Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (4), 77-98 (2016).
Series of Physical-Technical and Mathematical Sciences.

(p, q)-admissible multilinear fractional integral operators and their commutators in product generalized local Morrey spaces

Ahmet Eroglu · Amil A. Hasanov · Mehriban N. Omarova

Received: 28.12.2015 / Revised: 03.05.2016 / Accepted: 27.06.2016

Abstract. In this paper we prove the boundedness of the $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators $T_{\alpha,m}$ from product generalized local Morrey space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$. We find the sufficient conditions on $(\varphi_1, \ldots, \varphi_m, \varphi)$ which ensures the boundedness of the commutators of $(\mathbf{p}, q)$-admissible multilinear fractional integral operators $T^{\overrightarrow{b}}_{\alpha,m}$ from $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$. In all cases the conditions for the boundedness of $T_{\alpha,m}$ are given in terms of Zygmund-type integral inequalities on $(\varphi_1, \ldots, \varphi_m, \varphi)$, which do not require any assumption on monotonicity of $\varphi_1, \ldots, \varphi_m, \varphi$ in r.

Keywords. Product generalized local Morrey space; $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operator; commutator.

Mathematics Subject Classification (2010): 42B20, 42B35, 47G10

1 Introduction

The multilinear theory has been well developed in the past twenty years. Multilinear fractional integral operator was studied first by Grafakos [13], subsequently, by Kenig and Stein [31], Grafakos and Kalton [15].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_n)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

In 1992, Grafakos [14] gave the following multilinear integrals defined by

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} f_1\left(x - \theta_1 y\right) \ldots f_m\left(x - \theta_m y\right) dy,$$

The research of M. Omarova was partially supported by the grant of Presidium Azerbaijan National Academy of Science 2015.

Ahmet Eroglu
Omer Halisdemir University, Department of Mathematics, Nigde, Turkey
E-mail: aeroglu@ohu.edu.tr

Amil A. Hasanov
Gandja State University, Gandja, Azerbaijan
E-mail: amilhasanov@gmail.com

Mehriban N. Omarova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
Baku State University
E-mail: mehriban_omarova@yahoo.com

where $\theta_i, (i = 1, \dots, m)$ are fixed distinct and nonzero real numbers and $0 < \alpha < n$. Grafakos proved that the operator $I_{\alpha,m}$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $0 < 1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n < 1$, which can be regarded as an extension result for the classical fractional integral on Lebesgue spaces. In [18,19] by Guliyev, Nazirova a certain O'Neil type inequality for dilated multi-linear convolution operators, including permutations of functions was proved. This inequality was used to extend Grafakos's result to more general multi-linear operators of potential type and the relevant maximal operators.

Let $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$. In 1999, Kenig and Stein [31] studied the following multilinear fractional integral

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\ldots f_m(y_m)}{|(x-y_1,\ldots,x-y_m)|^{nm-\alpha}}\, dy_1 \ldots\ldots dy_m$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n > 0$ for each $p_i > 1, (i = 1, \dots, m)$. If some $p_i = 1$, then $I_{\alpha,m}$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$, where $WL_q(\mathbb{R}^n)$ denotes the weak L_q-space of measurable functions on $\mathbb{R}^n$. Obviously, the multilinear fractional integral operator $I_{\alpha,m}$ is a natural generalization of the classical fractional integral operator $I_\alpha \equiv I_{\alpha,1}$.

Let $T_{\alpha,m}$ be a multi-sublinear operator.

Definition 1.1 *($(\mathbf{p},q)$-admissible multi-sublinear fractional integral operator). Let multi-sub-linear operator $T_{\alpha,m}$ will be called $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operator, if:*

1) *$T_{\alpha,m}$ satisfies the size condition of the form*

$$\chi_{B(x,r)}(z)\left|T_{\alpha,m}\Big(f_1\chi_{\mathbb{R}^n\setminus B(x,2r)},\ldots,f_m\chi_{\mathbb{R}^n\setminus B(x,2r)}\Big)(z)\right| \\ \leq C\chi_{B(x,r)}(z)\int_{\left(\mathbb{R}^n\setminus B(x,2r)\right)^m}\frac{\left|f_1(y_1)\ldots f_m(y_m)\right|}{|(z-y_1,\ldots,z-y_m)|^{nm-\alpha}}dy \tag{1.1}$$

for $x \in \mathbb{R}^n$ and $r > 0$;

2) *$T_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

Definition 1.2 *(weak $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operator). Let multi-sublinear operator $T_{\alpha,m}$ will be called the weak $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operator, if:*

1) *$T_{\alpha,m}$ satisfies the size condition* (1.2).

2) *$T_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to the weak $WL_q(\mathbb{R}^n)$.*

Let $\overrightarrow{b} = (b_1, \dots, b_m)$ be a finite family of locally integrable functions. The commutators generated by the m-th fractional integral $I_{\alpha,m}$ and $\overrightarrow{b}$ are defined by

$$I_{\alpha,m}^{\overrightarrow{b}}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^{m}\big(b_i(x)-b_i(y_i)\big)f_i(y_i)}{|(x-y_1,\ldots,x-y_m)|^{nm-\alpha}}\, dy_1 dy_2 \ldots dy_m.$$

Let $T_{\alpha,m}^{\overrightarrow{b}}$ be a multi-sublinear operator.

Definition 1.3 *(commutator of $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operator). Let multi-sublinear operator $T^{\overrightarrow{b}}_{\alpha,m}$ will be called commutator of $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operator, if:*

1) *$T^{\overrightarrow{b}}_{\alpha,m}$ satisfies the size condition of the form*

$$\chi_{B(x,r)}(z)\left|T^{\overrightarrow{b}}_{\alpha,m}\Big(f_1\chi_{\mathbb{R}^n\setminus B(x,2r)},\ldots,f_m\chi_{\mathbb{R}^n\setminus B(x,2r)}\Big)(z)\right|$$
$$\le C\chi_{B(x,r)}(z)\int_{\left(\mathbb{R}^n\setminus B(x,2r)\right)^m}\frac{\prod\limits_{i=1}^m\left|b_i(x)-b_i(y_i)\right||f_i(y_i)|}{|(z-y_1,\ldots,z-y_m)|^{nm-\alpha}}dy \tag{1.2}$$

for $x\in\mathbb{R}^n$ and $r>0$;

2) *$T^{\overrightarrow{b}}_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n)\times\ldots\times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

In this study, we prove the boundedness of the $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operators $T_{\alpha,m}$ from product generalized local Morrey space $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$, if $1<p_1,\ldots,p_m<n/\alpha$ and $1/p_1+\cdots+1/p_m-1/q=\alpha/n$. Also we prove the boundedness of the weak $(\mathbf{p},q)$-admissible multi-sublinear fractional integral operators $T_{\alpha,m}$ from the space $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to the weak space $WLM^{\{x_0\}}_{q,\varphi}$, if $1\le p_1,\ldots,p_m<n/\alpha$, $1/p_1+\cdots+1/p_m-1/q=\alpha/n$ and at least one exponent p_i equals one. In the case $b_i\in CBMO^{\{x_0\}}_{q_i,\lambda_i}$, for $0<\lambda_i<1/n$, $i=1,2,\ldots,m$, we find the sufficient conditions on $(\varphi_1,\ldots,\varphi_m,\varphi)$ which ensures the boundedness of the $(\mathbf{p},q)$-admissible multilinear fractional integral operators $T^{\overrightarrow{b}}_{\alpha,m}$ from $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$, $1<p,p_i,q_i<\infty$, for $i=1,2,\ldots,m$ such that $1/p=1/p_1+\ldots+1/p_n+1/q_1+\ldots+1/q_n$.

By $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2 Preliminaries

For $x\in\mathbb{R}^n$ and $r>0$, we denote by $B(x,r)$ the open ball centered at x of radius r, by $|B(x,r)|$ the Lebesgue measure of the ball $B(x,r)$, and by ${}^{\complement}B(x,r)$ its complement. Morrey spaces $M_{p,\lambda}\equiv M_{p,\lambda}(\mathbb{R}^n)$ introduced by C. Morrey [36] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}}:=\sup_{x\in\mathbb{R}^n,\ r>0}r^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,r))},$$

where $0\le\lambda<n$, $1\le p<\infty$.

We also denote by $WM_{p,\lambda}\equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f\in WL^{\rm loc}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}}=\sup_{x\in\mathbb{R}^n,\ r>0}r^{-\frac{\lambda}{p}}\|f\|_{WL_p(B(x,r))}<\infty,$$

where WL_p denotes the weak L_p-space. These spaces play an important role in the study of local properties of the solutions of partial differential equations, together with weighted Lebesgue spaces, see [12], [32].

In [35], Mizuhara introduced the generalized Morrey spaces $M_{p,\varphi}\equiv M_{p,\varphi}(\mathbb{R}^n)$ which was later extended and studied by many authors (see [16,20,37]). We define the generalized local Morrey spaces as follows.

Definition 2.1 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi}$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.$$

Recall that the local Morrey-type space $LM_{p\theta,w(\cdot)}$ is introduced by Guliyev in his doctoral thesis [16] (see, also [17]). In [16] the sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in local Morrey-type space $LM_{p\theta,w(\cdot)}$ are given. In a series of papers by V. Burenkov, H.V. Guliyev, V.S. Guliyev, etc. (see, for example [2]-[6]) some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,w(\cdot)}$ are obtained. Note that, the global Morrey-type spaces $GM_{p\theta,w(\cdot)}$ introduced by Burenkov and Guliyev in [2] (see also, [3]-[6]). In [2]-[6] the sufficient conditions for the boundedness of integral operators of harmonic analysis, including the maximal operator, the fractional maximal operators, the fractional integral operator and the singular integral operators in global Morrey-type space $GM_{p\theta,w(\cdot)}$ are given.

Definition 2.2 *[16], [17] Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 2.3 *[23], [24] Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty.$$

The following statements, containing results obtained in [35], [37] was proved in [16], [20] (see also [1], [3]-[6], [17]).

Theorem A. [16,17] *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 \le p < \infty$, $1/p - 1/q = \alpha/n$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(x_0, t)\, t^{\alpha-1}\, dt \le C\, \varphi_2(x_0, r), \tag{2.1}$$

where C does not depend on x_0 and r. Then the operator I_α is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi_2}$ for $p = 1$.

The following statements, containing results in Theorem A was proved in [1], see also [21].

Theorem B. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 \le p < \infty$, $1/p - 1/q = \alpha/n$ and (φ_1, φ) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C\, \varphi(x_0, r), \tag{2.2}$$

where C does not depend on x_0 and r. Let the operator I_α is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi}$ for $p = 1$.

3 $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators in the product spaces $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$

In this section, we prove the boundedness of the $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators $T_{\alpha,m}$ from product generalized local Morrey space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$, if $1 < p_1, \ldots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m - 1/q = \alpha/n$. Also we prove the boundedness of the weak $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators $T_{\alpha,m}$ from the space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to the weak space $WLM^{\{x_0\}}_{q,\varphi}$, if $1 \le p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m - 1/q = \alpha/n$ and at least one exponent p_i equals one.

We will use the following statements on the boundedness of the weighted Hardy operators

$$H_w g(r) := \int_r^\infty g(t) w(t) dt,\ 0 < t < \infty$$

and

$$H_w^* g(r) := \int_r^\infty \big(1 + \ln \frac{t}{r}\big)\, g(t) w(t) dt,\ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0, \infty)$.
The following theorem was proved in [23,24].

Theorem 3.1 *[23,24] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0, \infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}\limits_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (3.1).

The following theorem was proved in [22].

Theorem 3.2 *[22] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0, \infty)$. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \leq C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{3.2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \big(1 + \ln \frac{t}{r}\big) \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{3.3}$$

Moreover, the value $C = B$ is the best constant for (3.1).

Remark 3.1 In (3.1) – (3.3) it is assumed that $0 \cdot \infty = 0$.

The following lemma was proved in [26].

Lemma 3.1 *Let $x_0 \in \mathbb{R}^n$, $m \geq 2$, $0 < \alpha < mn$, $1 \leq p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$, $1/q = 1/q_1 + \ldots + 1/q_m = 1/p - \alpha/n$, $1/q_i = 1/p_i - \alpha/(nm)$, $1 \leq i \leq m$.*

If $T_{\alpha,m}$ be a $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators, then for $p_1, \ldots, p_m > 1$ the inequality

$$\|T_{\alpha,m}(\overrightarrow{f})\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} \, t^{-\frac{n}{q_i}} \, \frac{dt}{t} \tag{3.4}$$

holds for any $B(x_0, r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

If $T_{\alpha,m}$ be a weak $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators and at least one exponent p_i equals one, then the inequality

$$\|T_{\alpha,m}(\overrightarrow{f})\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} \, t^{-\frac{n}{q_i}} \, \frac{dt}{t} \tag{3.5}$$

holds for any $B(x_0, r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. Let $p_1, \ldots, p_m \in (1, \infty)$, $1/p = \sum\limits_{k=1}^m 1/p_k$, $1/q = 1/p - \alpha/n$. For arbitrary $x \in \mathbb{R}^n$, set $B = B(x, r)$ for the ball centered at x with a radius r, $2B = B(x, 2r)$. We represent $\overrightarrow{f} = (f_1, \ldots, f_m)$ as

$$f_j = f_j^0 + f_j^\infty, \quad f_j^0 = f_j \chi_{2B}, \quad f_j^\infty = f_j \chi_{\complement_{(2B)}}, \quad j = 1, \ldots, m. \tag{3.6}$$

Then we write

$$\prod_{i=1}^{m} f_i(y_i) = \prod_{i=1}^{m} \left(f_i^0(y_i) + f_i^\infty(y_i)\right)$$
$$= \sum_{\beta_1,\dots,\beta_m \in \{0,\infty\}} f_1^{\beta_1}(y_1) \dots f_m^{\beta_m}(y_m)$$
$$= \prod_{i=1}^{m} f_i^0(y_i) + {\sum_{\beta_1,\dots,\beta_m}}' f_1^{\beta_1}(y_1) \dots f_m^{\beta_m}(y_m),$$

where each term in $\sum'$ contains at least one $\beta_i \neq 0$. Since $T_{\alpha,m}$ is an m-linear operator, then we split $T_{\alpha,m}(\overrightarrow{f})$ as follows:

$$\left|T_{\alpha,m}(\overrightarrow{f})(y)\right| \le \left|T_{\alpha,m}(f_1^0,\dots,f_m^0)(y)\right| + \left|{\sum_{\beta_1,\dots,\beta_m}}' T_{\alpha,m}(f_1^{\beta_1},\dots,f_m^{\beta_m})(y)\right|,$$

where $\beta_1,\dots,\beta_m \in \{0,\infty\}$ and each term in $\sum'$ contains at least one $\beta_i \neq 0$. Then,

$$\|T_{\alpha,m}(\overrightarrow{f})\|_{L_q(B(x,r))} \le \|T_{\alpha,m}(f_1^0,\dots,f_m^0)\|_{L_q(B(x,r))}$$
$$+ \|{\sum_{\beta_1,\dots,\beta_m}}' T_{\alpha,m}(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_q(B(x,r))} \le J^0 + {\sum}' J^{\beta_1,\dots,\beta_m}.$$

Thus,

$$J^0 = \|T_{\alpha,m}(\overrightarrow{f^0})\|_{L_q(B(x,r))} \le \|T_{\alpha,m}(\overrightarrow{f^0})\|_{L_q(\mathbb{R}^n)}$$
$$\lesssim \prod_{i=1}^{m} \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,2r))}.$$

On the other hand, we have

$$\prod_{i=1}^{m} \|f_i\|_{L_{p_i}(2B)} \approx |B|^{m(1-\frac{\alpha}{nm})} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{nm+1-\alpha}}$$
$$\le |B|^{m(1-\frac{\alpha}{nm})} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t^{nm+1-\alpha}}$$

$$\lesssim \prod_{i=1}^{m} |B|^{\frac{1}{q_i}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i(B(x,t))}} |B(x,t)|^{-\frac{1}{p_i'}} \frac{dt}{t^{1-\alpha}}$$
$$\approx |B|^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} \, |B(x,t)|^{-\frac{1}{q_i}} \frac{dt}{t}. \tag{3.7}$$

Thus

$$J^0 \lesssim |B(x,r)|^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} \, |B(x,t)|^{-\frac{1}{q_i}} \frac{dt}{t}. \tag{3.8}$$

For the other terms, let us first deal with the case when $\beta_1 = \cdots = \beta_m = \infty$.

When $|x-y_i| \le r$, $|z-y_i| \ge 2r$, we have $\frac{1}{2}|z-y_i| \le |x-y_i| \le \frac{3}{2}|z-y_i|$, and therefore,

$$|T_{\alpha,m}(f_1^\infty,\dots,f_m^\infty)(z)| \lesssim \int_{\left(^{\complement}B(x,2r)\right)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{(|(x-y_1|+\ldots+|x-y_m|)^{mn-\alpha}} d\vec{y}$$
$$\lesssim \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i$$

and

$$\|T_{\alpha,m}(f_1^\infty,\dots,f_m^\infty)\|_{L_q(B(x,r))} \le \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \, \|\chi_{B(x,r)}\|_{L_q(\mathbb{R}^n)}$$
$$\lesssim |B(x,r)|^{1/q} \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i.$$

By Fubini's theorem we have

$$\int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \approx \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m |f_i(y_i)| dy_i \int_{|x-y_i|}^\infty \frac{dt}{t^{n+1-\alpha/m}}$$
$$\approx \int_{2r}^\infty \prod_{i=1}^m \int_{2r\le|x-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1-\alpha/m}} \lesssim \int_{2r}^\infty \prod_{i=1}^m \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1-\alpha/m}}.$$

Applying Hölder's inequality, we get

$$\int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \lesssim \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \, |B(x,t)|^{-1/p_i} \, \frac{dt}{t^{n+1-\alpha/m}}$$
$$\lesssim \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \, |B(x,t)|^{-1/q_i} \, \frac{dt}{t}, \tag{3.9}$$

where $1/p_i - 1/q_i = \alpha/(nm)$, $i = 1,\dots,m$.

Moreover, for all $p_i \in [1,\infty)$, $1/p_i - 1/q_i = \alpha/(nm)$, $i = 1,\dots,m$ the inequality

$$\|T_{\alpha,m}(f_1^\infty,\dots,f_m^\infty)\|_{L_q(B(x,r))} \lesssim r^{n/q} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \, t^{-n/q_i} \, \frac{dt}{t}. \tag{3.10}$$

is valid.

We now consider the cases when exactly l of the β_l's are ∞ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. To this end we may assume that $\beta_1 = \ldots = \beta_l = \infty$ and $\beta_{l+1} = \ldots = \beta_m = 0$. Recall the fact that $|x-y_i| \approx |z-y_i|$

for $z \in B(x,r)$, $y_i \in {}^{\complement}B(x,2r)$ and $1 \le i \le l$. We have

$$\begin{aligned}
&\big|T_{\alpha,m}(f_1^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)(z)\big| \\
&\lesssim \int_{\left({}^{\complement}B(x,2r)\right)^l} \int_{\left(B(x,2r)\right)^{m-l}} \frac{|f_1(y_1)\cdots f_m(y_m)|\, d\overrightarrow{y}}{(|x-y_1|+\ldots+|x-y_m|)^{mn-\alpha}} \\
&\lesssim \int_{\left({}^{\complement}B(x,2r)\right)^l} \frac{|f_1(y_1)\cdots f_l(y_l)|\, dy_1\ldots dy_l}{(|x-y_1|+\ldots+|x-y_l|)^{mn-\alpha}} \\
&\times \int_{\left(B(x,2r)\right)^{m-l}} |f_{l+1}(y_{l+1})\cdots f_m(y_m)|\, dy_{l+1}\ldots dy_m \\
&\lesssim \int_{\left({}^{\complement}B(x,2r)\right)^l} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i.
\end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned}
&\int_{\left({}^{\complement}B(x,2r)\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/l}} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
&\approx \int_{\left({}^{\complement}B(x,2r)\right)^l} \prod_{i=1}^{l} |f_i(y_i)| dy_i \int_{|x-y_i|}^{\infty} \frac{dt}{t^{n+1-\alpha/m}} \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
&\approx \int_{2r}^{\infty} \prod_{i=1}^{l} \int_{2r\le|x-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1-\alpha/m}} \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
&\lesssim \int_{2r}^{\infty} \prod_{i=1}^{l} \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1-\alpha/m}} \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i.
\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}
&\int_{{}^{\complement}(2B)} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
&\lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x,t))} |B(x,t)|^{1/p_i'} \frac{dt}{t^{n+1-\alpha/l}} \\
&\le \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-1/q_i} \frac{dt}{t}. \qquad (3.11)
\end{aligned}$$

From (3.11) we get

$$\begin{aligned}
&\|T_{\alpha,m}(f_1^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)\|_{L_q(B(x,r))} \\
&\lesssim |B(x,r)|^{1/q} \prod_{i=1}^{l} \int_{{}^{\complement}B(x,2r)} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \prod_{i=l+1}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
&\lesssim |B(x,r)|^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i}(B(x,t))} |B(x,t)|^{-1/q_i} \frac{dt}{t}.
\end{aligned}$$

We now proof the second part. For any ball $B = B(x,r) \subset \mathbb{R}^n$, decompose $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{2B}$, $2B = B(x,2r)$, $i = 1,\ldots,m$. Then for any given $\lambda > 0$, we can write

$$\begin{aligned}
&\left(\left\{y \in B(x,r) : \left|T_{\alpha,m}(\overrightarrow{f})(y)\right| > \lambda\right\}\right)^{1/q} \\
&\leq \left(\left\{y \in B(x,r) : \left|T_{\alpha,m}(f_1^0,\ldots,f_m^0)(y)\right| > \lambda/2^m\right\}\right)^{1/q} \\
&+ \sum{}' \left(\left\{y \in B(x,r) : \left|T_{\alpha,m}(f_1^{\beta_1},\ldots,f_m^{\beta_m})(y)\right| > \lambda/2^m\right\}\right)^{1/q} \\
&= J_*^0 + \sum{}' J_*^{\beta_1,\ldots,\beta_m},
\end{aligned}$$

where each term in $\sum'$ contains at least one $\beta_i \neq 0$. We have

$$\begin{aligned}
J_*^0 &= \|T_{\alpha,m}(\overrightarrow{f^0})\|_{WL_q(B(x,r))} \leq \|T_{\alpha,m}(\overrightarrow{f^0})\|_{WL_q(\mathbb{R}^n)} \\
&\lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} = \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,2r))}.
\end{aligned}$$

We have the following estimate:

$$\|T_{\alpha,m}(f_1^{\beta_1},\ldots,f_m^{\beta_m})\|_{L_q(B(x,r))} \lesssim r^{n/q} \int_{2r}^\infty \prod_{i=1}^m \|f\|_{L_{p_i}(B(x,t))} \, t^{-n/q_i} \, \frac{dt}{t}.$$

Then

$$\begin{aligned}
&\sum{}' J_*^{\beta_1,\ldots,\beta_m} = \sum{}' \|T_{\alpha,m}(f_1^{\beta_1},\ldots,f_m^{\beta_m})\|_{WL_q(B(x,r))} \\
&\leq \sum{}' \|T_{\alpha,m}(f_1^{\beta_1},\ldots,f_m^{\beta_m})\|_{L_q(B(x,r))} \lesssim r^{n/q} \int_{2r}^\infty \prod_{i=1}^m \|f\|_{L_{p_i}(B(x,t))} \, t^{-n/q_i} \, \frac{dt}{t}.
\end{aligned}$$

Now we give the boundedness of multi-sublinear fractional integral operators in product generalized local Morrey spaces.

Theorem 3.3 *Let* $x_0 \in \mathbb{R}^n$, $m \geq 2$, $0 < \alpha < mn$, $1 \leq p_1,\ldots,p_m < \infty$ *with* $1/p = 1/p_1 + \ldots + 1/p_m$, $1/q = 1/q_1 + \ldots + 1/q_m = 1/p - \alpha/n$, $1/q_i = 1/p_i - \alpha/(nm)$, $1 \leq i \leq m$ *and* $(\varphi_1,\ldots,\varphi_m,\varphi)$ *satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{q}+1}} dt \lesssim \varphi(x_0,r), \tag{3.12}$$

where the implicit constant does not depend on r.

If $T_{\alpha,m}$ *be a* $(\mathbf{p},q)$*-admissible multi-sublinear fractional integral operators, then the operator* $T_{\alpha,m}$ *is bounded from product space* $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ *to* $LM_{q,\varphi}^{\{x_0\}}$ *for* $p_i > 1$, $i = 1,\ldots,m$.

If $T_{\alpha,m}$ *be a weak* $(\mathbf{p},q)$*-admissible multi-sublinear fractional integral operators, then the operator* $T_{\alpha,m}$ *is bounded from product space* $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ *to* $WLM_{q,\varphi}^{\{x_0\}}$ *for* $p_i \geq 1$, $i = 1,\ldots,m$, $\min\{p_1,\ldots,p_m\} = 1$.

Proof. Let $1 < p_1, \ldots, p_m < \infty$ and $\overrightarrow{f} \in LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$. By Theorem 3.1 and Lemma 3.1 with $v_2(r) = \varphi(x_0, r)^{-1}$, $v_1(r) = \varphi(x_0, r)^{-1}$, we have

$$\|T_{\alpha,m}(\overrightarrow{f})\|_{LM^{\{x_0\}}_{q,\varphi}} \lesssim \sup_{r>0} \varphi(x_0, r)^{-1} \int_r^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))}\, t^{-\frac{n}{q_i}}\, \frac{dt}{t}$$
$$\lesssim \sup_{r>0} \prod_{i=1}^m \varphi_i(x_0, r)^{-1}\, r^{-\frac{n}{p_i}}\, \|f_i\|_{L_{p_i}(B(x_0,r))} = \prod_{i=1}^m \|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.$$

When $p_i \geq 1$, $i = 1, \ldots, m$, $\min\{p_1, \ldots, p_m\} = 1$, the proof is similar and we omit the details here.

From Theorem 3.3 we get the following corollary about boundedness of $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators on product generalized Morrey space.

Corollary 3.1 *Let $m \geq 2$, $0 < \alpha < mn$, $1 \leq p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$, $1/q = 1/q_1 + \ldots + 1/q_m = 1/p - \alpha/n$, $1/q_i = 1/p_i - \alpha/(nm)$, $1 \leq i \leq m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x, s) s^{\frac{n}{p_i}}}{t^{\frac{n}{q}}}\, \frac{dt}{t} \lesssim \varphi(x, r), \tag{3.13}$$

where the implicit constant does not depend on x and r.

If $T_{\alpha,m}$ be a $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators, then the operator $T_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{q,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$.

If $T_{\alpha,m}$ be a weak $(\mathbf{p}, q)$-admissible multi-sublinear fractional integral operators, then the operator $T_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{q,\varphi}$ for $p_i \geq 1$, $i = 1, \ldots, m$, $\min\{p_1, \ldots, p_m\} = 1$.

From Corollary 3.1 we get the following corollary proven in [26] (see also [27]) about boundedness of multilinear fractional integral operators on product generalized Morrey space.

Corollary 3.2 *Let $m \geq 2$, $0 < \alpha < mn$, $1 \leq p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$, $1/q = 1/q_1 + \ldots + 1/q_m = 1/p - \alpha/n$, $1/q_i = 1/p_i - \alpha/(nm)$, $1 \leq i \leq m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition* (3.13). *Then the operator $I_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{q,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$, and from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{q,\varphi}$ for $p_i \geq 1$, $i = 1, \ldots, m$, $\min\{p_1, \ldots, p_m\} = 1$.*

4 Commutators of $(\mathbf{p}, q)$-admissible multilinear fractional integral operators in the product spaces $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$

Let T be a linear operator. For a function b we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x)\, Tf(x) - T(bf)(x)$$

for any suitable function f. If $\widetilde{T}$ is a Calderón-Zygmund singular integral operator, a well known result of Coifman, Rochberg and Weiss [10] states that the commutator $[b, \widetilde{T}]f = b\widetilde{T}f - \widetilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [8, 9,11]). In [7], Chanillo proved that the commutator $[b,I_\alpha]f=b\,I_\alpha f-I_\alpha(bf)$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $(1<p<q<\infty,\ \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n})$ if and only if $b\in BMO(\mathbb{R}^n)$.

The definition of local Campanato space as follows.

Definition 4.1 *Let $1\le q<\infty$ and $0\le\lambda<\frac{1}{n}$. A function $f\in L_q^{\rm loc}(\mathbb{R}^n)$ is said to belong to the $CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (central Campanato space), if*

$$\|f\|_{CBMO_{q,\lambda}^{\{x_0\}}}=\sup_{r>0}\Big(\frac{1}{|B(x_0,r)|^{1+\lambda q}}\int_{B(x_0,r)}|f(y)-f_{B(x_0,r)}|^q dy\Big)^{1/q}<\infty,$$

where

$$f_{B(x_0,r)}=\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}f(y)dy.$$

Define

$$CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)=\{f\in L_q^{\rm loc}(\mathbb{R}^n)\ :\ \|f\|_{CBMO_{q,\lambda}^{\{x_0\}}}<\infty\}.$$

In [33], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n)=CBMO_{q,0}^{\{0\}}(\mathbb{R}^n)$. Note that, $BMO(\mathbb{R}^n)\subset CBMO_q^{\{x_0\}}(\mathbb{R}^n)$, $1\le q<\infty$. The space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1\le q<\infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{r>0}\Big(\frac{1}{|B|}\int_B|f(y)-f_B|^q dy\Big)^{1/q}<\infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ depends on q. If $q_1<q_2$, then $CBMO_{q_2}^{\{x_0\}}(\mathbb{R}^n)\subsetneqq CBMO_{q_1}^{\{x_0\}}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Lemma 4.1 *[23,24,34] Let b be a function in $CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1\le q<\infty$, $0\le\lambda<\frac{1}{n}$ and $r_1,r_2>0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}}\int_{B(x_0,r_1)}|b(y)-b_{B(x_0,r_2)}|^q dy\right)^{\frac{1}{q}}\le C\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_{CBMO_{q,\lambda}^{\{x_0\}}},$$

where $C>0$ is independent of b, r_1 and r_2.

In [30], in the case $b_i\in CBMO_{q_i,\lambda_i}^{\{x_0\}}$, for $0<\lambda_i<1/n$, $i=1,2,\dots,m$, the sufficient conditions on the pair $(\varphi_1,\dots,\varphi_m,\varphi)$ which ensures the boundedness of the commutator of multilinear singular integral operators $T_m^{\vec{b}}$ from $LM_{p_1,\varphi_1}^{\{x_0\}}\times\ldots\times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$, $1<p,p_i,q_i<\infty$, for $i=1,2,\dots,m$ such that $1/p=1/p_1+\ldots+1/p_n+1/q_1+\ldots+1/q_n$ were found.

In this section, in the case $b_i\in CBMO_{q_i,\lambda_i}^{\{x_0\}}$, for $0<\lambda_i<1/n$, $i=1,2,\dots,m$, we find the sufficient conditions on the pair $(\varphi_1,\dots,\varphi_m,\varphi)$ which ensures the boundedness of the commutator of $(\mathbf{p},q)$-admissible multilinear fractional integral operators $T_{\alpha,m}^{\mathbf{b}}$ from $LM_{p_1,\varphi_1}^{\{x_0\}}\times\ldots\times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{q,\varphi}^{\{x_0\}}$, $1<p,p_i,q_i<\infty$, for $i=1,2,\dots,m$ such that $1/q=1/p_1+\ldots+1/p_n+1/q_1+\ldots+1/q_n-\alpha/n$.

Lemma 4.2 *Let* $x_0 \in \mathbb{R}^n$, $m \geq 2$, $0 < \alpha < mn$, $1 < p, p_i, q_i < \infty$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ *for* $0 \leq \lambda_i < \frac{1}{n}$, $i = 1, 2, \ldots, m$ *and* $\frac{1}{q} = \frac{1}{p_1} + \ldots + \frac{1}{p_m} + \frac{1}{q_1} + \ldots + \frac{1}{q_m} - \frac{\alpha}{n}$. *Then, for the commutator of* $(\mathbf{p}, q)$*-admissible multilinear fractional integral operators* $T^{\mathbf{b}}_{\alpha,m}$ *the following inequality*

$$\|T^{\mathbf{b}}_{\alpha,m}(\mathbf{f})\|_{L^q(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big)^m \times t^{n\sum\limits_{i=1}^m \lambda_i + n\sum\limits_{i=1}^m \frac{1}{q_i} - \frac{n}{q} - 1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt$$

holds for all $B(x_0, r)$ *and all* $f_i \in L^{p_i}_{loc}(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$.

Proof. Without loss of generality, it is suffice for us to show that the conclusion holds for $m = 2$.

Let $B = B(x_0, r)$, $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where $f_i^0 = f_i \chi_{2B}$ and $f_i^\infty = f_i \chi_{(2B)^c}$, for $i = 1.2$. Thus, we have

$$T^{(b_1,b_2)}_{\alpha,2}(f_1, f_2)(x) = T^{(b_1,b_2)}_{\alpha,2}(f_1^0, f_2^0)(x) + T^{(b_2,b_2)}_{\alpha,2}(f_1^0, f_2^\infty)(x) + \\ + T^{(b_1,b_2)}_{\alpha,2}(f_1^\infty, f_2^0)(x) + T^{(b_1,b_2)}_{\alpha,2}(f_1^\infty, f_2^\infty)(x).$$

So,

$$\begin{aligned}\|T^{(b_1,b_2)}_{\alpha,2}(f_1, f_2)\|_{L_q(B)} &\leq \|T^{(b_1,b_2)}_{\alpha,2}(f_1^0, f_2^0)\|_{L_q(B)} + \|T^{(b_1,b_2)}_{\alpha,2}(f_1^0, f_2^\infty)\|_{L_q(B)} \\ &+ \|T^{(b_1,b_2)}_{\alpha,2}(f_1^\infty, f_2^0)\|_{L_q(B)} + \|T^{(b_1,b_2)}_{\alpha,2}(f_1^\infty, f_2^\infty)\|_{L_q(B)} \\ &=: I + II + III + IV.\end{aligned}$$

Let us estimate I, II, III and IV, respectively.
Since

$$\begin{aligned}&(b_1(x) - b_1(y))(b_2(x) - b_2(y)) \\ &= (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B) - (b_1(x) - (b_1)_B)(b_2(y) - (b_2)_B) \\ &- (b_1(y) - (b_1)_B)(b_2(x) - (b_2)_B) + (b_1(y) - (b_1)_B)(b_2(y) - (b_2)_B),\end{aligned} \tag{4.1}$$

then

$$\begin{aligned}&\|T^{(b_1,b_2)}_{\alpha,2}(f_1^0, f_2^0)\|_{L_q(B)} = \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)T_2(f_1^0, f_2^0)\|_{L_q(B)} \\ &+ \|(b_1 - (b_1)_B)T_{\alpha,2}(f_1^0, (b_2 - (b_2)_B)f_2^0)\|_{L_q(B)} \\ &+ \|(b_2 - (b_2)_B)T_2((b_1 - (b_1)_B)f_1^0, f_2^0)_{L_q(B)} \\ &+ \|T_{\alpha,2}((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^0)\|_{L_q(B)} =: T_1 + T_2 + T_3 + T_4.\end{aligned} \tag{4.2}$$

Let $1 < \overline{p}, \overline{q} < \infty$, such that $\frac{1}{\overline{p}} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{\overline{q}} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q} = \frac{1}{r} + \frac{1}{\overline{q}}$ and $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, using the Hölder's inequality and by the boundedness of $T_{\alpha,2}$ from product

$L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n)$ to $L_{\overline{q}}(\mathbb{R}^n)$, $0 < \alpha < 2n$ with $1/\overline{q} = 1/p_1 + 1/p_2 - \alpha/n$ for each $p_i > 1 (i = 1, 2)$, we have

$$\begin{aligned}
T_1 &\lesssim \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)\|_{L_{\overline{\tau}}(B)} \|T_{\alpha,2}(f_1^0, f_2^0)\|_{L_{\overline{q}}(B)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L_{q_2}(B)} \|b_2 - (b_2)_B\|_{L_{q_2}(B)} \|f_1\|_{L_{p_1}(2B)} \|f_2\|_{L_{p_1}(2B)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|b_2 - (b_2)_B\|_{L_{q_2}(B)} r^{\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}\right)n} \\
&\times \int_{2r}^{\infty} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} \frac{dt}{t^{\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}\right)n+1}} \\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\
&\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.
\end{aligned} \tag{4.3}$$

Let $1 < \tau < \infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\tau}$. Then similar to the estimate of (4.3), we have

$$\begin{aligned}
T_2 &\lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|T_{\alpha,2}(f_1^0, (b_2 - (b_2)_B) f_2^0)\|_{L_\tau(B)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|f_1^0\|_{L_{p_1}(\mathbb{R}^n)} \|(b_2 - (b_2)_{2B}) f_2^0\|_{L_s(\mathbb{R}^n)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|b_2 - (b_2)_B\|_{L_{q_2}(2B)} \|f_1\|_{L_{p_1}(2B)} \|f_2\|_{L_{p_2}(2B)},
\end{aligned} \tag{4.4}$$

where $1 < s < \frac{2n}{\alpha}$, such that $\frac{1}{s} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{\tau} - \frac{1}{p_1} + \frac{\alpha}{n}$.

From Lemma 4.1, it is easy to see that

$$\|b_i - (b_i)_B\|_{L_{q_i}(B)} \leq C r^{\frac{n}{q_i} + n\lambda_i} \|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}},$$

and

$$\begin{aligned}
\|b_i - (b_i)_B\|_{L_{q_i}(B)} &\leq \|b_i - (b_i)_{2B}\|_{L_{q_i}(2B)} + \|(b_i)_B - (b_i)_{2B}\|_{L_{q_i}(2B)} \\
&\leq C r^{\frac{n}{q_i} + n\lambda_i} \|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}},
\end{aligned} \tag{4.5}$$

for $i = 1, 2$.

Then,

$$\begin{aligned}
T_2 &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\
&\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
T_3 &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\
&\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.
\end{aligned}$$

Moreover, let $1 < \tau_1, \tau_2 < \frac{2n}{\alpha}$, such that $\frac{1}{\tau_1} = \frac{1}{p_1} + \frac{1}{q_1}$ and $\frac{1}{\tau_2} = \frac{1}{p_2} + \frac{1}{q_2}$. It is easy to see that $\frac{1}{q} = \frac{1}{\tau_1} + \frac{1}{\tau_2} - \frac{\alpha}{n}$. Then by the boundedness of $T_{\alpha,2}$ from product $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n)$

to $L_{\overline{q}}(\mathbb{R}^n)$, $0<\alpha<2n$ with $1/\overline{q}=1/p_1+1/p_2-\alpha/n$ for each $p_i>1 (i=1,2)$, Hölder's inequality and inequality (4.5), we obtain

$$\begin{aligned} T_4 &\lesssim \|(b_1-(b_1)_B)f_1^0\|_{L_{\tau_1}(\mathbb{R}^n)}\|(b_2-(b_2)_B)f_2^0\|_{L_{\tau_2}(\mathbb{R}^n)} \\ &\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(2B)}\|b_2-(b_2)_B\|_{L_{q_2}(2B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_2}(2B)} \\ &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \\ &\times \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned} \tag{4.6}$$

Therefore, combining the estimates of T_1, T_2, T_3 and T_4, we have

$$\begin{aligned} I &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Let us estimate II.

Similarly with (4.2), we have

$$\begin{aligned} &\|T^{(b_1,b_2)}_{\alpha,2}(f_1^0,f_2^\infty)\|_{L_p(B)} = \|(b_1-(b_1)_B)(b_2-(b_2)_B)T_{\alpha,2}(f_1^0,f_2^\infty)\|_{L_p(B)} \\ &+\|(b_1-(b_1)_B)T_{\alpha,2}(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \\ &+\|(b_2-(b_2)_B)T_{\alpha,2}((b_1-(b_1)_B)f_1^0,f_2^\infty)_{L_p(B)} \\ &+\|T_{\alpha,2}((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} =: II_1+II_2+II_3+II_4. \end{aligned} \tag{4.7}$$

Let $1<\overline{p},\overline{q}<\infty$, such that $\frac{1}{\overline{p}}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{\overline{q}}=\frac{1}{q_1}+\frac{1}{q_2}$. Then, using Hölder's inequality we have

$$\begin{aligned} II_1 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_{2B})\|_{L_{\overline{q}}(B)}\|T_{\alpha,2}(f_1^0,f_2^\infty)\|_{L_{\overline{p}}(B)} \\ &\lesssim \|b_1-(b_1)_B\|_{L_{q_2}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_1}(2B)} \\ &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{(\frac{1}{q_1}+\frac{1}{q_2})n+(\lambda_1+\lambda_2)n} r^{\big(\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}\big)n} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-\big(\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}\big)n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt \\ &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \\ &\times \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned} \tag{4.8}$$

Moreover, we have

$$\begin{aligned} &|T_{\alpha,2}(f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \\ &\lesssim \int_{2B}|f_1(y_1)|dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n-\alpha}}dy_2. \end{aligned}$$

It's obvious that

$$\int_{2B} |f_1(y_1)|dy_1 \lesssim \|f_1\|_{L_{p_1}(2B)} |2B|^{1-1/p_1} \tag{4.9}$$

and

$$\begin{aligned}
&\int_{(2B)^c} \frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n-\alpha}} dy_2 \\
&\lesssim \int_{(2B)^c} |b_2(y_2)-(b_2)_B||f_2(y_2)| \Big[\int_{|x_0-y_2|}^{\infty} \frac{dt}{t^{2n-\alpha+1}} \Big] dy_2 \\
&\lesssim \int_{2r}^{\infty} \|b_2(y_2)-(b_2)_{B(x_0,t)}\|_{L_{q_2}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} |B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})} \frac{dt}{t^{2n-\alpha+1}} \\
&+ \int_{2r}^{\infty} \||(b_2)_{B(x_0,t)} - (b_2)_{B(x_0,r)}|\|f_2\|_{L_{p_2}(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
&\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} |B(x_0,t)|^{\frac{1}{q_2}+\lambda_2} \|f_2\|_{L_{p_2}(B(x_0,t))} |B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})} \frac{dt}{t^{2n-\alpha+1}} \\
&+ \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) |B(x_0,t)|^{\lambda_2} \|f_2\|_{L_{p_2}(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
&\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{-n+n\lambda_2-\frac{n}{p_2}-1+\alpha} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.
\end{aligned} \tag{4.10}$$

Therefore, from (4.9) and (4.10) it follows that

$$\begin{aligned}
&|T_{\alpha,2}(f_1^0, (b_2-(b_2)_B)f_2^{\infty})(x)| \\
&\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \|f_1\|_{L_{p_1}(2B)} |2B|^{1-\frac{1}{p_1}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{-n+n\lambda_2-\frac{n}{p_2}-1+\alpha} \|f_2\|_{L_{p_2}(B(x_0,t))} dt \\
&\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1+\alpha} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.
\end{aligned}$$

Thus, let $1<\tau<\infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\tau}$, then similarly to the estimate of (4.3), we have

$$\begin{aligned}
II_2 &= \|(b_1-(b_1)_B)T_{\alpha,2}(f_1^0, (b_2-(b_2)_B)f_2^{\infty})\|_{L_q(B)} \\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)} \|T_2(f_1^0, (b_2-(b_2)_B)f_2^{\infty})\|_{L_\tau(B)} \\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} |B|^{\lambda_1+\frac{1}{q_1}+\frac{1}{\tau}} \\
&\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1+\alpha} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt \\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \\
&\int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \\
&\times \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.
\end{aligned} \tag{4.11}$$

Similarly, we have

$$II_3 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Let us estimate II_4.
Since,

$$|T_{\alpha,2}((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \lesssim \int_{2B} |b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1 \int_{(2B)^c} \frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n-\alpha}} dy_2,$$

and

$$\int_{2B} |b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} |B|^{\lambda_1+1-\frac{1}{p_1}} \|f_1\|_{L_{p_1}(2B)}. \tag{4.12}$$

Then, by (4.10) and (4.13), we get

$$|T_{\alpha,2}((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n})-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Therefore,

$$II_4 = \|T_{\alpha,2}((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Combining the estimates of II_1-II_4, we have

$$II \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Similarly,

$$III \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

For IV we have

$$\begin{aligned}
&\|T_{\alpha,2}^{(b_1,b_2)}(f_1^\infty, f_2^\infty)\|_{L_p(B)} \\
&\le \|(b_1-(b_1)_B)(b_2-(b_2)_B)T_{\alpha,2}(f_1^\infty, f_2^\infty)\|_{L_p(B)} \\
&+ \|(b_1-(b_1)_B)T_{\alpha,2}(f_1^\infty,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \\
&+ \|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^\infty, f_2^\infty)_{L_p(B)} \\
&+ \|T_{\alpha,2}((b_1-(b_1)_B)f_1^\infty,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \\
&=: IV_1 + IV_2 + IV_3 + IV_4.
\end{aligned}$$

Let us estimate IV_1, IV_2, IV_3 and IV_4, respectively.

Let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{\tau}$. Then, from Hölder's inequality and inequality (3.5), we get

$$\begin{aligned}
IV_1 &\lesssim \|(b_1-(b_1)_B)\|_{L_{q_1}(B)}\|(b_2-(b_2)_B)\|_{L_{q_2}(B)}\|T_{\alpha,2}(f_1^\infty, f_2^\infty)\|_{L_\tau(B)} \\
&\lesssim \|b_1\|_{CBMO_{q_1,\lambda_1}^{\{x_0\}}}\|b_2\|_{CBMO_{q_2,\lambda_2}^{\{x_0\}}}|B|^{(\lambda_1+\lambda_2)+(\frac{1}{q_1}+\frac{1}{q_2})+\frac{1}{\tau}} \\
&\times \int_{2r}^\infty \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}t^{-n(\frac{1}{p_1}+\frac{1}{p_2})-1+\alpha}dt \\
&\lesssim \|b_1\|_{CBMO_{q_1,\lambda_1}^{\{x_0\}}}\|b_2\|_{CBMO_{q_2,\lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&|T_{\alpha,2}(f_1^\infty,(b_2-(b_2)_B)f_2^\infty(x)| \\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|}{(|x_0-y_1|+|x_0-y_2|)^{2n-\alpha}}dy_1dy_2 \\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}|f_1(y_1)||b_2(y_2)-(b_2)_B||f_2(y_2)|\Big[\int_{|x_0-y_1|+|x_0-y_2|}^\infty\frac{dt}{t^{2n+1}}\Big]dy_1dy_2 \\
&\lesssim \int_{2r}^\infty\Big[\int_{B(x_0,t)}|f_1(y_1)|dy_1\Big]\Big[\int_{B(x_0,t)}|b_2(y_2)-(b_2)||f_2(y_2)|dy_2\Big]\frac{dt}{t^{2n-\alpha+1}}.
\end{aligned}$$

Since,

$$\int_{B(x_0,t)}|f_1(y_1)|dy_1 \lesssim \|f_1\|_{L_{p_1}(B(x_0,t))}t^{n(1-\frac{1}{p_1})},$$

and

$$\int_{B(x_0,t)} |b_2(y_2) - (b_2)_B||f_2(y_2)|\, dy_2$$
$$\lesssim \|b_2(y_2) - (b_2)_{B(x_0,t)}\|_{L_{q_2}(B(x_0,t))} \|f_2\|_{L_{p_2}} |B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}$$
$$+ |(b_2)_{B(x_0,t)} - (b_2)_{B(x_0,r)}| \|f_2\|_{L_{p_2}} |B(x_0,t)|^{1-\frac{1}{p_2}}$$
$$\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} |B(x_0,t)|^{\frac{1}{q_2}+\lambda_2} \|f_2\|_{L_{p_2}} |B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}$$
$$+ \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \Big(1+\ln\frac{t}{r}\Big) |B(x_0,t)|^{\lambda_2} \|f_2\|_{L_{p_2}} |B(x_0,t)|^{1-\frac{1}{p_2}}$$
$$\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-\frac{n}{p_2}+n} \|f_2\|_{L_{p_2}(B(x_0,t))}.$$

Then,

$$|T_{\alpha,2}(f_1^\infty, (b_2-(b_2)_B)f_2^\infty)(x)|$$
$$\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}$$
$$\times \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt. \tag{4.13}$$

Let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{\tau}$. Then, from Hölder's inequality and inequality (4.13), we have

$$IV_2 \lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B)} \|T_{\alpha,2}(f_1^\infty, (b_2-(b_2)_B)f_2^\infty)\|_{L_\tau(B)}$$
$$\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}$$
$$\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Similarly,

$$IV_3 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}$$
$$\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{q_1}+\frac{1}{q_2}\big)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Similar to the estimate of (4.13), we have

$$|T_{\alpha,2}(b_1-(b_1)_B)f_1^\infty, (b_2-(b_2)_B)f_2^\infty(x)|$$
$$\lesssim \int_{(2B)^c} \int_{(2B)^c} |b_1(y_1) - (b_1)_B||b_2(y_2) - (b_2)_B||f_1(y_1)||f_2(y_2)|$$
$$\times \Big[\int_{|x_0-y_1|+|x_0-y_2|}^\infty \frac{dt}{t^{2n-\alpha+1}}\Big] dy_1 dy_2 \lesssim \int_{2r}^\infty \Big[\int_{B(x_0,t)} |f_1(y_1) - (b_1)_B||f_1(y_1)| dy_1\Big]$$
$$\times \Big[\int_{B(x_0,t)} |b_2(y_2) - (b_2)_B||f_2(y_2)| dy_2\Big] \frac{dt}{t^{2n-\alpha+1}}$$
$$\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}$$
$$\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)-n\big(\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Thus,

$$IV_4 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} r^{\frac{n}{q}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Then, from the estimate of IV_1-IV_4, we deduced that

$$IV \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

So, combining the estimates of I, II, III and IV, we have

$$\|T^{(b_1,b_2)}_{\alpha,2}(f_1,f_2)\|_{L_p(B)} \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{q_1}+\frac{1}{q_2}\right)-\frac{n}{q}-1}$$
$$\times \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \qquad (4.14)$$

Therefore, we complete the proof of Theorem 4.2.

Now we give the boundedness of the commutator of $(\mathbf{p},q)$-admissible multilinear fractional integral operators $T^{\mathbf{b}}_{\alpha,m}$ in product generalized local Morrey spaces.

Theorem 4.1 *Let $x_0 \in \mathbb{R}^n$, $m \geq 2$, $0 < \alpha < mn$, $1 < p_1,\ldots,p_m < nm/\alpha$ with $1/p = 1/p_1+\ldots+1/p_m$, $1/q = 1/q_1+\ldots+1/q_m = 1/p-\alpha/n$, $1/q_i = 1/p_i-\alpha/(nm)$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ for $0 \leq \lambda_i < \frac{1}{n}$, $1 \leq i \leq m$ and $(\varphi_1,\ldots,\varphi_m,\varphi)$ satisfies the condition*

$$\int_r^{\infty} \Big(1+ln\frac{t}{r}\Big)^m t^{n\sum\limits_{i=1}^m \lambda_i + n\sum\limits_{i=1}^m \frac{1}{q_i}-\frac{n}{q}} \operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0,s)s^{\frac{n}{p_i}}\frac{dt}{t} \lesssim \varphi(x_0,r), \qquad (4.15)$$

where the implicit constant does not depend on r.

Then the commutator of $(\mathbf{p},q)$-admissible multilinear fractional integral operator $T^{\mathbf{b}}_{\alpha,m}$ is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$.

Proof. The statement of Theorem 4.1 follows from Lemma 4.2 and Theorem 3.2 in the same manner as Theorem 3.3.

From Theorem 4.1 we get the boundedness of multilinear fractional integral operators in product generalized local Morrey spaces.

Corollary 4.1 *Let $x_0 \in \mathbb{R}^n$, $m \geq 2$, $0 < \alpha < mn$, $1 < p_1,\ldots,p_m < nm/\alpha$ with $1/p = 1/p_1+\ldots+1/p_m$, $1/q = 1/q_1+\ldots+1/q_m = 1/p-\alpha/n$, $1/q_i = 1/p_i-\alpha/(nm)$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ for $0 \leq \lambda_i < \frac{1}{n}$, $1 \leq i \leq m$ and $(\varphi_1,\ldots,\varphi_m,\varphi)$ satisfies the condition* (4.15). *Then the operator $I^{\overrightarrow{b}}_{\alpha,m}$ is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1}\times\ldots\times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$.*

References

1. Akbulut, A., Guliyev, V.S., Mustafayev, R.: *Boundedness of the maximal operator and singular integral operator in generalized Morrey spaces*, Math. Bohem. **137** (1), 27–43 (2012).
2. Burenkov, V.I., Guliyev, H.V.: *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*. Studia Math. **163** (2), 157–176 (2004).
3. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces*, J. Comput. Appl. Math. **208** (1), 280-301 (2007).
4. Burenkov, V.I., Guliyev, V.S.: *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **30** (3), 211–249 (2009).
5. Burenkov, V.I., Gogatishvili, A., Guliyev, V.S., Mustafayev, R.: *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55** (8-10), 739–758 (2010).
6. Burenkov, V.I., Gogatishvili, A., Guliyev, V.S., Mustafayev, R.: *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35** (1), 67–87 (2011).
7. Chanillo, S.: *A note on commutators*, Indiana Univ. Math. J. **23**, 7–16 (1982).
8. Chiarenza, F., Frasca, M., Longo, P.: *Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40**, 149-168 (1991).
9. Chiarenza, F., Frasca, M., Longo, P.: *$W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336**, 841–853 (1993).
10. Coifman, R., Rochberg, R., Weiss, G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (2), 611–635 (1976).
11. Di Fazio, G., Ragusa, M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241–256 (1993).
12. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. *Princeton Univ. Press, Princeton, NJ* (1983).
13. Grafakos, L.: Classical and Modern Fourier Analysis, *Pearson Education, Inc. Upper Saddle River, New Jersey* (2004).
14. Grafakos, L.: *On multilinear fractional integrals*, Studia Math. **102**, 49–56 (1992).
15. Grafakos, L., Kalton, N.: *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math., **52**, 169–179 (2001).
16. Guliyev, V.S.: *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$*, Doctor's degree dissertation, Mat. Inst. Steklov, Moscow (1994), 329p. (in Russian)
17. Guliyev, V.S.: Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications, *Casioglu, Baku*, 332 p. (1999) (in Russian)
18. Guliyev, V.S., Nazirova, Sh.A.: *A rearrangement estimate for the generalized multilinear fractional integrals*, Siberian Math. J, **48**, 463–470 (2007).
19. Guliyev, V.S., Nazirova, Sh.A.: *O'Neil inequality for multilinear convolutions and some applications*, Integral Equations Operator Theory, **60**, 485–497 (2008).
20. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. (2009), Art. ID 503948, 20 p.
21. Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.S.: *Boundedness of sublinear operators and commutators on generalized Morrey Space*, Integral Equations Operator Theory. **71** (3), 327–355 (2011).

22. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3), 33–61 (2012).
23. Guliyev, V.S.: *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3** (2), 79–94 (2013).
24. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193** (2), 211–227 (2013).
25. Guliyev, V.S., Ismayilova, A.F.: *Multi-sublinear maximal operator and multilinear singular integral operators on generalized Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2), 65–77 (2014).
26. Guliyev, V.S., Ismayilova, A.F.: *Multi-sublinear fractional maximal operator and multilinear fractional integral operators on generalized Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2), 22–33 (2014).
27. Guliyev, V.S., Omarova, M.N.: *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math. **5** (1), 104–132 (2015).
28. Iida, T., Sato, E., Sawano, Y., Tanaka, H.: *Multilinear fractional integrals on Morrey spaces*, Acta Math. Sinica (Engl. Ser) **28**, 1375–1384 (2012).
29. Iida, T., Sato, E., Sawano, Y., Tanaka, H.: *Sharp bounds for multilinear fractional integral operators on Morrey type spaces*, Positivity **16**, 339–358 (2012).
30. Hasanov, A.A., Muradova, Sh.A., Gadjieva, E.A.: *Commutators of multilinear singular integral operators on generalized local Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **35** (4), 61–74 (2015).
31. Kenig, C.E., Stein, E.M.: *Multilinear estimates and fractional integration*, Math. Res. Lett. **6**, 1–15 (1999).
32. Kufner, A., John, O., Fuçik, S.: Function Spaces, *Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague* (1977).
33. Lu, S.Z., Yang, D.C.: *The central BMO spaces and Littlewood-Paley operators*, Approx. Theory Appl. (N.S.) **11**, 72–94 (1995).
34. Lu, S.Z., Wu, Q.: *CBMO estimates for commutators and multilinear singular integrals*, Math. Nachr. **276**, 75–88 (2004).
35. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183–189 (1991).
36. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126–166 (1938).
37. Nakai, E.: *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95–103 (1994).
38. Sawano, Y.: *Generalized Morrey space for non-doubling measures*, Nonlinear Differential Equations Appl. **15**, 413–425 (2008).