Transactions of NAS of Azerbaijan, Issue Mathematics, **36** (4), 99-107 (2016).
Series of Physical-Technical and Mathematical Sciences.

Multilinear rough fractional integral on product modified Morrey spaces

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Received: 12.12.2015 / Revised: 23.07.2016 / Accepted: 10.08.2016

Abstract. We will study the boundedness of multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernels $\Omega \in L^s(\mathbb{S}^{mn-1})$, $1 < s \leq \infty$ on product modified Morrey spaces. We find for the operator $I_{\Omega,\alpha,m}$ necessary and sufficient conditions on the parameters of the boundedness on product modified Morrey spaces $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey spaces $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

Keywords. Multilinear fractional integral operator; modified Morrey spaces.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35.

1 Introduction

Multilinear analysis is considered as a very efficacious research area in studying harmonic analysis, geometric functions theory and univalent functions theory. Recently, fractional calculus in complex domains has confirmed delectable enforcements in the geometric function theory.

The classical Morrey spaces, introduced by Morrey [15] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. They appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See [3–5] for details. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca *et al.* [2]. In [2], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces. The fractional type operators and their boundedness theory play important roles in harmonic analysis and other fields, and the multilinear operators arise in numerous situations involving product-like operations.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Also for $\overrightarrow{x} = (x_1, \ldots, x_m) \in \mathbb{R}^{mn}$ and $r > 0$, we denote by $B(\overrightarrow{x}, r)$ the open ball centered at $\overrightarrow{x} \in \mathbb{R}^{mn}$ of radius r. We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_m)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

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Definition 1.1 *Let $1 \le p < \infty$, $0 \le \lambda \le n$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{WL_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))}$$

respectively.

Definition 1.2 *Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[r]_1 = \min\{1, r\}$. We denote by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, and by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the weak modified Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{W\widetilde{L}_p(B(x,r))}$$

respectively.

Note that

$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset\!\succ L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \quad \max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \le \|f\|_{\widetilde{L}_{p,\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In 1999, Kenig and Stein [14] studied the following multilinear fractional integral,

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{nm-\alpha}} dy_1 dy_2 \dots dy_m,$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n > 0$ for each $p_i > 1 (i = 1, \dots, m)$. If some $p_i = 1$, then $I_{\alpha,m}$ is bounded $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$. Obviously, the multilinear fractional integral $I_{\alpha,m}$ is a natural generalization of the classical fractional integral $I_\alpha \equiv I_{\alpha,1}$.

Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multi-sublinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ with rough kernels Ω is defined by

$$\mathcal{M}_{\Omega,\alpha,m}(\overrightarrow{f})(x) = \sup_{r>0} \frac{1}{r^{nm-\alpha}} \int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^{m} |f_i(x-y_i)| d\overrightarrow{y}, \qquad 0 \le \alpha < nm.$$

If $m = 1$, then $M_{\Omega,\alpha} \equiv \mathcal{M}_{\Omega,\alpha,1}$ is the fractional maximal operator with rough kernel Ω. When $m = 1$ and $\Omega \equiv 1$, then $M_\alpha \equiv \mathcal{M}_{1,\alpha,1}$ is the classical fractional maximal operator.

Note that, in [8], [9] Guliyev, Ismayilova was study the boundedness of multi-sublinear fractional maximal operator, multilinear fractional integral operator and multilinear singular integral opertors on product generalized Morrey spaces.

In [13] we proved the boundedness of the multi-sublinear fractional maximal operator with rough kernels $\mathcal{M}_{\Omega,\alpha,m}$ from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \dots, p_m < \infty$,

$1/p = 1/p_1 + \ldots + 1/p_m$, $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$ and from the space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \ldots + 1/p_m$, $\alpha/n \le 1/s' - 1/q \le \alpha/(n-\lambda)$ and at least one exponent p_i, $1 \le i \le m$ equals one.

In this work, we prove the boundedness of the multilinear fractional maximal operator with rough kernels $I_{\Omega,\alpha,m}$ from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \ldots + 1/p_m$, $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$ and from the space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \ldots + 1/p_m$, $\alpha/n \le 1/s' - 1/q \le \alpha/(n-\lambda)$ and at least one exponent p_i, $1 \le i \le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2 Multilinear fractional integral operator $I_{\Omega,\alpha,m}$ on product modified Morrey spaces

In this part, we investigate the boundedness of multilinear fractional integral operator $I_{\Omega,\alpha,m}$ on product modified Morrey spaces.

Spanne and Adams obtained two remarkable results on Morrey spaces (see Definition 1.1 of the Morrey spaces in Section 1) for I_α. Their results can be summarized as follows.

Theorem 2.1 *[10, 16] **(Spanne, but published by Peetre)** Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$ and $\mu/q = \lambda/p$. Then for $p > 1$, the operators M_α and I_α are bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, I_α is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.*

Theorem 2.2 *[1, 7] Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) If $p > 1$, then condition $1/p - 1/q = \alpha(n - \lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.
(ii) If $p = 1$, then condition $1 - 1/q = \alpha/(n - \lambda)$ s necessary and sufficient for the boundedness of the operator I_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.

If $\lambda = 0$, then the statement of Theorems 2.1 and 2.2 reduces to the well known Hardy-Littlewood-Sobolev inequality.

On the other hand, in 2011, Guliyev, Hasanov and Zeren [7] found this inequality in modified Morrey spaces (see Definition 1.2 in Section 1) is also valid and proved that

Theorem 2.3 *[7] Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) If $p > 1$, then condition $\alpha/n \le 1/p - 1/q \le \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator M_α from $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.
(ii) If $p = 1$, then condition $\alpha/n \le 1 - 1/q \le \alpha/(n - \lambda)$ s necessary and sufficient for the boundedness of the operator M_α from $\widetilde{L}^{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

When $m \ge 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, in [11] was find out $\mathcal{M}_{\Omega,m}$ also have the same properties by providing the following multi-version result of the Guliyev, Hasanov and Zeren [7].

Theorem 2.4 *[11] Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \;\; for \;\; 0 \le \lambda_j < n. \tag{2.1}$$

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{W\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

Lemma 2.1 *[17] Let $0 < \alpha < mn$, $1 \leq s' < mn/\alpha$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$ and $f \in L^{p_1}(\mathbb{R}^n) \times \ldots \times L^{p_m}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ for any $x \in \mathbb{R}^n$*

$$\Big|I_{\Omega,\alpha,m}\boldsymbol{f}(x)\Big| \leq C\Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\boldsymbol{f}(x)\Big]^{\frac{1}{2}} \Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\boldsymbol{f}(x)\Big]^{\frac{1}{2}}. \tag{2.2}$$

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, in [13] was proved that for $I_{\Omega,\alpha,m}$ also have the same properties by providing the following multi-version of the Theorem 2.2.

Theorem 2.5 *Let $0 < \alpha < mn$, $1 < s \leq \infty$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$. Let also $\sum_{j=1}^{m} \frac{\lambda_j}{p_j} = \frac{\lambda}{p}$, $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{m(n-\lambda_j)}$ and $0 \leq \lambda_j < n - \frac{\alpha p_j}{m}$, $j = 1, \ldots, m$.*

(i) If $p > s'$ and $\sum_{j=1}^{m} \frac{\lambda_j}{q_j} = \frac{\lambda}{q}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{L^{q,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$ and $\lambda \sum_{j=1}^{m} \frac{1}{p_j q_j} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j q_j}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak Morrey space $WL^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{WL^{q,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, we find out $I_{\Omega,\alpha,m}$ also have the same properties by providing the following multi-version of the Theorem 2.3.

Theorem 2.6 *Let $0 < \alpha < mn$, $1 < s \leq \infty$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$. Let also $\sum_{j=1}^{m} \frac{\lambda_j}{p_j} = \frac{\lambda}{p}$, $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{m(n-\lambda_j)}$ and $0 \leq \lambda_j < n - \frac{\alpha p_j}{m}$, $j = 1, \ldots, m$.*

(i) If $p > s'$ *and* $\sum_{j=1}^{m} \frac{\lambda_j}{q_j} = \frac{\lambda}{q}$*, then the condition* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $I_{\Omega,\alpha,m}$ *from product modified Morrey space* $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$*. Moreover, there exists a positive constant C such that for all* $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{\widetilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = s'$ *and* $\lambda \sum_{j=1}^{m} \frac{1}{p_j q_j} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j q_j}$*, then the condition* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* $I_{\Omega,\alpha,m}$ *from product modified Morrey space* $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ *to the weak Morrey space* $W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$*. Moreover, there exists a positive constant C such that for all* $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|I_{\Omega,\alpha,m}\boldsymbol{f}\|_{W\widetilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

Proof. (i) *Sufficiency*. Following the method used in [6], we choose a small positive number ε with $0 < \varepsilon < \min\{\alpha, \frac{m(n-\lambda_j)}{p_j} - \alpha, \frac{n-\lambda}{p} - \alpha\}$. One can then see from the condition of Theorem 2.5 that $1 \le s' < p_j < \frac{m(n-\lambda_j)}{\alpha+\varepsilon}$ and $1 \le s' < p_j < \frac{m(n-\lambda_j)}{\alpha-\varepsilon}$, and we let

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha+\varepsilon}{n-\lambda} \le \frac{1}{\widetilde{q}_1} \le \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha+\varepsilon}{n},$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha-\varepsilon}{n-\lambda} \le \frac{1}{\widetilde{q}_2} \le \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - \frac{\alpha-\varepsilon}{n}.$$

Now if each $p_j > s'$, then from [13], Theorem 1.1(i) implies that

$$\|\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}, \quad \|\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

A simple calculation yields $\frac{q}{2\widetilde{q}_1} + \frac{q}{2\widetilde{q}_2} = 1$. Hence, using Lemma 2.1, the Holder inequality and the above inequalities, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^\lambda} \int_{B(x,t)} |I_{\Omega,\alpha,m}f(y)|^q dy \Big)^{1/q}$$

$$\le C \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^\lambda} \int_{B(x,t)} \Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\Big]^{\frac{q}{2}} \Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\Big]^{\frac{q}{2}} dy \Big)^{\frac{1}{q}}$$

$$\le C \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^\lambda} \int_{B(x,t)} \Big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\Big]^{\widetilde{q}_1} dy \Big)^{\frac{1}{2\widetilde{q}_1}}$$

$$\times \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^\lambda} \Big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\Big]^{\widetilde{q}_2} dy \Big)^{\frac{1}{2\widetilde{q}_1}}$$

$$\le C \|\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}\|^{1/2}_{L^{\widetilde{q}_1,\lambda}} \|\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}\|^{1/2}_{L^{\widetilde{q}_2,\lambda}} = C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},$$

Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$. Define $\mathbf{f}_\varepsilon(x)=\big(f_1(\varepsilon x),\ldots,f_m(\varepsilon x)\big)$ for $\varepsilon>0$. Then it is easy to show that

$$I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y)=\varepsilon^{-\alpha}I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y). \tag{2.3}$$

Let $[t]_{1,+}=\max\{1,t\}$. Then by (2.3) we get

$$\begin{aligned}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{\widetilde{L}^{q,\lambda}}&=\varepsilon^{-\alpha}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{B(x,t)}|I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)|^q dy\Big)^{\frac{1}{q}}\\&=\varepsilon^{-\alpha-\frac{n}{q}}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{B(\varepsilon x,\varepsilon t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{\frac{1}{q}}\\&=\varepsilon^{-\alpha-\frac{n}{q}}\sup_{t>0}\Big(\frac{[\varepsilon t]_1}{[t]_1}\Big)^{\frac{\lambda}{q}}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[\varepsilon t]_1^\lambda}\int_{B(\varepsilon x,\varepsilon t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{\frac{1}{q}}\\&=\varepsilon^{-\alpha-\frac{n}{q}}\,[\varepsilon]_{1,+}^{\frac{\lambda}{q}}\,\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}}.\end{aligned}$$

Since $I_{\Omega,\alpha,m}$ is bounded from $\widetilde{L}^{p_1,\lambda_1}\times\ldots\times\widetilde{L}^{p_m,\lambda_m}$ to $\widetilde{L}^{q,\lambda}$, we have

$$\begin{aligned}\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}}&=\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\,\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{\widetilde{L}^{q,\lambda}}\le C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\|f_j(\varepsilon\cdot)\|_{\widetilde{L}^{p_j,\lambda_j}}\\&=C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^{\lambda_j}}\int_{B(x,t)}|f_j(\varepsilon y)|^{p_j}dy\Big)^{\frac{1}{p_j}}\\&=C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\varepsilon^{-\frac{n}{p_j}}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{\frac{1}{p_j}}\\&=C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\varepsilon^{-\frac{n}{p_j}}\sup_{t>0}\Big(\frac{[\varepsilon t]_1}{[t]_1}\Big)^{\frac{\lambda_j}{p_j}}\\&\times\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon[t]_1)^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{\frac{1}{p_j}}\\&=C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\varepsilon^{-\frac{n}{p_j}}\,[\varepsilon]_{1,+}^{\frac{\lambda_j}{p_j}}\,\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},\end{aligned}$$

where C is independent of ε.
If $n/p<n/q+\alpha$, then for all $\mathbf{f}\in\widetilde{L}^{p_1,\lambda_1}\times\ldots\times\widetilde{L}^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}}=0$ as $\varepsilon\to 0$.
If $(n-\lambda)/p>(n-\lambda)/q+\alpha$, then for all $\mathbf{f}\in\widetilde{L}^{p_1,\lambda_1}\times\ldots\times\widetilde{L}^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{\widetilde{L}^{q,\lambda}}=0$ as $\varepsilon\to\infty$.
Therefore we get $\frac{n-\lambda}{p}-\frac{n-\lambda}{q}\le\alpha\le\frac{n}{p}-\frac{n}{q}$.

(ii) *Sufficiency*. If $p_i=s'$ for some i, we take $\eta^2=\beta^{2-\frac{q}{q_2}}\Big(\prod\limits_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\Big)^{\frac{q}{q_2}-1}$ for any $\beta>0$, then applying Lemma 2.1 and Theorem 2.6 in [13], we get

$$\begin{aligned}&\Big|\big\{y\in B(x,t):\big|I_{\Omega,\alpha,m}\mathbf{f}(y)\big|>\beta\big\}\Big|\\&\le C\Big|\big\{y\in B(x,t):C\big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}}\,\big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}}>\beta\big\}\Big|\end{aligned}$$

$$\le C\Big|\big\{y\in B(x,t):\sqrt{C}\big[\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}}>\eta\big\}\Big|$$
$$+\Big|\big\{y\in B(x,t):\sqrt{C}\big[\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)\big]^{\frac{1}{2}}>\beta/\eta\big\}\Big|$$
$$\le C\Big|\big\{y\in B(x,t):\mathcal{M}_{\Omega,\alpha+\varepsilon,m}\mathbf{f}(y)>C\eta^2\big\}\Big|$$
$$+\Big|\big\{y\in B(x,t):\mathcal{M}_{\Omega,\alpha-\varepsilon,m}\mathbf{f}(y)>C\beta^2/\eta^2\big\}\Big|$$
$$=Ct^\lambda\Big[\Big(\frac{1}{\eta^2}\prod_{j=1}^m\|f_j\|_{L^{\widetilde{p}_j,\lambda_j}}\Big)^{\widetilde{q}_1}+\Big(\frac{\eta^2}{\beta^2}\prod_{j=1}^m\|f_j\|_{L^{\widetilde{p}_j,\lambda_j}}\Big)^{\widetilde{q}_2}\Big]$$
$$=Ct^\lambda\left(\frac{1}{\beta}\prod_{j=1}^m\|f_j\|_{L^{\widetilde{p}_j,\lambda_j}}\right)^q.$$

Hence, we obtain the following inequality

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}}=\sup_{\beta>0}\beta\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\Big|\Big\{y\in B(x,t):|I_{\Omega,\alpha,m}\mathbf{f}(y)|>\beta\Big\}\Big|\Big)^{\frac{1}{p}}$$
$$\le C\prod_{j=1}^m\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

This is the conclusion (ii) of Theorem 2.6.
Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$. From equality (2.3) we get

$$\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{W\widetilde{L}^{q,\lambda}}=\sup_{\tau>0}\tau\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{\big\{y\in B(x,t):I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y)>\tau\big\}}dy\Big)^{1/q}$$
$$=\sup_{\tau>0}\tau\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{\big\{y\in B(x,t):I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\big\}}dy\Big)^{1/q}$$
$$=\varepsilon^{-\frac{n}{q}}\sup_{\tau>0}\tau\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^\lambda}\int_{\big\{y\in B(\varepsilon x,\varepsilon t):I_{\Omega,\alpha,m}\mathbf{f}(y)>\tau\varepsilon^\alpha\big\}}dy\Big)^{1/q}$$
$$=\varepsilon^{-\alpha-\frac{n}{q}}\sup_{t>0}\Big(\frac{[\varepsilon t]_1}{[t]_1}\Big)^{\frac{\lambda}{q}}\sup_{\tau>0}\tau\varepsilon^\alpha$$
$$\times\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon[t]_1)^\lambda}\int_{\big\{y\in B(x,\varepsilon t):I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\big\}}dy\Big)^{1/q}$$
$$=\varepsilon^{-\alpha-\frac{n}{q}}\,[\varepsilon]_{1,+}^{\frac{\lambda}{q}}\,\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}}.$$

By the boundedness of the operator $I_{\Omega,\alpha,m}$ from $\widetilde{L}^{p_1,\lambda_1}\times\ldots\times\widetilde{L}^{p_m,\lambda_m}$ to $W\widetilde{L}^{q,\lambda}$, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}}=\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\,\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{W\widetilde{L}^{q,\lambda}}$$
$$\le C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\|f_j(\varepsilon\cdot)\|_{\widetilde{L}^{p_j,\lambda_j}}$$
$$=C\varepsilon^{\alpha+\frac{n}{q}}\,[\varepsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{[t]_1^{\lambda_j}}\int_{B(x,t)}|f_j(\varepsilon y)|^{p_j}dy\Big)^{1/p_j}$$

$$= C\varepsilon^{\alpha+\frac{n}{q}} \, [\varepsilon]_{1,+}^{-\frac{\lambda}{q}} \prod_{j=1}^{m} \varepsilon^{-\frac{n}{p_j}} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{[t]_1^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy \Big)^{1/p_j}$$

$$= C\varepsilon^{\alpha+\frac{n}{q}} \, [\varepsilon]_{1,+}^{-\frac{\lambda}{q}} \prod_{j=1}^{m} \varepsilon^{-\frac{n}{p_j}} \sup_{t>0} \Big(\frac{[\varepsilon t]_1}{[t]_1} \Big)^{\frac{\lambda_j}{p_j}} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon [t]_1)^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy \Big)^{1/p_j}$$

$$= C\varepsilon^{\alpha+\frac{n}{q}} \, [\varepsilon]_{1,+}^{-\frac{\lambda}{q}} \prod_{j=1}^{m} \varepsilon^{-\frac{n}{p_j}} \, [\varepsilon]_{1,+}^{\frac{\lambda_j}{p_j}} \, \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},$$

where C is independent of ε.

If $1/p < 1/q+\alpha/n$, then for all $\mathbf{f} \in \widetilde{L}^{p_1,\lambda_1}\times\ldots\times\widetilde{L}^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $1/p > 1/q+\alpha/(n-\lambda)$, then for all $\mathbf{f} \in \widetilde{L}^{p_1,\lambda_1}\times\ldots\times\widetilde{L}^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{W\widetilde{L}^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$.

Acknowledgements

The authors would like to express their gratitude to the referees for his (her) very valuable comments and suggestions.

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