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On absence of solutions of a semi-linear elliptic equation with biharmonic operator in the exterior of a ball

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Abstract. *We study absence of global solutions of a semi-linear elliptic equation with a biharmonic operator* $\Delta^2 u + \frac{c}{|x|^2}\Delta u - |x|^\sigma |u|^q = 0$ *in the exterior of a ball. Sufficient condition absence of global solutions is obtained.The proof is based on the method of a test functions.*

Keywords. Semi-linear elliptic equation, biharmonic operator, global solution, critical exponent, method of test functions.

Mathematics Subject Classification (2010): 35A01, 35J61, 35J91

1 Introduction

Introduce the following denotations. Let $R > 0$, $B_R = \{x;\ |x| < R\}$, $B_R^c = \{x;\ |x| > R\}$, $B_{R_1,\,R_2} = \{x;\ R_1 < |x| < R_2\}$, $\partial B_R = \{x;\ |x| = R\}$, where $x = (x_1,\ ...,\ x_n) \in R^n$, $r = |x| = \sqrt{x_1^2 + ... + x^2}$.

In B_R^c consider the equation

$$\Delta^2 u + \frac{c}{|x|^2}\Delta u - |x|^\sigma |u|^q = 0, \tag{1.1}$$

where $0 \le c \le \frac{(n-2)^2}{4}$, $q > 1$, $\sigma > -4$, $\Delta^2 u = \Delta(\Delta u)$, Δ is $n-$dimensional Laplace operator.

Denote $\alpha_\pm = -\frac{n-2}{2} \pm \sqrt{D}$, $D = \frac{(n-2)^2}{4} - c$.

We will study the existence of global solutions of equation (1.1) in B_R^c satisfying the condition

$$\int_{\partial B_R} \Delta u dx \le 0. \tag{1.2}$$

Under the global solution of problem (1.1), (1.2) we understand the function $u(x) \in C^4(B_R^c)$,satisfying condition (1.2) on the boundary and equation (1.1) at every point of B_R^c.

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The issue on the existence of global solutions of weakly nonlinear elliptic equations occupies a significant place in theory of such equations, and a lot of papers have been devoted to it. For review of such papers see the monograph [4]. Weakly nonlinear equations with a biharmonic operator were considered by many authors. In the paper [1] for $c = 0$ equation (1.1) is considered in the ball B_R with the boundary conditions $\int_{\partial B_R} \Delta u dx \leq 0$, $\int_{\partial B_R} \Delta u dx \geq 0$ and it is proved that if $\sigma \geq 4$, $q > 1$, then the solution is absent. In the paper [5] for $\sigma = 0$, $c = 0$ equation (1.1) is considered in B_R^c with different boundary conditions on ∂B_R and critical exponent of absence of solution of the problems under consideration are found. In the paper [6] for $\sigma \geq 0$, similar problems are considered and it is shown how the results of the paper [5] may generalized for an arbitrary $\sigma \geq 0$.

In this paper we consider problem (1.1), (1.2) for $0 \leq c \leq \frac{(n-2)^2}{4}$, $\sigma > -4$ and also in the papers [1],[5],[6] using the technique of test function, worked out Mitidieri and Pohozaev in the papers [2]-[4] find an exact exponent of absence of a global solutions.

2 Formulation of the main result and proof

The main result of this paper is the following theorem:

Theorem 2.1 *Let $q > 1$, $0 \leq c \leq \frac{(n-2)^2}{4}$, $(q-1)(2+\alpha_-) + 4 + \sigma \geq 0$. If $u(x)$ is the solution of problem (1.1), (1.2) in B_R^c, then $u \equiv 0$.*

Proof. For simplicity of notation, we consider the equation in B_1^c.

Multiply equation (1.1) by the function $\varphi \in C_0^\infty(B_{1,2\rho})$ and integrate by parts. We get the following:

$$\int_{B_{1,2\rho}} |u|^q |x|^\sigma \varphi dx = \int_{B_{1,2\rho}} \Delta(\Delta u)\varphi dx + \int_{B_{1,2\rho}} \frac{c}{|x|^2} \Delta u \varphi dx$$

$$= \int_{\partial B_{1,2\rho}} \frac{\partial(\Delta u)}{\partial n} \varphi ds - \int_{B_{1,2\rho}} (\nabla(\Delta u), \nabla \varphi) dx + \int_{B_{1,2\rho}} \frac{c}{|x|^2} \Delta u \varphi dx$$

$$= -\int_{\partial B_{1,2\rho}} \Delta u \frac{\partial \varphi}{\partial n} ds + \int_{B_{1,2\rho}} \Delta u \Delta \varphi dx + \int \frac{c}{|x|^2} \Delta u \varphi dx. \tag{2.1}$$

Take $\varphi(x) = \xi(x)\psi(x)$,

$$\psi(x) = \begin{cases} 1, & \text{for } 1 \leq |x| \leq \rho, \\ \left(2 - \frac{r}{\rho}\right)^\beta, & \text{for } \rho \leq |x| \leq 2\rho, \\ 0, & \text{for } |x| \geq 2, \end{cases}$$

$$\xi(x) = |x|^{\alpha_+} - |x|^{\alpha_-},$$

where β is a rather large positive number. It is easy to verify that $\xi(x)$ is the solution of the equation

$$\Delta \xi + \frac{c}{|x|^2} \xi = 0,$$

and $\xi/_{|x|=1} = 0$.

As for $|x| = 1$, $\frac{\partial \varphi}{\partial n} = \frac{\partial \xi}{\partial n} = -\frac{\partial \xi}{\partial r} = -\left(\alpha_+ r^{\alpha_+ - 1} - \alpha_- r^{\alpha_- - 1}\right) = -(\alpha_+ - \alpha_-) = -2\sqrt{D}$, then $\int_{\partial B_1} \Delta u \frac{\partial \varphi}{\partial n} ds = -2\sqrt{D} \int_{\partial B_1} \Delta u dx \geq 0$.

From (2.1) we get

$$\int_{B_{1,\,2\rho}} |u|^q\, |x|^\sigma\, \xi\psi dx \le \int_{B_{1,\,2\rho}} \Delta u \Delta\left(\xi\psi\right) dx + \int_{B_{1,\,2\rho}} \frac{c}{|x|^2} \Delta u \xi\psi dx$$

$$= \int_{B_{1,\,2\rho}} \Delta u \psi \left(\Delta\xi + \frac{c}{|x|^2}\xi\right) dx + \int_{B_{1,\,2\rho}} \Delta u \left(2\left(\nabla\xi,\, \nabla\psi\right) + \xi\Delta\psi\right) dx$$

$$= \int_{B_{1,\,2\rho}} \Delta u \left(2\left(\nabla\xi,\, \nabla\psi\right) + \xi\Delta\psi\right) dx = \int_{\partial B_{1,\,2\rho}} \frac{\partial u}{\partial v} \left(2\left(\nabla\xi,\, \nabla\psi\right) + \xi\Delta\psi\right) ds$$

$$- \int_{B_{1,\,2\rho}} \left(\nabla u,\, \nabla\left(2\left(\nabla\xi,\, \nabla\psi\right) + \xi\Delta\psi\right)\right) dx$$

$$= - \int_{\partial B_{1,\,2\rho}} u \left(2\frac{\partial}{\partial v}\left(\nabla\xi,\, \nabla\psi\right) + \frac{\partial}{\partial v}\left(\xi\Delta\psi\right)\right) ds$$

$$+ \int_{B_{1,\,2\rho}} u \left(2\Delta\left(\nabla\xi,\, \nabla\psi\right) + \Delta\left(\xi\Delta\psi\right)\right) dx$$

$$= \int_{B_{\rho,\,2\rho}} u \left(2\Delta\left(\nabla\xi,\, \nabla\psi\right) + \Delta\left(\xi\Delta\psi\right)\right) dx$$

$$\le \left(\int_{B_{\rho,\,2\rho}} |u|^q\, |x|^\sigma\, \xi\psi dx\right)^{\frac{1}{q}} \left(\int_{B_{\rho,\,2\rho}} \frac{\left|2\Delta\left(\nabla\xi,\, \nabla\psi\right) + \Delta\left(\xi\Delta\psi\right)\right|^{q'}}{|x|^{\sigma(q'-1)}\, \xi^{q'-1}\psi^{q'-1}} dx\right)^{\frac{1}{q'}}, \qquad (2.2)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Here we used the fact that on $\partial B_{1,\,2\rho}$ all derivatives ψ equal zero.

As a result we have

$$\int_{B_{1,\,2\rho}} |u|^q\, |x|^\sigma\, \xi\psi dx \le \int_{B_{\rho,\,2\rho}} \frac{\left|2\Delta\left(\nabla\xi,\, \nabla\psi\right) + \Delta\left(\xi\Delta\psi\right)\right|^{q'}}{|x|^{\sigma(q'-1)}\, \xi^{q'-1}\psi^{q'-1}} dx. \qquad (2.3)$$

Estimate the last integral.

Denote $J\left(\xi,\, \psi\right) = 2\Delta\left(\nabla\xi,\, \nabla\psi\right) + \Delta\left(\xi\Delta\psi\right)$. At first we calculate each addend of $J\left(\xi,\, \psi\right)$ separately

$$2\Delta\left(\nabla\xi,\, \nabla\psi\right) = 2\frac{\partial^2}{\partial r^2}\left(\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}\right) + \frac{n-1}{r}\frac{\partial}{\partial r}\left(\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}\right)$$

$$= 2\frac{\partial\xi}{\partial r}\frac{\partial^3\psi}{\partial r^3} + 4\frac{\partial^2\xi}{\partial r^2}\frac{\partial^2\psi}{\partial r^2} + 2\frac{\partial^3\xi}{\partial r^3}\frac{\partial\psi}{\partial r}$$

$$+ \frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}\frac{\partial\psi}{\partial r} + \frac{n-1}{r}\frac{\partial\xi}{\partial r}\frac{\partial^2\psi}{\partial r^2}, \qquad (2.4)$$

$$\Delta\left(\xi\Delta\psi\right) = \frac{\partial^2}{\partial r^2}\left(\xi\Delta\psi\right) + \frac{n-1}{r}\frac{\partial}{\partial r}\left(\xi\Delta\psi\right)$$

$$= \frac{\partial^2\xi}{\partial r^2}\Delta\psi + 2\frac{\partial\xi}{\partial r}\frac{\partial}{\partial r}\left(\Delta\psi\right) + \xi\frac{\partial^2}{\partial r^2}\left(\Delta\psi\right) + \frac{n-1}{r}\frac{\partial\xi}{\partial r}\Delta\psi$$

$$+ \frac{n-1}{r}\xi\frac{\partial}{\partial r}\left(\Delta\psi\right) = \frac{\partial^2\xi}{\partial r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}\frac{\partial\psi}{\partial r}$$

$$+2\frac{\partial\xi}{\partial r}\frac{\partial}{\partial r}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\psi}{\partial r}\right)+\xi\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\psi}{\partial r}\right)$$
$$+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\frac{\partial^2\psi}{\partial r^2}+\frac{(n-1)^2}{r^2}\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}+\frac{n-1}{r}\xi\frac{\partial}{\partial r}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\psi}{\partial r}\right)$$
$$=\frac{\partial^2\xi}{\partial r^2}\frac{\partial^2\psi}{\partial r^2}+\frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}\frac{\partial\psi}{\partial r}+2\frac{\partial\xi}{\partial r}\frac{\partial^3\psi}{\partial r^3}+2\frac{\partial\xi}{\partial r}\frac{n-1}{r}\frac{\partial^2\psi}{\partial r^2}$$
$$-2\frac{n-1}{r^2}\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}+\frac{n-1}{r}\xi\frac{\partial^3\psi}{\partial r^3}+2\frac{n-1}{r^3}\xi\frac{\partial\psi}{\partial r}-2\frac{n-1}{r^2}\xi\frac{\partial^2\psi}{\partial r^2}$$
$$+\xi\frac{\partial^4\psi}{\partial r^4}+\frac{n-1}{r}\frac{\partial\xi}{\partial r}\frac{\partial^2\psi}{\partial r^2}+\frac{(n-1)^2}{r^2}\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}+\frac{n-1}{r}\xi\frac{\partial^3\psi}{\partial r^3}-\frac{(n-1)^2}{r^3}\xi\frac{\partial\psi}{\partial r}$$
$$+\frac{(n-1)^2}{r^2}\xi\frac{\partial^2\psi}{\partial r^2}=\xi\frac{\partial^4\psi}{\partial r^2}+2\frac{\partial\xi}{\partial r}\frac{\partial^3\psi}{\partial r^3}+2\frac{n-1}{r}\xi\frac{\partial^3\psi}{\partial r^3}$$
$$+\frac{\partial^2\xi}{\partial r^2}\frac{\partial^2\psi}{\partial r^2}+3\frac{n-1}{r}\frac{\partial\xi}{\partial r}\frac{\partial^2\psi}{\partial r^2}-2\frac{n-1}{r^2}\xi\frac{\partial^2\psi}{\partial r^2}+\frac{(n-1)^2}{r^2}\xi\frac{\partial^2\psi}{\partial r^2}$$
$$+\frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}\frac{\partial\psi}{\partial r}-2\frac{n-1}{r^2}\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}+\frac{(n-1)^2}{r^2}\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}$$
$$+2\frac{n-1}{r^3}\xi\frac{\partial\psi}{\partial r}-\frac{(n-1)^2}{r^3}\xi\frac{\partial\psi}{\partial r}. \tag{2.5}$$

From (2.4) and (2.5) we get

$$J(\xi,\,\psi)=\xi\frac{\partial^4\psi}{\partial r^4}+4\frac{\partial\xi}{\partial r}\frac{\partial^3\psi}{\partial r^3}+2\frac{n-1}{r}\xi\frac{\partial^3\psi}{\partial r^3}+5\frac{\partial^2\xi}{\partial r^2}\frac{\partial^2\psi}{\partial r^2}$$
$$+2\frac{\partial^3\xi}{\partial r^3}\frac{\partial\psi}{\partial r}+4\frac{n-1}{r}\frac{\partial\xi}{\partial r}\frac{\partial^2\psi}{\partial r^2}-2\frac{n-1}{r^2}\xi\frac{\partial^2\psi}{\partial r^2}+\frac{(n-1)^2}{r^2}\xi\frac{\partial^2\psi}{\partial r^2}$$
$$+2\frac{n-1}{r}\frac{\partial^2\xi}{\partial r^2}\frac{\partial\psi}{\partial r}+\frac{(n-1)(n-3)}{r^2}\frac{\partial\xi}{\partial r}\frac{\partial\psi}{\partial r}-\frac{(n-1)(n-3)}{r^3}\xi\frac{\partial\psi}{\partial r}.$$

If in the right side of (2.3) we pass to polar coordinates, we get

$$\int_{B_{1,\,2\rho}}|u|^q\,|x|^\sigma\,\xi\psi dx\le c_1\int_{\rho\le r\le 2\rho}\frac{|J(\xi,\,\psi)|^{q'}}{r^{\sigma(q'-1)}\xi^{q'-1}\psi^{q'-1}}r^{n-1}dr. \tag{2.6}$$

In the right integral we make a substitution:

$$t=\frac{r}{\rho},r=t\rho,\tilde{\psi}(t)=\psi(t\rho)=\psi(r)\,,\tilde{\xi}(t)=\xi(t\rho)=\xi(r)\,.$$

Then we get

$$\int_{\rho\le r\le 2\rho}\frac{|J(\xi,\,\psi)|^{q'}\,r^{n-1}}{r^{\sigma(q'-1)}\xi^{q'-1}\psi^{q'-1}}dr$$
$$=\int_{1\le t\le 2}\frac{\rho^{(\alpha_+-4)q'}\left|\tilde{J}\left(\tilde{\xi},\,\tilde{\psi}\right)\right|^{q'}t^{n-1}\rho^n}{\rho^{(\sigma+\alpha_+)(q'-1)}t^{(\sigma+\alpha_+)q'-1}\left(-\rho^{-2\sqrt{D}}t^{-2\sqrt{D}}\right)^{q'-1}\tilde{\psi}^{q'-1}}dt$$
$$=\rho^{-(4q'-\alpha_+-n+\sigma(q'-1))}A\left(\tilde{\xi},\,\tilde{\psi}\right), \tag{2.7}$$

where

$$A\left(\tilde{\xi},\,\tilde{\psi}\right)=\int_{1\le t\le 2}\frac{\left|\tilde{J}\left(\tilde{\xi},\,\tilde{\psi}\right)\right|^{q'}t^{n-1}}{t^{(\sigma+\alpha_{+})(q'-1)}\left(1-\rho^{-2\sqrt{D}}t^{-2\sqrt{D}}\right)^{q'-1}\tilde{\psi}^{q'-1}}dt, \qquad (2.8)$$

$$\tilde{J}\left(\tilde{\xi},\,\tilde{\psi}\right)=t^{\alpha_{+}}\left(1-(t\rho)^{-2\sqrt{D}}\right)\frac{\partial^{4}\tilde{\psi}}{\partial t^{4}}+4t^{\alpha_{+}-1}\left(\alpha_{+}-\alpha_{-}(t\rho)^{-2\sqrt{D}}\right)\frac{\partial^{3}\tilde{\psi}}{\partial t^{3}}$$

$$+2\,(n-1)\,t^{\alpha_{+}-1}\left(1-(t\rho)^{-2\sqrt{D}}\right)\frac{\partial^{3}\tilde{\psi}}{\partial t^{3}}$$

$$+5t^{\alpha_{+}-2}\left(\alpha_{+}\left(\alpha_{+}-1\right)-\alpha_{-}\left(\alpha_{-}-1\right)(t\rho)^{-2\sqrt{D}}\right)\frac{\partial^{2}\tilde{\psi}}{\partial t^{2}}$$

$$+2t^{\alpha_{+}-3}\left(\alpha_{+}\left(\alpha_{+}-1\right)\left(\alpha_{+}-2\right)-\alpha_{-}\left(\alpha_{-}-1\right)\left(\alpha_{-}-2\right)(t\rho)^{-2\sqrt{D}}\right)\frac{\partial\tilde{\psi}}{\partial t}$$

$$+4\,(n-1)\,t^{\alpha_{+}-2}\left(\alpha_{+}-\alpha_{-}(t\rho)^{-2\sqrt{D}}\right)\frac{\partial\tilde{\psi}}{\partial t^{2}}-2\,(n-1)\,t^{\alpha_{+}-2}\left(1-(t\rho)^{-2\sqrt{D}}\right)\frac{\partial^{2}\tilde{\psi}}{\partial t^{2}}$$

$$+\,(n-1)^{2}\,t^{\alpha_{+}-2}\left(1-(t\rho)^{-2\sqrt{D}}\right)\frac{\partial^{2}\tilde{\psi}}{\partial t^{2}}$$

$$+2\,(n-1)\,t^{\alpha_{+}-3}\left(\alpha_{+}\left(\alpha_{+}-1\right)-\alpha_{-}\left(\alpha_{-}-1\right)(t\rho)^{-2\sqrt{D}}\right)\frac{\partial\tilde{\psi}}{\partial t}$$

$$+\,(n-1)\,(n-3)\,t^{\alpha_{+}-3}\left(\alpha_{+}-\alpha_{-}(t\rho)^{-2\sqrt{D}}\right)\frac{\partial\tilde{\psi}}{\partial t}$$

$$-\,(n-1)\,(n-3)\,t^{\alpha_{+}-3}\left(1-(t\rho)^{-2\sqrt{D}}\right)\frac{\partial\tilde{\psi}}{\partial t}.$$

From here

$$\left|\tilde{J}\left(\tilde{\xi},\,\tilde{\psi}\right)\right|\le t^{\alpha_{+}}\left|\frac{\partial^{4}\tilde{\psi}}{\partial t^{4}}\right|+c_{2}t^{\alpha_{+}-1}\left|\frac{\partial^{3}\tilde{\psi}}{\partial t^{3}}\right|+c_{3}t^{\alpha_{+}-2}\left|\frac{\partial^{2}\tilde{\psi}}{\partial t^{2}}\right|+c_{4}t^{\alpha_{+}-3}\left|\frac{\partial\tilde{\psi}}{\partial t}\right|,$$

where $c_2,\ c_3,\ c_4$ are dependent on $n,\ \alpha_+,\ \alpha_-$.

Taking all these into account in (2.8), we get that for large ρ

$$A\left(\tilde{\xi},\,\tilde{\psi}\right)\le\int_{1\le t\le 2}\frac{\left|t^{\alpha_{+}}\left|\frac{\partial^{4}\tilde{\psi}}{\partial t^{4}}\right|+c_{2}t^{\alpha_{+}-1}\left|\frac{\partial^{3}\tilde{\psi}}{\partial t^{3}}\right|+c_{3}t^{\alpha_{+}-2}\left|\frac{\partial^{3}\tilde{\psi}}{\partial t^{2}}\right|+c_{4}t^{\alpha_{+}-3}\left|\frac{\partial\tilde{\psi}}{\partial t}\right|\right|}{t^{(\sigma+\alpha_{+})(q'-1)}\tilde{\psi}^{q'-1}}dt.$$

It easy to see that for large β

$$A\left(\tilde{\xi},\,\tilde{\psi}\right)<c_{5}<\infty,$$

where c_5 is independent of ρ.

Then from (2.6), (2.7) and (2.8) it follows that

$$\int_{B_{1,\,2\rho}}|u|^{q}\,|x|^{\sigma}\,\xi\psi dx\le c_{1}\rho^{-(4q'-\alpha_{+}-n+\sigma(q'-1))}A\left(\tilde{\xi},\,\tilde{\psi}\right)$$

$$\le c_{1}c_{5}\rho^{-\left(4\frac{q}{q-1}-\alpha_{+}-(n-2)-2+\sigma\frac{1}{q-1}\right)}=c_{6}\rho^{-\frac{(q-1)\left(2+\alpha_{-}\right)+4+\sigma}{q-1}}. \qquad (2.9)$$

If $(q-1)(2+\alpha_-)+4+\sigma>0$, then tending $\rho\to+\infty$ from (2.9), we get

$$\int_{B_1^c}|u|^q|x|^\sigma\xi dx\le 0.$$

Consequently $u\equiv 0$ in B_1^c.

Let now $(q-1)(2+\alpha_-)+4+\sigma=0$.

Then from (2.9) it follows

$$\int_{B_1^c}|u|^q|x|^\sigma\xi dx\le c_6.$$

Then
$\int_{B_{\rho,\,2\rho}}|u|^q|x|^\sigma\xi dx\to 0$ as $\rho\to+\infty$.
From (2.2) and (2.7) it follow that

$$\int_{B_{1,\,2\rho}}|u|^q|x|^\sigma\xi\psi dx\le\left(\int_{B_{\rho,\,2\rho}}|u|^q|x|^\sigma\xi\psi dx\right)^{\frac{1}{q}}\left(\int_{B_{\rho,\,2\rho}}\frac{|J(\xi,\,\psi)|^{q'}}{|x|^{\sigma(q'-1)}\xi^{q'-1}\psi^{q'-1}}dx\right)^{\frac{1}{q'}}$$

$$\le c_5^{\frac{1}{q'}}\int_{B_{\rho,\,2\rho}}|u|^q|x|^\sigma\xi\psi dx\le c_5\left(\int_{B_{\rho,\,2\rho}}|u|^q|x|^\sigma\xi dx\right)^{\frac{1}{q}}\to 0,$$

as $\rho\to+\infty$, i.e. in the limit

$$\int_{B_1^c}|u|^q|x|^\sigma\xi dx=0.$$

So, in this case $u\equiv 0\ \ B_1^c$. This proves the theorem.

Now let us show that estimation on absence of a global solution is exact, i.e., if $(q-1)(2+\alpha_-)+4+\sigma<0$, then there exists a global solution of problem (1.1), (1.2).We will look for this solution in the form $A|x|^{-\mu}$. Substituting this function in the equation, we get

$$\mu(\mu+2)(\mu+2-n)(\mu+4-n)|x|^{-\mu-4}+c\mu(\mu+2-n)|x|^{-\mu-4}$$
$$-A^{q-1}|x|^{\sigma-\mu q}=0.$$

Hence $\mu=\frac{4+\sigma}{q-1}$,

$$A^{q-1}=\mu(\mu+2-n)\left((\mu+2)^2+(\mu+2)(2-n)+c\right)$$

$$=\mu(\mu+2-n)(\mu+2+\alpha_-)(\mu+2+\alpha_+).$$

As

$$\int_{\partial B_1}\Delta u dx=\int_{\partial B_1}A\mu(\mu+2-n)dx=\mu(\mu+2-n)c_6\le 0,$$

then from condition (1.2)

$$\mu+2-n\le 0.$$

As $(q-1)(2+\alpha_-)+4+\sigma<0$, then

$$\frac{4+\sigma}{q-1}+2<\frac{n-2}{2}+\sqrt{\frac{(n-2)^2}{4}-c},$$

i.e.

$$\mu + 2 < \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - c}.$$

Obviously, then $\mu + 2 - n \leq 0$. Coming back the expression A, we get that for

$$\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - c} < \frac{4+\sigma}{q-1} + 2 < \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - c}$$

the function $u(x) = A\,|x|^{-\frac{4+\sigma}{q-1}}$ will be a positive solution of problem (1.1), (1.2), where

$$A = \left[\frac{4+\sigma}{q-1}\left(\frac{4+\sigma}{q-1} + 2 - n\right)\left(\frac{4+\sigma}{q-1} + 2 + \alpha_-\right)\left(\frac{4+\sigma}{q-1} + 2 + \alpha_+\right)\right]^{\frac{1}{q-1}}.$$

We can write the estimation $(q-1)(2+\alpha_-) + 4 + \sigma \geq 0$ as follows:

$$\frac{4+\sigma}{q-1} \geq -\alpha_- - 2.$$

Hence

$$q \leq 1 + \frac{4+\sigma}{-\alpha_- - 2} = 1 + \frac{4+\sigma}{-2 + \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - c}} = q^*.$$

The value q^* is said to the an exact critical exponent on absence of global solution of problem (1.1), (1.2).

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