $\overline{e^-(x,\lambda)}$ form two fundamental solutions of the problem (1.1), (1.2). Therefore, for $\lambda \in \mathbb{R}$, the representations

$$e^+(x,\lambda) = b(\lambda)e^-(x,\lambda) + a(\lambda)\overline{e^-(x,\lambda)}, \tag{3.2}$$

$$e^-(x,\lambda) = -\overline{b(\lambda)}e^+(x,\lambda) + a(\lambda)\overline{e^+(x,\lambda)}, \tag{3.3}$$

hold, and also by virtue of (3.1)

$$a(\lambda) = \frac{W[e^+(x,\lambda),\ e^-(x,\lambda)]}{2i}, \quad b(\lambda) = -\frac{W[e^+(x,\lambda),\ \overline{e^-(x,\lambda)}]}{2i}. \tag{3.4}$$

Assuming that

$$u^\pm(x,\lambda) = e^\mp(x,\lambda)\frac{1}{a(\lambda)}, \quad r^+(\lambda) = -\frac{\overline{b(\lambda)}}{a(\lambda)},$$

$$r^-(\lambda) = -\frac{b(\lambda)}{a(\lambda)}, \quad t(\lambda) = \frac{1}{a(\lambda)}, \tag{3.5}$$

equalities (3.2), (3.3) one can rewrite in the form

$$u^\pm(x,\lambda) = r^\pm(\lambda)e^\pm(x,\lambda) + \overline{e^\pm(x,\lambda)}. \tag{3.6}$$

From (3.5), (3.6) by virtue of (2.3) we get the following asymptotic formulae

$$u^\pm(x,\lambda) = r^\mp(\lambda)\begin{pmatrix}1\\ \mp i\end{pmatrix}e^{\pm i\lambda x} + \begin{pmatrix}1\\ \pm i\end{pmatrix}e^{\mp i\lambda x} + 0(1), \ \ x \to \pm\infty,$$

$$u^{\pm}(x,\lambda)= \ t(\lambda)\begin{pmatrix}1\\ \pm i\end{pmatrix}e^{\mp i\lambda x}+0(1),\ \ x\to\mp\infty.$$

Solutions $u^{\pm}(x,\lambda)$ are called eigen-functions of the left $(u^{-}(x,\lambda))$ and the right $(u^{+}(x,\lambda))$ scattering problems, and coefficients $r^{-}(\lambda)$, $r^{+}(\lambda)$ and $t(\lambda)$ are called respectively, the left, the right coefficients of reflection and the coefficient of transmission.

It is easy to show that in case $\Omega(x)\equiv 0$ we have

$$a_0(\lambda)=\frac{W[e^{+}(x,\lambda),\ e^{-}(x,\lambda)]}{2i}=\alpha^{(+)}+i\beta,$$

$$b_0(\lambda)=\frac{W[e_0^{+}(x,\lambda),\ \overline{e_0^{-}(x,\lambda)}]}{2i}=\left(\alpha^{(-)}-i\beta\right)e^{2i\lambda a},\ \ \alpha^{(\pm)}=\frac{1}{2}\left(\alpha\pm\alpha^{-1}\right), \quad (3.7)$$

$$r_0^{+}(\lambda)=-\frac{\overline{b_0(\lambda)}}{a_0(\lambda)}=-\frac{\alpha^{(-)}+i\beta}{\alpha^{(+)}+i\beta}e^{-2i\lambda a},$$

$$r_0^{-}(\lambda)=\frac{\alpha^{(-)}-i\beta}{\alpha^{(+)}+i\beta}e^{2i\lambda a}.$$

Lemma 3.1 *Functions $a(\lambda)$, $b(\lambda)$ definined by formulas (3.4) possess the following properties:*

$$1)\ \ a(\lambda)=a_0(\lambda)+\int\limits_0^{+\infty}\varphi(t)e^{i\lambda t}dt;$$

$$2)\ \ b(\lambda)=b_0(\lambda)+\int\limits_{-\infty}^{+\infty}\psi(t)e^{i\lambda t}dt,$$

where $\varphi(t)\in L_1(0,+\infty)$, $\psi(t)\in L_1(-\infty,+\infty)$;

$$3)\ |a(\lambda)|^2-|b(\lambda)|^2=1;$$

4) function $a(\lambda)$ has no zeros on the half-plane $Im\lambda\geq 0$.

It can be show that one of the coefficient of reflection is uniquely defined through the other. Indeed, it follows from the formula (3.5)

$$r^{-}(\lambda)=-\overline{r^{+}(\lambda)}\,\frac{\overline{a(\lambda)}}{a(\lambda)}, \quad (3.8)$$

and function $a(z)$ can be regenerated by $r^{+}(\lambda)$:

$$a(z)=\left(\alpha^{(+)}+i\beta\right)\exp\left\{-\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{ln\left[\left(1-|r^{+}(\lambda)|^2\right)\left((\alpha^{(+)})^2+\beta^2\right)\right]}{\lambda-z}d\lambda\right\}. \quad (3.9)$$

Inverse scattering problem for the problem (1.1), (1.2) is to regenerate the potential $\Omega(x)$ by left and right coefficients of reflection.

4 Basic equations of inverse scattering problem

According to (3.5) and lemma 3.1 we get that functions $r^{\pm}(\lambda)$ are continuous on $\mathbb{R}$ and

$$\left|r^{\pm}(\lambda)\right| < 1, \quad \lambda \in \mathbb{R}, \tag{4.1}$$

$$r^{\pm}(\lambda) - r_0^{\pm}(\lambda) = \int\limits_{-\infty}^{+\infty} \varphi^{\pm}(x) e^{-i\lambda x} dx, \tag{4.2}$$

where

$$r_0^{\pm}(\lambda) = \mp \frac{\alpha^{(-)} \pm i\beta}{\alpha^{(+)} + i\beta} e^{\mp 2\lambda a i}, \quad \varphi^{\pm}(x) \in L_1(-\infty, +\infty). \tag{4.3}$$

Theorem 4.1 *For every $x \neq a$ kernels $K^{\pm}(x,y)$ of representations $(2.3)_{\pm}$ satisfy the functionally integral equations*

$$\begin{gathered} R_1^{\pm}(x,y) + K^{\pm}(x,y) + K^{\pm}(x, 2a-y) \\ \times \left[\frac{\alpha^{(+)}\alpha^{(-)} \pm \beta^2}{\left(\alpha^{(+)}\right)^2 + \beta^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pm \frac{\beta \alpha^{\mp 1}}{\left(\alpha^{(+)}\right)^2 + \beta^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\ \pm \int\limits_{x}^{\pm\infty} K^{\pm}(x,t) R^{\pm}(t+y) dt = 0, \ \ \pm y > \pm x, \end{gathered} \tag{4.4}$$

where

$$R_1^{\pm} = \begin{cases} R^{\pm}(x,y), & \pm x > \pm a, \\ (M^{\pm 1})^{(-)} R^{\pm}(x+y) + (M^{\pm 1})^{(+)} R^{\pm}(2a - x + y), & \pm x < \pm a, \end{cases}$$

$$R^{\pm}(x) = \frac{1}{2\pi} Re \int\limits_{-\infty}^{+\infty} \left[r^{\pm}(\lambda) - r_0^{\pm}(\lambda)\right] \begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix} e^{\pm i\lambda x} d\lambda. \tag{4.5}$$

Proof. To derive the equation $(4.4)_{+}$, we use identity (3.3), which can be written in the form

$$\begin{gathered} \left(\frac{1}{a(\lambda)} - \frac{1}{a_0(\lambda)}\right) e^{-}(x,\lambda) = (r^{+}(\lambda) - r_0^{+}(\lambda)) e^{+}(x,\lambda) \\ + \overline{e^{+}(x,\lambda)} + r_0^{+}(\lambda) e^{+}(x,\lambda) - \frac{1}{a_0(\lambda)} e^{-}(x,\lambda). \end{gathered}$$

Multiply both sides of this equality by $\frac{1}{2\pi}(1,-i)\, e^{i\lambda y}$ where $y > x$ and then integrating with respect to λ in limits from $-\infty$ to $+\infty$ we get

$$\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left(\frac{1}{a(\lambda)} - \frac{1}{a_0(\lambda)}\right) e^{-}(x,\lambda)\,(1,-i)\, e^{i\lambda y} d\lambda$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} (r^{+}(\lambda) - r_0^{+}(\lambda)) e^{+}(x,\lambda)\,(1,-i)\, e^{i\lambda y} d\lambda$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}(\overline{e^{+}(x,\lambda)}+r_0^{+}(\lambda)e^{+}(x,\lambda)-\frac{1}{a_0(\lambda)}e^{-}(x,\lambda))\,(1,-i)\,e^{i\lambda y}d\lambda. \qquad (4.6)$$

As function $\frac{1}{a(\lambda)}-\frac{1}{a_0(\lambda)}$ is analytical in the upper half-plane, tends to zero as $|\lambda|\to\infty$ $(Im\lambda\geq 0)$ and function $e^{-}(x,\lambda)e^{i\lambda y}\,(1,-i)$ for $y>x$ is uniformly bounded in the half-plane $Im\lambda\geq 0$, then using Jordan's Lemma, we get for $y>x$

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\frac{1}{a(\lambda)}-\frac{1}{a_0(\lambda)}\right)e^{-}(x,\lambda)\,(1,-i)\,e^{i\lambda y}d\lambda=0,\ \ y>x, \qquad (4.7)$$

Further, using $(2.3)_+$ for solution $e^{+}(x,\lambda)$, we have

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}(r^{+}(\lambda)-r_0^{+}(\lambda))e^{+}(x,\lambda)\,(1,-i)\,e^{i\lambda y}d\lambda$$

$$=\widehat{R}_1^{+}(x,y)+\int\limits_{x}^{+\infty}K^{+}(x,t)\widehat{R}^{+}(t+y)dt, \qquad (4.8)$$

$$\widehat{R}_1^{+}(x,y)=\begin{cases}\widehat{R}^{+}(t+y), & x>a,\\ M^{(-)}\widehat{R}^{+}(x+y)+M^{(+)}\widehat{R}^{+}(2a-x+y), & x<a,\end{cases}$$

$$\widehat{R}^{+}(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}(r^{+}(\lambda)-r_0^{+}(\lambda))e^{i\lambda x}d\lambda\cdot\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix},$$

and also, taking into consideration the identity

$$\overline{e_0^{+}(x,\lambda)}+r_0^{+}(\lambda)e_0^{+}(x,\lambda)-\frac{1}{a_0(\lambda)}e_0^{-}(x,\lambda)\equiv 0,$$

for the second summed of the right side (4.6) we obtain

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\overline{e_0^{+}(x,\lambda)}+r_0^{+}(\lambda)e^{+}(x,\lambda)-\frac{1}{a_0(\lambda)}e_0^{-}(x,\lambda)\right)(1,-i)e^{i\lambda y}d\lambda$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\int\limits_{x}^{+\infty}K^{+}(x,t)\begin{pmatrix}1\\ i\end{pmatrix}e^{-i\lambda t}dt+r_0^{+}(\lambda)\int\limits_{x}^{+\infty}K^{+}(x,t)\begin{pmatrix}1\\ -i\end{pmatrix}e^{i\lambda t}dt\right.$$

$$\left.-\frac{1}{a_0(\lambda)}\int\limits_{-\infty}^{x}K^{-}(x,t)\begin{pmatrix}1\\ i\end{pmatrix}e^{-i\lambda t}dt\right)(1,-i)e^{i\lambda y}d\lambda$$

$$=K^{+}(x,t)\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}-\frac{\alpha^{(-)}+i\beta}{\alpha^{(+)}+i\beta}K^{+}(x,2a-y)\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}, \qquad (4.9)$$

where $K^{-}(x,y)=0$ for $y>x$. Therefore, according to (4.7)-(4.9), equality (4.6) takes the form

$$0=\widehat{R}_1^{+}(x,y)+\int\limits_x^{+\infty}K^{+}(x,t)\widehat{R}^{+}(t+y)dt$$

$$+K^{+}(x,y)\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}-\frac{\alpha^{(-)}+i\beta}{\alpha^{(+)}+i\beta}K^{+}(x,2a-y)\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}.$$

It follows the validity of $(4.4)_{+}$ and

$$R_1^{+}(x,y)=Re\widehat{R}_1^{+}(x,y),\;\;R^{+}(x)=Re\widehat{R}^{+}(x).$$

Similarly, using the identity (3.2) we get $(4.4)_{-}$. The theorem has been proved.

It follows from the basic equations $(4.4)_{\pm}$ that coefficients of reflection $r^{\pm}(\lambda)$ of the considered problem satisfy the following conditions:

I. Functions $r^{\pm}(\lambda)$ are continuous for all $\lambda\in R,\;|r^{\pm}(\lambda)|<1$ and

$$r^{\pm}(\lambda)-r_0^{\pm}(\lambda)=\int\limits_{-\infty}^{+\infty}\varphi^{\pm}(t)e^{-i\lambda t}dt,\quad \varphi^{\pm}(t)\in L_1(-\infty,+\infty)$$

functions

$$R^{\pm}(x)=\frac{1}{2\pi}Re\int\limits_{-\infty}^{+\infty}\left[r^{\pm}(\lambda)-r_0^{\pm}(\lambda)\right]\begin{pmatrix}1 & \mp i\\ \mp i & -1\end{pmatrix}e^{i\lambda x}d\lambda$$

are continuous and

$$\int\limits_{x'}^{+\infty}\left|R^{\pm}(\pm x)\right|dx<+\infty,\quad \text{for all}\quad x'>-\infty.$$

5 Inverse scattering problem

Now we state and prove the following theorem.

Theorem 5.1 *If the conditions I are fulfilled then the equations* $(4.4)_{+}$ *and* $(4.4)_{-}$ *have unique solutions respectively* $K^{+}(x,\cdot)$ *(with components from* $L_1(x,\infty)$*) and* $K^{-}(x,\cdot)$ *(with components from* $L_1(-\infty,x)$*) for every fixed respectively* $x>-\infty$ *and* $x<+\infty$.

Proof. For every fixed $x\to-\infty$ the operator

$$\left(M_x^{+}\right)(y)=\begin{cases} f(y)I_2, & x>a,\\ f(y)I_2+f(2a-y)\left[\frac{\alpha^{(+)}\alpha^{(-)}+\beta^2}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\frac{\beta\alpha^{-1}}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right], & x<a,\end{cases}$$

acting in space $L_1(x,+\infty)$ (and also, in space $L_2(x,\infty)$) is reversible. Therefore, the basis equation $(4.4)_{+}$ is equivalent to the equation

$$K^{+}(x,y)+\left(M_x^{+}\right)^{-1}R_1^{+}(x,y)+\left(M_x^{+}\right)^{-1}R^{+}K^{+}(x,\cdot)(y)=0,\;\;y>x,$$

i.e. the equation with the completely continuous operator $\left(M_x^+\right)^{-1} R^+$ (concerning the completely continuity of R^+ see Lemma 3.3.1 in [11]). Thus, to prove the theorem it is sufficient to show the homogeneous equation

$$f_x(y) + f_x(2a-y)\left[\frac{\alpha^{(+)}\alpha^{(-)}+\beta^2}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\beta\alpha^{-1}}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right],$$

$$+\int\limits_x^{+\infty} f_x(t)R^+(t+y)dt = 0, \quad y > x, \tag{5.1}$$

has only trivial solution $f_x(\cdot)$ with components from $L_1(x,\infty)$. According to the condition I of the theorem the components of solution $f_x(\cdot)$ belong to the $L_\infty(x,\infty)$, and hence to the space $L_2(x,\infty)$ (see Lemma 3.3.2 in [1]).

Now we scalar multiply the equation (5.1) by $f_x(\cdot)$ and then integrate with respect to y in the interval (x,∞). As a result we get

$$\int\limits_x^{+\infty} \left(f_x(y), f_x(y)\right) dy$$

$$+\int\limits_x^{+\infty}\left(f_x(2a-y)\left[\frac{\alpha^{(+)}\alpha^{(-)}+\beta^2}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\beta\alpha^{-1}}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right], f_x(y)\right) dy$$

$$+\int\limits_x^{+\infty}\left(f_x(t), R^+(t+y), f_x(y)\right) dtdy = 0. \tag{5.2}$$

According to Parseval's equality

$$\int\limits_x^{+\infty} \left(f_x(y), f_x(y)\right) dy = \int\limits_{-\infty}^{+\infty} \left|\Phi(\lambda)\right|^2 d\lambda,$$

and also

$$\int\limits_x^{+\infty}\left(f_x(2a-y)\left[\frac{\alpha^{(+)}\alpha^{(-)}+\beta^2}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\beta\alpha^{-1}}{\left(\alpha^{(+)}\right)^2+\beta^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right], f_x(y)\right) dy$$

$$= Re\int\limits_{-\infty}^{+\infty} r_0^+(\lambda)\Phi(\lambda)d\lambda,$$

where

$$\Phi(\lambda) = \int\limits_x^{+\infty} f_x(y)\begin{pmatrix}1\\ -i\end{pmatrix} e^{i\lambda y}dy,$$

and

$$\int\limits_x^{+\infty}\left(f_x(t)R^+(t+y), f_x(y)\right) dy = Re\int\limits_{-\infty}^{+\infty}[r^+(\lambda) - r_0^+(\lambda)]\Phi^2(\lambda)d\lambda,$$

(see $(4.5)_+$) hence, it follows from the equation (5.2) that

$$\int_{-\infty}^{+\infty} |\Phi(\lambda)|^2 \, d\lambda + Re \int_{-\infty}^{+\infty} r^+(\lambda)\Phi^2(\lambda) d\lambda = 0.$$

From here we have

$$\int_{-\infty}^{+\infty} |\Phi(\lambda)|^2 \, d\lambda \leq \int_{-\infty}^{+\infty} \left| r^+(\lambda) \right| |\Phi(\lambda)|^2 \, d\lambda,$$

i.e.

$$\int_{-\infty}^{+\infty} (1 - \left| r^+(\lambda) \right|) \, |\Phi(\lambda)|^2 \, d\lambda \leq 0.$$

It follows that $\Phi(\lambda) \equiv 0$ since $1 - |r^+(\lambda)| > 0$ for all $\lambda \in (-\infty, +\infty)$. Thus, the homogeneous equation (5.1) has only the trivial solution.

Unique solvability of the equations $(4.4)_-$ is proved similarly.

Corollary 5.1 *The potential $\Omega(x)$ is uniquely defined by either the left $(r^-(\lambda))$ or the right $(r^+(\lambda))$ coefficients of reflection.*

References

1. Agranovich, Z.S., Marchenko, V.A.: The inverse problem of scattering theory. *Kharkiv, University*, 268 p. (1960).
2. Amirov, R.Kh.: *On system of differential Dirac equations with conditions of discontinuity of the interval.* Ukr. Mat.J. (6), 601-613 (2005).
3. Frolov, I.S.: *An inverse problem of scattering theory for a system of Dirac equations on the whole axis.* Dokl. Akad. Nauk SSSR, **207** (1), 44-47 (1972) (in Russian).
4. Gasymov, M.G.: *An inverse problem of scattering theory for a system of Dirac equations of order 2n.* Trudy Moscov, Mat. Obsc., **19**, 41-112 (1968) (in Russian).
5. Gasymov, M.G., Dzhabiev, T.T.: *Solution of the inverse problem by two spectra for the Dirac equation on a finite interval.* Dokl. Akad. Nauk SSR, **22** (7), 3-6 (1966) (in Russian).
6. Gasymov, M.G., Levitan, B.M.: *Determination of the Dirac system from scattering phase.* Dokl. Nauk SSSR, **167** (6), 1219-1222 (1966) (in Russian).
7. Gasymov, M.G., Levitan, B.M.: *The inverse problem for a dirac system Dirac system* Dokl. Nauk SSSR, **167**, 967-970 (1966) (in Russian).
8. Huseynov, H.M.: *About Levinson type formula sytem of Dirac equations.* Izvestiya Akad.Nauk Az SSR, (2), 31-36 (1978) (in Russian).
9. Huseynov, H.M., Latifova, A.D.: *The main equation for the system of Dirac equations with discontinuity conditions interior to interval*. Trans. Natl. Acad Sci. Azaerb. **28** (1), 63-76 (2008).
10. Huseynov, H.M.: *An inverse scattering problem for a system of dirav equations with discontinuity conditions*. Proc. of IMM NAS of Azerb., **40**, 215-225 (2014).
11. Marchenko, V.A.: Sturm-Liouville operators and their applications. *Naukova Dumka, Kiev*, 232 p. (1977).