**Transactions of NAS of Azerbaijan, Issue Mathematics**, **36** (4), 108-113 (2016).
Series of Physical-Technical and Mathematical Sciences.

# Some properties of a class of singular surface integral operators

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Received: 08.06.2016 / Revised: 28.09.2016 / Accepted: 03.10.2016

**Abstract.** Boundedness and compactness of a class of improper surface integral operators in the space of continuous functions is proved.

**Keywords.** Lyapunov surface, improper surface integral, Holder space.

**Mathematics Subject Classification (2010):** 45E05; 31B10

## 1 Introduction and problem statement.

It is known that study of some theoretical and applied problems of mathematics, physics and mechanics (for example, boundary value problems for the Helmholtz equation and so on) reduces to different classes of integral equations dependent on improper surface integral (see [4]). Therefore, study of some basic properties of a improper surface integral is of paramount importance.

Let us consider a improper surface integral of the form

$$W_\rho\ (x) = \int_S \frac{K(x,\, y)}{|x-y|^2}\,\rho(y)dS_y, \quad x \in S, \tag{1.1}$$

where $S \subset R^3$ is the Lyapunov surface, $K\,(x, y)$ is a continuous function on $S \times S$, while $\rho \in C\,(S)$ ($C\,(S)$ denotes the space of continuous functions on $S$ with the norm $\|\,\rho\,\|_\infty = \max\limits_{x\in S}\,|\,\rho\,(x)\,|$). Besides, assume that there exists a function

$$f \in \mathrm{E} = \left\{ \varphi \ : \quad \varphi \uparrow\,,\ \lim_{\delta\to 0} \varphi\,(\delta) = 0\,,\ \varphi\,(\delta_1 + \delta_2) \le \varphi\,(\delta_1) + \varphi\,(\delta_2) \right\}$$

such that

$$|K\,(x, y)| \le M^1 f\ (|x-y|)\,, \quad \forall x, y \in S, \tag{1.2}$$

and there exists a natural number $m$ such that

$$|K\,(x_1, y) - K\,(x_2, y)|$$

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[1] Here and in the sequel, positive constants different in different inequalities are denoted by $M$.

$$\leq M \sum_{j=1}^{m} |x_1 - x_2|^{\alpha_j} \ |x_1 - y|^{\beta_j} \ |x_2 - y|^{\gamma_j}, \quad \forall x_1, x_2, y \in S, \tag{1.3}$$

where the notation $\varphi \uparrow$ means increase of the function $\varphi$, $0 < \alpha_j \leq 1$, $\beta_j \geq 0$, $\gamma_j \geq 0$ and $\beta_j + \gamma_j > 0$, $j = \overline{1, m}$.

Note that, in particular the compactness of a weak singular operator in the space $C(S)$ was proved in the paper [4], some properties of the operator of a double layer acoustic potential in generalized Holder spaces were studied in [2]. Naturally, there arises an interest to investigate basic properties of the operator $(A\rho)(x) = W_\rho(x)$, $x \in S$ in a more general case, and this paper is devoted to this issue.

## 2 Basic results.

Let $S_\varepsilon(x)$ and $\Gamma_\varepsilon(x)$ be the parts of the surfaces and tangential plane $\Gamma(x)$ at the point $x \in S$ contained in the sphere $B_\varepsilon(x)$ of radius $\varepsilon > 0$ centered at the point $x$. Besides, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $x$. Then

$$|x - \tilde{y}| \leq |x - y| \leq C_1(S)|x - \tilde{y}| \text{ and } mes S_\varepsilon(x) \leq C_2(S)\, mes \Gamma_\varepsilon(x) \tag{2.1}$$

where $C_1(S)$ and $C_2(S)$ are positive constants dependent only on $S$ (see [5]).

For the function $\rho \in C(S)$ introduce the modulus of continuity of the form

$$\omega(\rho, \delta) = \max_{\substack{|x - y| \leq \delta \\ x, y \in S}} |\rho(x) - \rho(y)|, \delta > 0.$$

It is known that $\omega \in$ E (see [1]).

**Theorem 2.1** *Let $\rho \in C(S)$ and $\int\limits_0^{diam\, S} f(t)\, t^{-1} dt < \infty$. Then the integral $W_\rho(x)$ converges as a improper one and the function $W_\rho(x)$ is continuous on S, moreover*

$$\omega(W_\rho, h) \leq M \, \|\rho\|_\infty \left( h^\alpha + \int_0^h \frac{f(t)}{t} dt + h \int_h^d \frac{f(t)}{t^2} dt \right), \tag{2.2}$$

*where* $\alpha = \min\limits_{j=\overline{1,m}} \alpha_j$.

**Proof.** Since

$$W_\rho(x) = \int_{S_d(x)} \frac{K(x, y)}{|x - y|^2} \rho(y)\, dS_y + \int_{S \setminus S_d(x)} \frac{K(x, y)}{|x - y|^2} \rho(y)\, dS_y, \ x \in S,$$

then, taking (1.2), (2.1) into account and reducing the integral to double one, for the first addend we have

$$\left| \int_{S_d(x)} \frac{K(x, y)}{|x - y|^2} \rho(y)\, dS_y \right| \leq M \|\rho\|_\infty \int_0^d \frac{f(t)}{t} dt,$$

while the second addend is a improper integral. Therefore

$$|W_\rho(x)| \leq M \, \|\rho\|_\infty, \ x \in S.$$

Now let us prove the continuity of the function $W_\rho(x)$ on $S$. Take any points $x_1, x_2 \in S$ such that $|x_1 - x_2| = h < d/2$, where $d$ is the radius of a standard sphere for $S$ (see [6]). Obviously,

$$W_\rho(x_1) - W_\rho(x_2) = \int\limits_{S_{h/2}(x_1)} \frac{K(x_1, y)}{|x_1 - y|^2} \rho(y)\, dS_y + \int\limits_{S_{h/2}(x_2)} \frac{K(x_1, y)}{|x_1 - y|^2} \rho(y)\, dS_y$$

$$- \int\limits_{S_{h/2}(x_2)} \frac{K(x_2, y)}{|x_2 - y|^2} \rho(y)\, dS_y - \int\limits_{S_{h/2}(x_1)} \frac{K(x_2, y)}{|x_2 - y|^2} \rho(y)\, dS_y$$

$$+ \int\limits_{S \backslash \left(S_{h/2}(x_1) \bigcup S_{h/2}(x_2)\right)} \left( \frac{K(x_1, y)}{|x_1 - y|^2} - \frac{K(x_2, y)}{|x_2 - y|^2} \right) \rho(y)\, dS_y. \tag{2.3}$$

Denote the addends in equality (2.3) by $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$, respectively.

Using (1.2), (2.1) and the formula for reducing the surface integral to double one, we get

$$|I_1| \le M\, \|\rho\|_\infty \int\limits_0^h f(t)\, t^{-1} dt.$$

It is known that if $f \in \mathrm{E}$, then the function $\frac{f(x)}{x}$ is almost decreasing (see [1]). Then taking into account the inequality

$$h/2 \le |y - x_1| \le 3h/2, y \in S_{h/2}(x_2),$$

we find

$$|I_2| \le M\, \|\rho\|_\infty f(3h/2) \frac{4}{h^2} \int\limits_{S_{h/2}(x_2)} dS_y \le M\, \|\rho\|_\infty f(2h)$$

$$= 2M\, \|\rho\|_\infty h \frac{f(2h)}{2h} \le M\, \|\rho\|_\infty \int\limits_0^h f(t)\, t^{-1} dt.$$

In the similar way we can show that

$$|I_3| \le M\, \|\rho\|_\infty \int\limits_0^h f(t)\, t^{-1} dt, |I_4| \le M\, \|\rho\|_\infty \int\limits_0^h f(t)\, t^{-1} dt.$$

Let us represent $I_5$ in the form $I_5 = I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)}$, where

$$I_5^{(1)} = \int\limits_{S_d(x_1) \backslash \left(S_{h/2}(x_1) \bigcup S_{h/2}(x_2)\right)} \frac{K(x_1, y) - K(x_2, y)}{|x_1 - y|^2} \rho(y)\, dS_y,$$

$$I_5^{(2)} = \int\limits_{S_d(x_1) \backslash \left(S_{h/2}(x_1) \bigcup S_{h/2}(x_2)\right)} K(x_2, y) \left( \frac{1}{|x_1 - y|^2} - \frac{1}{|x_2 - y|^2} \right) \rho(y)\, dS_y,$$

$$I_5^{(3)} = \int\limits_{S\backslash S_d(x_1)} \frac{K(x_1,\, y) - K(x_2,\, y)}{|x_1 - y|^2} \rho\,(y)\, dS_y,$$

$$I_5^{(4)} = \int\limits_{S\backslash S_d(x_1)} K(x_2,\, y) \left( \frac{1}{|x_1 - y|^2} - \frac{1}{|x_2 - y|^2} \right) \rho\,(y)\, dS_y.$$

It is clear that

$$\left| I_5^{(1)} \right| \le \|\rho\|_\infty \int\limits_{S_d(x_1)\backslash\left(S_{h/2}(x_1)\bigcup S_{h/2}(x_2)\right)} \frac{|\, K(x_1,\, y) - K(x_2,\, y)\,|}{|x_1 - y|^2} dS_y$$

$$\le M\ \|\rho\|_\infty h^\alpha \int\limits_{S_d(x_1)\backslash S_{h/2}(x_1)} \frac{1}{|x_1 - y|^{2-p}} dS_y \le M\ \|\rho\|_\infty h^\alpha,$$

where $p = \min\limits_{j=\overline{1,m}} \{\alpha_j + \beta_j + \gamma_j - \alpha\}$.

Taking into account the inequalities

$$|x_1 - y| \le |x_1 - x_2| + |x_2 - y| \le 3\ |x_2 - y|\,, y \in S\backslash \left( S_{h/2}\,(x_1) \cup S_{h/2}\,(x_2) \right)$$

and

$$|x_2 - y| \le 3\ |x_1 - y|\,, y \in S\backslash \left( S_{h/2}\,(x_1) \cup S_{h/2}\,(x_2) \right),$$

we have

$$\left| \frac{1}{|x_1 - y|^2} - \frac{1}{|x_2 - y|^2} \right| \le \frac{M\,h}{|x_1 - y|^3}.$$

Hence we find

$$\left| I_5^{(2)} \right| = M\ \|\rho\|_\infty\, h \int\limits_{S_d(x_1)\backslash S_{h/2}(x_1)} \frac{f\,(|x_2 - y|)}{|x_1 - y|^3} dS_y \le M\,\|\rho\|_\infty\ h \int\limits_h^d \frac{f\,(t)}{t^2}\, dt.$$

As the integrals $I_5^{(3)}$ and $I_5^{(4)}$ are improper, therefore

$$\left| I_5^{(3)} \right| \le M\ \|\rho\|_\infty\, h^\alpha$$

and

$$\left| I_5^{(4)} \right| \le M\ \|\rho\|_\infty\, h^\alpha.$$

As a result, summing the obtained estimates, we get

$$|\, W_\rho\,(x_1) - W_\rho\,(x_2)\,| \le M\ \|\rho\|_\infty \left( h^\alpha + \int\limits_0^h \frac{f\,(t)}{t} dt + h \int\limits_h^d \frac{f\,(t)}{t^2}\, dt \right). \tag{2.4}$$

Using the de L'Hospital rule, we get

$$\lim_{h\to 0} h \int\limits_h^d \frac{f\,(t)}{t^2}\, dt = \lim_{h\to 0} \frac{\int_h^d \frac{f(t)}{t^2}\, dt}{\frac{1}{h}} = \lim_{h\to 0} h^2 \left( \frac{f\,(h)}{h^2} - \frac{f\,(d)}{d^2} \right) = 0.$$

Consequently,

$$\lim_{h\to 0}\left(h^{\alpha}+\int_{0}^{h}\frac{f\left(t\right)}{t}dt+h\int_{h}^{d}\frac{f\left(t\right)}{t^{2}}\,dt\right)=0.$$

Hence we get $W_{\rho}\in C\left(S\right)$. Besides, taking into account increase of the function $\psi\left(h\right)=h^{\alpha}+\int\limits_{0}^{h}\frac{f(t)}{t}dt+h\int_{h}^{d}\frac{f(t)}{t^{2}}\,dt$, from inequality (2.4) we get validity of inequality (2.2). The theorem is proved.

Theorems 2.2 and 2.3 follow from theorem 2.1.

**Theorem 2.2** *Let* $\int\limits_{0}^{diam\,S}f\left(t\right)\,t^{-1}dt<\infty$. *Then the operator* $A$ *boundedly acts from* $C\left(S\right)$ *to* $C\left(S\right)$.

Denote by $H_{\beta}\left(S\right)$ the Holder space with the index $0<\beta\leq 1$.

**Theorem 2.3** *Let* $f\in H_{\beta}\left(S\right)$, *then*

*(a) if* $\beta<1$, *then the operator* $A$ *boundedly acts from* $C\left(S\right)$ *to* $H_{\min\{\alpha,\beta\}}\left(S\right)$*;*

*(b) if* $\alpha=\beta=1$, *then the operator* $A$ *boundedly acts from* $C\left(S\right)$ *to* $H_{\gamma}\left(S\right)$, *where* $\gamma\in\left(0,1\right)$.

**Theorem 2.4** *Let* $\int\limits_{0}^{diam\,S}f\left(t\right)\,t^{-1}dt<\infty$. *Then the operator* $A$ *is a compact operator in the space* $C\left(S\right)$.

**Proof.** Let $\rho\in C\left(S\right)$. Then by theorem 2.2 $A\rho\in C\left(S\right)$.

Let us consider the sequence of operators

$$\left(A_{n}\rho\right)\left(x\right)=\int\limits_{S}k_{n}\left(x,\,y\right)\,\rho\left(y\right)\,dS_{y},\ \ x\in S\left(n=1,\,2,\,3,...\right),$$

where

$$k_{n}\left(x,\,y\right)=\begin{cases}0\,, & \text{if}\quad \left|x-y\right|\leq\frac{1}{2n},\\ \frac{\left(2n\,\left|x-y\right|-1\right)K\left(x,y\right)}{\left|x-y\right|^{2}}\,, & \text{if}\ \frac{1}{2n}\leq\left|x-y\right|\leq\frac{1}{n}\,,\\ \frac{K\left(x,y\right)}{\left|x-y\right|^{2}}\,, & \text{if}\ \frac{1}{n}\leq\left|x-y\right|\,.\end{cases}$$

As the functions $k_{n}\left(x,\,y\right)$ $\left(n=1,\,2,\,3,...\right)$ are continuous on $S\times S$, then the operators $A_{n}$ $\left(n=1,\,2,\,3,...\right)$ are compact on $C\left(S\right)$ (see [3]). Besides, reducing the surface integral to double one, we have:

$$\left|\,\left(A\rho\right)\left(x\right)-\left(A_{n}\rho\right)\left(x\right)\,\right|\leq\left\|\,\rho\right\|_{\infty}\int\limits_{S_{1/(2\,n)}(x)}\frac{\left|\,K(x,\,y)\right|}{\left|x-y\right|^{2}}\,dS_{y}$$

$$+2\,\left\|\,\rho\right\|_{\infty}\int\limits_{S_{1/n}(x)\backslash S_{1/(2\,n)}(x)}\frac{\left|\,K(x,\,y)\,\right|}{\left|x-y\right|^{2}}\,dS_{y}$$

$$+2\,n\,\|\rho\|_\infty \int\limits_{S_{1/n}(x)\backslash S_{1/(2\,n)}(x)} \frac{|K(x,\,y)|}{|x-y|}\,dS_y \le M\,\|\rho\|_\infty \int\limits_0^{1/n} \frac{f\,(t)}{t}\,dt, x \in S.$$

And this means that

$$\|A - A_n\|_\infty \le M \int\limits_0^{1/n} \frac{f\,(t)}{t}\,dt \to 0 \text{ as } n \to \infty,$$

consequently, the operator $A$ is compact (see [3]). The theorem is proved.

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