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Componentwise equiconvergence theorems for a fourth order differential operator

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Abstract. In the paper on a compact we study componentwise uniform equiconvergence of othtogonal expansion of a vector-function from the class $W_{1,m}^{1}(G)$ in eigen vector-functions of a fourth order differential operator with expansion in trigonometric series. Rate of componentwise uniform equiconvergence on a compact is established.

Keywords. eigen vector-function, equiconvergence, spectral expansion.

Mathematics Subject Classification (2010): 34L10, 42A20

1 Main notion and formulation of results.

On the interval $G=(0,1)$ we consider the operator

$$L\Psi=\psi^{(4)}+U_2(x)\,\psi^{(2)}+U_3(x)\,\psi^{(1)}+U_4(x)\,\psi$$

with matrix coefficients $U_l(x)=(u_{lij}(x))_{i,j=1}^{m}$, $l=\overline{2,4}$, where $u_{lij}(x)\in L_1(G)$ are real functions and $u_{lij}(x)=u_{lji}(x)$.

Let $L_p^m(G)$, $p\geq 1$ be the space of m- component vector-functions $f(x)=(f_1(x),f_2(x),...,f_m(x))^T$ with the norm

$$\|f\|_{p,m}=\left\{\int_G |f(x)|^p\,dx\right\}^{1/p}=\left\{\int_G\left(\sum_{j=1}^{m}|f_j(x)|^2\right)^{p/2}dx\right\}^{1/p}.$$

In the case $p=\infty$ the norm is determined by the equality $\|f\|_{\infty,m}=\sup_{x\in\overline{G}} vrai\,|f(x)|$.

Denote by $W_{p,m}^1(G)\,,p\geq 1$ the class of absolutely continuous on $\overline{G}$ vector-functions $f(x)$ for which $f'(x)\in L_p^m(G)$. The norm in $W_{p,m}^1(G)$ is determined by the equality $\|f\|_{W_{p,m}^1(G)}=\|f\|_{p,m}+\|f'\|_{p,m}$.

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Let $\{\lambda_k\}_{k=1}^{\infty}$, $\lambda_k \leq 0$, and $\{\Psi_k(x)\}_{k=1}^{\infty}$ be the system of eigen values and the system of complete orthonormalized in $L_2^m(G)$ eigen-functions of the operator L, i.e. $L\psi_k + \lambda_k\psi_k = 0$, where $\psi_k(x) = \left(\psi_k^1(x), \psi_k^2(x), ..., \psi_k^m(x)\right)^T$. Denote $\mu_k = \sqrt[4]{-\lambda_k}$ and for $f(x) \in W_{p,m}^1(G), p \geq 1$ introduce a partial sum of its spectral expansion in the system $\{\Psi_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x,f) = \left(\sigma_\nu^1(x,f), \sigma_\nu^2(x,f), ..., \sigma_\nu^m(x,f)\right)^T,$$

where

$$\sigma_\nu^j(x,f) = \sum_{\mu_k \leq \nu} (f,\psi_k)\,\psi_k^j(x), j = \overline{1,m};$$

$$f_k = (f,\psi_k) = \int_0^1 \langle f(x), \psi_k(x)\rangle\, dx = \int_0^1 \sum_{j=1}^m f_j(x)\,\overline{\psi_k^j(x)}dx.$$

Denote by $S_\nu(x,f_j)$, $j = \overline{1,m}$ the partial sum of trigonometric Fourier series of the function $f_j(x)$, i.e. $S_\nu(x,f_j) = \frac{a_0}{2} + \sum\limits_{2\pi k \leq \nu} (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$, where

$$a_k = 2\int_0^1 f_j(x)\cos 2\pi kx dx, \quad k = 0,1,...; \quad b_k = 2\int_0^1 f_j(x)\sin 2\pi kx dx, \quad k = 1,2,....$$

Consider the difference $\Delta_\nu^j(x,f) = \sigma_\nu^j(x,f) - S_\nu(x,f_j)$, $j = \overline{1,m}$.

If for any compact $K \subset G$ the difference $\Delta_\nu^j(x,f)$ tends to zero uniformly with respect to $x \in K$ as $\nu \to +\infty$, we say that the j-th component of spectral expansion of the vector-function $f(x)$ in the system $\{\Psi_k(x)\}_{k=1}^{\infty}$ uniformly equiconverges on any compact of the interval G with expansion in trigonometric series to appropriate j-th component $f_j(x)$ of the vector-function $f(x)$.

Let $\alpha_k = \begin{cases} \mu_k^{-1}, \text{ if } \{\Psi_k(x)\}_{k=1}^{\infty} & \text{is uniformly bounded,} \\ \mu_k^{-\frac{1}{2}}, \text{ if } \{\Psi_k(x)\}_{k=1}^{\infty} & \text{is not uniformly bounded,} \end{cases}$

$\alpha(\nu) = \begin{cases} \nu^{-1}, \text{ if } \{\Psi_k(x)\}_{k=1}^{\infty} & \text{is uniformly bounded,} \\ \nu^{-\frac{1}{2}}, \text{ if } \{\Psi_k(x)\}_{k=1}^{\infty} & \text{is not uniformly bounded.} \end{cases}$

In the paper the following results are established.

Theorem 1.1 *Let the elements $u_{2ij}(x)$, $j = \overline{1,m}$ of the i-th row of the matrix $U_2(x)$ belong to the class $L_r(G)$, $r > 1$, $U_l(x) \in L_1(G)$, $l = 3,4$ and $f(x) \in W_{1,m}^1(G)$. Then the i-th component of expansion of the vector-function $f(x)$ in the system $\{\Psi_k(x)\}_{k=1}^{\infty}$ uniformly equiconverges on any compact $K \subset G$ with expansion in trigonometric Fourier series to appropriate i-th component $f_i(x)$ of the vector-function $f(x)$, and the following estimation is valid:*

$$\left\|\Delta_\nu^i(\cdot,f)\right\|_{C(K)} = O(\alpha(\nu)), \quad \nu \to +\infty. \tag{1.1}$$

Note that at $m = 1$ for the Sturm-Liouville operator (in this case the system $\{\Psi_k(x)\}_{k=1}^{\infty}$ is uniformly bounded), estimation (1.1) was first established in the paper [1]. At $m = 1$ for an arbitrary even order operator, under the condition $\|\psi_k\|_{L_1(K)}\|\psi_k\|_{L_\infty(G)} \leq C_0(K)$, $k = 1,2,...$, the estimation $O\left(\nu^{-1}\right)$ was proved in the paper [10]. Theorem 1.1 itself for Sturm-Liouville vector operator was proved in [3] (in this case the system $\{\Psi_k(x)\}_{k=1}^{\infty}$ is uniformly bounded as well).

Theorem 1.2 *Let the elements* $u_{2ij}(x)$, $j=\overline{1,m}$ *of the* i*-th row of the matrix* $U_2(x)$ *belong to the class* $L_1(G)$, $U_l(x)\in L_1(G)$, $l=3,4$ *and* $f(x)\in W^1_{1,m}(G)$*. Then the* i*-th component of expansion of the vector-function* $f(x)$ *in the system* $\{\Psi_k(x)\}_{k=1}^{\infty}$ *uniformly equiconverges on any compact* $K\subset G$ *with expansion in trigonometric Fourier series to appropriate* i*-th component* $f_i(x)$ *of the vector-function* $f(x)$*, and the following estimation is valid:*

$$\left\|\Delta^i_\nu(\cdot,f)\right\|_{C(K)}=O\left(\alpha(\nu)\left(1+T_i(\nu)\right)\right),\quad \nu\to+\infty, \tag{1.2}$$

where

$$T_i(\nu)=\inf_{n\geq 2}\left\{\Omega_{1i}\left(U_2,n^{-1}\right)\ln\nu+\|U_2\|_{1i}\ln n\right\},$$

$$\Omega_{1i}(U_2,\delta)=\max_{1\leq j\leq m}\omega_1(u_{2ij},\delta),\quad \|U_2\|_{1i}=\max_{1\leq j\leq m}\|u_{2ij}\|_{L_1(G)}.$$

Note that for Sturm-Liouville's vector-operator, theorem 1.2 was proved in the paper [2].

Corollary 1.1 *If the conditions of theorem 1.2 are fulfilled and* $\Omega_{1i}(U_2,\delta)=O\left(\ln^{-\gamma}\delta^{-1}\right)$, $\delta\to+0$, $\gamma>0$*, then the following estimation is valid:*

$$\left\|\Delta^i_\nu(\cdot,f)\right\|_{C(K)}=O\left(\left(2^\gamma\gamma^{-\frac{\gamma}{1+\gamma}}+2\gamma^{\frac{1}{1+\gamma}}\right)\alpha(\nu)\ln^{1/(1+\gamma)}\nu,\nu\to+\infty\right), \tag{1.3}$$

In particular, for $\gamma=1$ *it holds the estimation*

$$\left\|\Delta^i_\nu(\cdot,f)\right\|_{C(K)}=O\left(\alpha(\nu)\sqrt{\ln\nu}\right),\quad \nu\to+\infty \tag{1.4}$$

Theorem 1.3 *Let the elements* $u_{2ij}(x)$, $j=\overline{1,m}$ *of the* i*-th row of the matrix* $U_2(x)$ *belong to the class* $L_r(G)$, $r\geq 1$, $U_l(x)\in L_1(G)$, $l=3,4$ *and the function* $f(x)\in W^1_{1,m}(G)$ *have a compact support. Then the* i*-th component of expansion of the vector-function* $f(x)$ *in the system* $\{\psi_k(x)\}_{k=1}^{\infty}$ *uniformly equiconverges on any compact* $K\subset G$ *with expansion in trigonometric Fourier series to appropriate* i*-th component* $f_i(x)$ *of the vector-function* $f(x)$*, and the following estimation is valid:*

$$\left\|\Delta^i_\nu(\cdot,f)\right\|_{C(K)}=\begin{cases} o\left(\alpha(\nu)\left(1+T_i(\nu)\right)\right) & for\ \ r=1\\ o\left(\alpha(\nu)\right) & for\ \ r>1\end{cases} \tag{1.5}$$

Note that theorem 1.3 at $m=1$, $r>1$ for the Sturm-Liouville operator was first proved in the paper [6]. In this case the orthonormalized system $\psi_k(x)$ knowingly is uniformly bounded.

2 Some auxiliary statements and proof of results.

For studying equiconvergence of spectral expansion with trigonometric expansion of the given function. E.I. Moseev-type [15] mean value formula is notably used for eigen functions $\psi_k(x)$ of the operator L. We used the following mean value formula for the eigen vector-function $\psi_k(x)$ of a fourth order operator with nonsmooth coefficients.

Lemma 2.1 *(see [13], [12]). For any sufficiently small* $R>0$ *there will be found* $\overline{R}$ *satisfying the condition* $2R\leq\overline{R}\leq C_0R$*, where* C_0 *is a constant dependent on order of the operator* L*, and the real numbers* $R_\alpha(\mu_k)$, $|R_\alpha(\mu_k)|\in\left[0,\overline{R}\right]$*, such that for any* $t\in[0,R]$

and $x \in G$, $dist\,(x, \partial G) > \overline{R}$, the following mean value formula ($\mu_k \geq \rho_0, \rho_0$ is a sufficiently large number) is valid

$$\frac{\psi_k(x-t)+\psi_k(x+t)}{2} = \psi_k(x)\cos\mu_k t + \int_x^{x+t} K_0(\xi - x, t)\, Q_{10}(\xi, \psi_k)\, d\xi$$

$$+\int_{x-t}^{x} K_0(x-\xi, t)\, Q_{20}(\xi, \psi_k)\, d\xi + \int_{t\leq \xi - x \leq \overline{R}} P_0(\xi - x, t)\, Q_{30}(\xi, \psi_k)\, d\xi$$

$$+\int_{t\leq x-\xi \leq \overline{R}} P_0(x-\xi, t)\, Q_{40}(\xi, \psi_k)\, d\xi + \int_{x-\overline{R}}^{x+\overline{R}} F_0(t, \xi - x)\, Q_{50}(\xi, \psi_k)\, d\xi$$

$$+\sum_{q=0}^{3}\sum_{\alpha=1}^{3} F_{q\alpha}(t, \mu_k)\, \psi_k^{(q)}(x+R_\alpha), \tag{2.1}$$

where

$$\left|\left[Q_{i0}(\xi, \psi_k)\right]_j\right| \leq const \left|\left[M(\xi, \psi_k)\right]_j\right|, \quad M(\xi, \psi_k) = 4\mu_k^{-3}\sum_{l=2}^{4} U_l(\xi)\, \psi_k^{(4-l)}(\xi),$$

$[b]_j$ *is the j-th component of the vector b, for the integrals*

$$J_0(r, R, \mu_k, \nu) = \int_r^R t^{-1}\sin\nu t K_0(r, t)\, dt, \quad 0 < r \leq R,$$

$$I_0(r, R, \mu_k, \nu) = \int_0^{\min\{r, R\}} t^{-1}\sin\nu t P_0(r, t)\, dt, \; r \in \left[0, \overline{R}\right],$$

$$K_1(R, \mu_k, r, \nu) = \int_0^R t^{-1}\sin\nu t F_0(t, r)\, dt, \; r \in \left[0, \overline{R}\right],$$

$$K_{q\alpha}(R, \mu_k, \nu) = \int_0^R t^{-1}\sin\nu t F_{q\alpha}(t, \mu_k)\, dt,$$

for $R_0/2 \leq R \leq R_0$, $R_0 > 0$ the following uniform estimations are valid

$$J_0 = \begin{cases} O\left(\min\left\{\nu\mu_k^{-1}, \mu_k\nu^{-1}\right\}\right) & for \quad |\mu_k - \nu| \geq \nu/2, \\ O\left(\ln\left\{\nu/\left|\nu - \mu_k\right|\right\}\right) & for \quad 2 \leq |\mu_k - \nu| \leq \nu/2, \\ O\left(\min\left\{\ln\nu, |\ln r|\right\}\right) & for \quad |\mu_k - \nu| \leq 2, \end{cases} \tag{2.2}$$

$$J_0 = O\left(\mu_k^{1-\varepsilon}\nu^{-1}r^{-\varepsilon}\right) \quad for \quad \rho_0 \leq \mu_k \leq \nu/2, \tag{2.3}$$

$$I_0 = O\left(\min\left\{\mu_k\nu^{-1}, \mu_k^{-1}\nu\right\}\right), \tag{2.4}$$

$$I_0 = \begin{cases} O\left(\left\{\mu_k^{1-\varepsilon}\nu^{-1}r^{-\varepsilon}\right\}\right) & for \quad \rho_0 \le \mu_k \le \nu/2, \\ O\left(\nu^{1-\varepsilon}\mu_k^{-1}r^{-\varepsilon}\right) & for \quad \mu_k \ge \nu/2, \end{cases} \tag{2.5}$$

$$K_1, K_{q,\alpha} = \begin{cases} O\left(\exp\left(-\delta\mu_k\right)\nu^{-1}\right) & for \quad \rho_0 \le \mu_k \le \nu/2, \\ O\left(\nu\exp\left(-\delta\mu_k\right)\right) & for \quad \mu_k \ge \nu/2, \end{cases} \tag{2.6}$$

$$\left\|J_0\left(\cdot, R, \mu_k, \nu\right)\right\|_{P,[0,R]} = O\left(\mu_k^{1-1/p}\nu^{-1}\right) \quad for \ \rho_0 \le \mu_k \le \nu/2, \tag{2.7}$$

$$\left\|I_0\left(\cdot, R, \mu_k, \nu\right)\right\|_{P,[0,\overline{R}]} = O\left(\mu_k^{1-1/p}\nu^{-1}\right) \quad for \ \rho_0 \le \mu_k \le \nu/2, \tag{2.8}$$

where $0 \le \varepsilon \le 1, \ \delta > 0.$

Lemma 2.2 *For any points* $x_1, x_2, 0 \le x_1 < x_2 \le 1$, *for the eigen vector-function* $\psi_k(x)$ *the following estimation is valid:*

$$\left|\int_{x_1}^{x_2} \psi_k(x)\,dx\right| \le const\ \alpha_k, \quad \mu_k \ge 1, \tag{2.9}$$

where const is independent of k, x_1, x_2.

Proof. For the eigen vector-function $\psi_k(x)$ the following representation is valid [10]:

$$\mu_k^{-l}\psi_k(t) = \sum_{j=1}^{3} X_{kj}(0)\left(-i\omega_j\right)^l \exp\left(-\omega_j\mu_k t\right) + \left(-i\omega_4\right)^l B_{k4}(0)\exp\left(i\omega_4\mu_k(1-t)\right)$$

$$+ (-i)^l \sum_{j=1}^{3}\omega_j^{l+1}\int_0^t M\left(\xi, \psi_k\right)\exp\left(i\omega_j\mu_k(\xi - t)\right)d\xi$$

$$- (-i)^l\,\omega_4^{l+1}\int_t^1 M\left(\xi, \psi_k\right)\exp\left(i\omega_4\mu_k(\xi - t)\right)d\xi, \quad l = \overline{0,3} \tag{2.10}$$

where $\omega_1 = -\omega_2 = -1$, $\omega_4 = -\omega_3 = -i$, $\mu_k \ne 0$. Furthermore, there hold (see [8])

$$\left|X_{kj}(0)\right| \le const\left\|\psi_k\right\|_{\infty,m}; \quad \left|B_{k4}(0)\right| \le const\left\|\psi_k\right\|_{\infty,m}, \tag{2.11}$$

$$\left|M\left(\xi, \psi_k\right)\right| \le const\mu_k^{-1}\left[\sum_{l=2}^{4}\left\|U_l(\xi)\right\|\mu_k^{2-l}\right]\left\|\psi_k\right\|_{\infty,m}. \tag{2.12}$$

Integrating formula (2.10) for $l = 0$ with respect to t from x_1 to x_2 and taking into account estimations (2.11), (2.12), for $\mu_k \ge 1$ we get

$$\left|\int_{x_1}^{x_2}\psi_k(t)\,dt\right| \le \sum_{j=1}^{3}\left|X_{kj}(0)\right|\left|\left.\frac{\exp\left(-i\omega_j\mu_k t\right)}{i\omega_j\mu_k}\right|_{x_1}^{x_2}\right|$$

$$+\left|B_{k4}(0)\right|\left|\int_{x_1}^{x_2}e^{i\omega_4\mu_k(1-t)}dt\right| + \sum_{j=1}^{3}\int_{x_1}^{x_2}\int_0^t\left|M\left(\xi,\psi_k\right)\right|d\xi dt + \int_{x_1}^{x_2}\int_t^1\left|M\left(\xi,\psi_k\right)\right|d\xi dt$$

$$\le \frac{const}{\mu_k}\left\|\psi_k\right\|_{\infty,m}.$$

Hence by virtue of the estimation $\|\psi_k\|_{\infty,m} \le const\mu_k^{\frac{1}{2}} \|\psi_k\|_{2,m} = const\mu_k^{\frac{1}{2}}$ we get

$$\left|\int_{x_1}^{x_2} \psi_k(t)\,dt\right| \le \frac{const}{\sqrt{\mu_k}}.$$

If the system $\{\psi_k(x)\}_{k=1}^{\infty}$ is uniformly bounded, then the following estimation will be valid:

$$\left|\int_{x_1}^{x_2} \psi_k(t)\,dt\right| \le \frac{const}{\mu_k}.$$

Lemma 2.2 is proved.

Denote by $D(x-y,\nu)$ the Dirichlet kernel of the trigonometric system $\{1, \sqrt{2}\cos 2\pi nx, \sqrt{2}\sin 2\pi nx, \ \ n = 1,2,...\}$, by $D^i(x-y,\nu)$ an m-component vector-column whose t-th component equals $D(x-y,\nu)$, the remaining components equal zero. We introduce the sum

$$\theta^i(x,y,\nu) = \sum_{\mu_k \le \nu} \psi_k^i(x)\,\overline{\psi_k(y)}, \ \ i = 1, m,$$

where $\psi_k(x) = \left(\psi_k^1(x), \psi_k^2(x), ..., \psi_k^m(x)\right)$. Without losing generality, we assume $\mu_k \ge 1$, $k = 1,2$.

Lemma 2.3 *Let $u_{2ij}(x) \in L_r(G)$, $j = \overline{1,m}$, $r \ge 1$, $U_l(x) \in L_1(G)$, $l = 3,4$. Then for any compact $K \subset G$, a estimation uniform with respect to $x \in K$*

$$\left|\int_{x_1}^{x_2} \left[\theta^i(x,y,\nu) - D^i(x-y,\nu)\right] dy\right| \le C_1(K) \begin{cases} \alpha(\nu) & for \ \ r > 1 \\ \alpha(\nu)(1+T_i(\nu)) & for \ \ r = 1 \end{cases} \tag{2.13}$$

for any $x_1 \le x_2$, $x_1, x_2 \in \overline{G}$. Therewith $C_1(K) = C_1(K, \|U_2\|_r, \|U_l\|_1 : l = 3,4)$.

Proof. We fix an arbitrary connected compact K of the interval G and an arbitrary number R_0, $dist(K,\partial G) > 4G_0R_0$, therewith C_0 is a constant from lemma 2.1. For any $R \in [R_0/2, R_0]$ and any $x \in K$ we consider m-component vector-function $W^i(x,y,u,\nu,R)$ whose i-th component equals the "cut off" Dirichlet kernel, i.e. equals

$$V(x,y,\mu,R) = \begin{cases} \frac{1}{\pi}\frac{\sin\nu(x-y)}{x-y} & \text{for} \ \ |x-y| \le R \\ 0 & \text{for} \ \ |x-y| > R \end{cases},$$

and the remaining components equal zero, where $y \in G$. As in the papers [4], [5], we introduce the averaging operation with respect to R of this vector-function:

$$\widetilde{W^i}(x,y,\nu,R_0) = S_{R_0}\left[W^i\right] = \frac{2}{R_0}\int_{R_0/2}^{R_0} W^i(x,y,\nu,R)\,dR.$$

The Fourier coefficients of this vector-function in the system $\overline{\{\psi_k(y)\}}_{k=1}^{\infty}$ are calculated by the formula

$$\widetilde{W_k^i}(x,\nu,R_0) = \left(\widetilde{W^i}(x,\cdot,\nu,R_0), \overline{\psi_k(\cdot)}\right) = S_{R_0}\left[W_k^i(x,\nu,R)\right],$$

where $W_k^i(x,\nu,R)$ are the Fourier coefficients of the vector-function $W^i(x,y,\nu,R)$, i.e. $W_k^i(x,\nu,R)=\left(W^i(x,\cdot,\nu,R),\overline{\psi_k(\cdot)}\right)$. Obviously,

$$W_k^i(x,\nu,R)=\frac{2}{\pi}\int\limits_0^R\frac{\sin\nu t}{t}\left[\frac{\psi_k^i(x-t)+\psi_k^i(x+t)}{2}\right]dt.$$

Taking into account the mean value formula (2.1) for $\mu_k>\rho_0$, and in the case $\mu_k\le\rho_0$ the mean value formula (see [10])

$$\frac{\psi_k(x-t)+\psi_k(x+t)}{2}=\psi_k(x)\cos\mu_k t+\frac{1}{2}\int\limits_{x-t}^{x+t}M(\xi,\psi_k)R_0^k(|\xi-x|-t)\,d\xi$$

$$+\sum_{j=2}^{3}A_{kj}(x)(\cos\mu_k\omega_j t-\cos\mu_k t),$$

where

$$R_0^k(z)=\sum_{j=1}^{4}\omega_j\exp(i\omega_j\mu_k z),\quad A_{kj}(x)=\sum_{l=0}^{1}\omega_j^{4-2l}(i\mu_k)^{-2l}\psi_k^{(2l)}(x),$$

we get (see [5], [10])

$$\widetilde{W}_k^i(x,\nu,R_0)=\delta_k^\nu\psi_k^i(x)+I(\nu,\mu_k,R_0)\psi_k^i(x)+D_k^i(x,\nu,R_0), \tag{2.14}$$

where

$$\delta_k^\nu=\frac{1}{2}(1+sgn(\nu-\mu_k)),\quad |I(\nu,\mu_k,R_0)|\le\frac{C(R_0)}{1+|\nu-\mu_k|^2}; \tag{2.15}$$

$$D_k^i(x,\nu,R_0)=\frac{1}{\pi}S_{R_0}\left[\int\limits_{x-R}^{x+R}[M(\xi,\psi_k)]_i I_k^{\rho_0}(|\xi-x|,R)\,d\xi\right]+\frac{2}{\pi}S_{R_0}\left[J_{ki}^{\rho_0}(R,x)\right]$$

for $\mu_k\le\rho_0$;

$$D_k^i(x,\nu,R_0)=\frac{2}{\pi}S_{R_0}\left[\int\limits_{x}^{x+R}[Q_{10}(\xi,\psi_k)]_i J_0(\xi-x,R,\mu_k,\nu)\,d\xi\right]$$

$$+\frac{2}{\pi}S_{R_0}\left[\int\limits_{x-R}^{x}[Q_{20}(\xi,\psi_k)]_i J_0(x-\xi,R,\mu_k,\nu)\,d\xi\right]$$

$$+\frac{2}{\pi}S_{R_0}\left[\int\limits_{x}^{x+\overline{R}}[Q_{30}(\xi,\psi_k)]_i I_0(\xi-x,R,\mu_k,\nu)\,d\xi\right]$$

$$+\frac{2}{\pi}S_{R_0}\left[\int\limits_{x-\overline{R}}^{x}[Q_{40}(\xi,\psi_k)]_i I_0(x-\xi,R,\mu_k,\nu)\,d\xi\right]$$

$$+\frac{2}{\pi}S_{R_0}\left[\int\limits_{x-\overline{R}}^{x+\overline{R}}\left[Q_{50}\left(\xi,\psi_k\right)\right]_i K_1\left(R,\mu_k,\left|\xi-x\right|,\nu\right)d\xi\right]$$

$$+\frac{2}{\pi}S_{R_0}\left[\sum_{q=0}^{3}\sum_{\alpha=1}^{3}\left[\psi_k^{(q)}\left(x+R_\alpha\right)\right]_i K_{q\alpha}\left(R,\mu_k,\nu\right)\right]$$

for $\mu_k > \rho_0$; $[b]_i$ is the i-th component of the vector b, for the expressions $I_k^{\rho_0}(r,R)$ and $J_{ki}^{\rho_0}$ the following estimations uniform for $R\in[R_0/2,R_0]$ are valid:

$$I_k^{\rho}=O\left(\nu^{-1}\mu_k^3\right),\quad J_{ki}^{\rho_0}=O\left(\nu^{-1}\sum_{s=0}^{1}\left|\left[u_k^{(2s)}(x)\right]_i\right|\right). \tag{2.16}$$

Because of basicity of the system $\overline{\{\psi_k(y)\}}$ in $L_2^m(G)$, belonging for each fixed $x\in K$ of the function $\widetilde{W}^i(x,y,\nu,R_0)$ in $L_2^m(G)$, relations (2.2)-(2.6), (2.14)-(2.16) and inequalities (see [10])

$$\sum_{\tau\le\mu_k\le\tau+1}1\le const,\quad \forall\tau\ge 0 \tag{2.17}$$

$$\left\|\psi_k^{(s)}\right\|_{\infty,K_1}1\le C\left(K_1,K_2\right)\left(1+\mu_k\right)^s\left\|\psi_k\right\|_{2,K_2},\quad K_1\subset intK_2,\ \ s=\overline{0,3}, \tag{2.18}$$

(see [7], [9]) with respect to the variable y we get in $L_2^m(G)$ the equalities (see [3])

$$\widetilde{W}^i(x,y,\nu,R_0)-\theta^i(x,y,\nu)=-\frac{1}{2}\sum_{\mu_k\le\nu}\psi_k^i(x)\overline{\psi_k(y)}$$

$$+\sum_{k=1}^{\infty}I\left(\nu,\mu_k,R_0\right)\psi_k^i(x)\overline{\psi_k(y)}+\sum_{k=1}^{\infty}D_k^i\left(x,\nu,R_0\right)\overline{\psi_k(y)}. \tag{2.19}$$

Therewith each series in (2.19) converges in $L_2^m(G)$ with respect to the variable y.

Integrating termwise the equalities (2.19) with respect to the variable y along the segment $[x_1,x_2]$, we find

$$\int\limits_{x_1}^{x_2}\widetilde{W}^i(x,y,\nu,R_0)\,dy-\int\limits_{x_1}^{x_2}\theta^i(x,y,\nu)\,dy=\sum_{l=1}^{10}S_l(x),$$

where

$$S_1(x)=-\frac{1}{2}\sum_{\mu_k=\nu}\psi_k^i(x)\int\limits_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_2(x)=\sum_{k=1}^{\infty}I\left(\nu,\mu_k,R_0\right)\psi_k^i(x)\int\limits_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_3(x)=\frac{1}{\pi}\sum_{1\le\mu_k\le\rho_0}^{\infty}S_{R_0}\left[\int\limits_{x-R}^{x+R}\left[M\left(\xi,\psi_k\right)\right]_i I_k^{\rho_0}\left(\left|x-\xi\right|,R\right)d\xi\right]\int\limits_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_4(x)=\frac{2}{\pi}\sum_{1\le\mu_k\le\rho_0}^{\infty}S_{R_0}\left[J_{ki}^{\rho_0}M(R,x)\right]\int_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_5(x)=\frac{2}{\pi}\sum_{\mu_k>\rho_0}S_{R_0}\left[\int_x^{x+R}[Q_{10}(\xi,\psi_k)]_i J_0(\xi-x,R,\mu_k,\nu)\,d\xi\right]\int_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_6(x)=\frac{2}{\pi}\sum_{\mu_k>\rho_0}S_{R_0}\left[\int_x^{x_2}[Q_{20}(\xi,\psi_k)]_i J_0(x-\xi,R,\mu_k,\nu)\,d\xi\right]\int_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_7(x)=\frac{2}{\pi}\sum_{\mu_k>\rho_0}S_{R_0}\left[\int_x^{x+\overline{R}}[Q_{30}(\xi,\psi_k)]_i I_0(\xi-x,R,\mu_k,\nu)\,d\xi\right]\int_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_8(x)=\frac{2}{\pi}\sum_{\mu_k>\rho_0}S_{R_0}\left[\int_{x-\overline{R}}^{x}[Q_{40}(\xi,\psi_k)]_i I_0(x-\xi,R,\mu_k,\nu)\,d\xi\right]\int_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_9(x)=\frac{2}{\pi}\sum_{\mu_k>\rho_0}S_{R_0}\left[\int_{x-\overline{R}}^{x+\overline{R}}[Q_{50}(\xi,\psi_k)]_i K_1(R,\mu_k,|\xi-x|,\nu)\,d\xi\right]\int_{x_1}^{x_2}\overline{\psi_k(y)}dy,$$

$$S_{10}(x)=\frac{2}{\pi}\sum_{\mu_k>\rho_0}S_{R_0}\left[\sum_{q=0}^{3}\sum_{\alpha=1}^{3}\left[\psi_k^{(q)}(x+R_\alpha)\right]_i K_{q\alpha}(R,\mu_k,\nu)\right].$$

For proving estimation (2.13), we should estimate the sum $S_l(x)$, $l=\overline{1,10}$ for $x\in K$. By lemma 2.2, inequalities (2.17), (2.18), for the sum $S_1(x)$ we get

$$|S_1(x)|\le\frac{1}{2}\sum_{\mu_k=\nu}\left|\psi_k^i(x)\right|\left|\int_{x_1}^{x_2}\overline{\psi_k(y)}dy\right|\le C(K)\sum_{\mu_k=\nu}\|\psi_k\|_{2,m}\alpha_k$$

$$\le C(K)\sum_{\mu_k=\nu}\alpha_k\le C(K)\,\alpha(\nu)\sum_{\mu_k=\nu}1\le C(K)\,\alpha(\nu).$$

By inequalities (2.9), (2.15), (2.17), (2.18) for the sum $S_2(x)$ we find

$$|S_2(x)|\le\sum_{k=1}^{\infty}|I(\nu,\mu_k,R_0)|\,\|\psi_k\|_2\left|\int_{x_1}^{x_2}\overline{\psi_k(y)}dy\right|\le C\sum_{\mu_k\ge1}|I(\nu,\mu_k,R_0)|\,\alpha_k$$

$$\le C\sum_{1\le\mu_k\le\nu/2}\alpha_k\left(1+|\nu-\mu_k|^2\right)^{-1}+C\sum_{|\mu_k-\nu|\le1}\alpha_k$$

$$+C\sum_{1\le|\mu_k-\nu|\le\nu/2}\alpha_k\left(1+|\nu-\mu_k|^2\right)^{-1}+C\sum_{\mu_k\ge3\nu/2}\alpha_k\left(1+|\mu_k-\nu|^2\right)^{-1}$$

$$\le C\left(1+\nu^2\right)^{-1}\sum_{1\le\mu_k\le\nu/2}\alpha_k+C\alpha(\nu)+C\alpha(\nu)\sum_{1\le|\mu_k-\nu|\le\nu/2}\left(1+|\nu-\mu_k|^2\right)^{-1}$$

$$+C\alpha(\nu)\sum_{\mu_k\geq 3\nu/2}\left(1+|\nu-\mu_k|^2\right)^{-1}\leq C\nu^{1/2}\left(1+\nu^2\right)^{-1}$$

$$+C\alpha(\nu)+C\alpha(\nu)+C\alpha(\nu)\nu^{-1}\leq C\alpha(\nu).$$

For estimating $S_3(x)$ we apply the first estimation of (2.16), estimations (2.17), (2.18), the Cauchy-Bunyakowsky inequality for the integral and take into account $\|\psi_k\|_{2,m}=1$:

$$|S_3(x)|\leq C\sum_{1\leq\mu_k\leq\rho_0}\frac{1}{4\mu_k^3}\int\limits_{x-R_0}^{x+R_0}$$

$$\times\left(\sum_{l=2}^{4}\|U_l(\xi)\|\left|\psi_k^{(4-l)}(\xi)\right|\mu_k^3\nu^{-1}\right)d\xi\,\|\psi_k\|_{2,m}(x_2-x_1)^{1/2}$$

$$\leq\nu^{-1}\sum_{1\leq\mu_k\leq\rho_0}\left(\int\limits_{x-R_0}^{x+R_0}\sum_{l=2}^{4}\|U_l(\xi)\|(1+\mu_k)^{4-l}d\xi\right)\|\psi_k\|_{2,m}^2$$

$$\leq C\nu^{-1}\sum_{1\leq\mu_k\leq\rho_0}(1+\mu_k)^2\leq C(\rho_0)\nu^{-1}\leq C\alpha(\nu).$$

The sum $S_4(x)$ is estimated in the same way with taking the second estimation of (2.16) into account. As a result, for $S_4(x)$ we find the estimation $|S_4(x)|\leq C(\rho_0)\nu^{-1}=O(\alpha(\nu))$.

By estimations (2.6), (2.9), (2.17), (2.18) the series $S_9(x)$ and $S_{10}(x)$ are overestimated by the quantity $\nu^{-1}=O(\alpha(\nu))$.

Estimate now $const$ the series $S_5(x)$. By estimation (2.9) and expressions $Q_{50}(\xi,\psi_k)$, $M(\xi,\psi_k)$ we have

$$|S_5(x)|\leq C\sum_{\mu_k\geq\rho_0}S_{R_0}$$

$$\times\left[\int\limits_{x}^{x+R}\left|\left[U_2(\xi)\psi_k^{(2)}(\xi)\right]_i\right||J_0(\xi-x,R,\mu_k,\nu)|d\xi\right]\mu_k^{-3}\alpha_k$$

$$+C\sum_{\mu_k\geq\rho_0}S_{R_0}\left[\int\limits_{x}^{x+R}\sum_{l=3}^{4}\|U_l(\xi)\|\left|\psi_k^{(4-l)}(\xi)\right||J_0(\xi-x,R,\mu_k,\nu)|d\xi\right]\mu_k^{-3}\alpha_k$$

$$=\gamma_1(x)+\gamma_2(x).$$

At first we estimate the series $\gamma_2(x)$. For that we use estimations (2.2), (2.17), (2.18)

$$\gamma_2(x)\leq C\sum_{\mu_k\geq\rho_0}S_{R_0}\left[\int\limits_{x}^{x+R}\left(\sum_{l=3}^{4}\|U_l(\xi)\|\mu_k^{4-l}\|\psi_k\|_{2,m}\right)\right.$$

$$\left.\times|J_0(\xi-x,R,\mu_k,\nu)|d\xi\right]\mu_k^{-3}\alpha_k$$

$$\leq C\sum_{\mu_k\geq\rho_0}\mu_k^{-2}\alpha_k S_{R_0}\left[\int\limits_{x}^{x+R}|J_0(\xi-x,R,\mu_k,\nu)|\sum_{l=3}^{4}\|U_l(\xi)\|d\xi\right]$$

$$\leq C\sum_{\mu_k\geq 1}\mu_k^{-2}\alpha_k S_{R_0}\left[\int\limits_x^{x+R}|J_0(\xi-x,R,\mu_k,\nu)|\sum_{l=3}^{4}\|U_l(\xi)\|\,d\xi\right]$$

$$\leq C\left\{\sum_{1\leq\mu_k\leq\nu/2}+\sum_{2\leq|\mu_k-\nu|\leq\nu/2}+\sum_{|\mu_k-\nu|\leq 2}+\sum_{\mu_k\geq 3\nu/2}\right\}$$

$$\leq C\left\{\sum_{1\leq\mu_k\leq\nu/2}\mu_k^{-1}\alpha_k\nu^{-1}+\sum_{2\leq|\mu_k-\nu|\leq\nu/2}\mu_k^{-2}\alpha_k\ln\left(\nu/|\mu_k-\nu|\right)\right.$$

$$\left.+\sum_{|\mu_k-\nu|\leq 2}\mu_k^{-2}\alpha_k\ln\nu+\sum_{\mu_k\geq 3\nu/2}\mu_k^{-1}\alpha_k\nu^{-1}\right\}\int\limits_x^{x+R_0}\sum_{l=3}^{4}\|U_l(\xi)\|\,d\xi$$

$$\leq C\left(K,\|U_l\|_1;l=3,4\right)\alpha(\nu)=O(\alpha(\nu)).$$

For estimating the series $\gamma_1(x)$ we represent it in the form

$$\gamma_1(x)=\sum_{\mu_k\geq\rho_0}\leq\sum_{\mu_k\geq 1}=\sum_{1\leq\mu_k\leq\nu/2}+\sum_{2<|\mu_k-\nu|\leq\nu/2}$$

$$+\sum_{|\mu_k-\nu|\leq 2}+\sum_{\mu_k\geq 3\nu/2}=\sum_{j=1}^{4}\gamma_1^j(x).$$

Prove that each sum $\gamma_1^j(x),\, j=\overline{2,4}$ has order $O(\alpha(\nu))$. For that we use estimations (2.2), (2.17) and (2.18):

$$\gamma_1^2(x)=C\sum_{2\leq|\mu_k-\nu|\nu/2}S_{R_0}$$

$$\times\left[\int\limits_x^{x+R}\left|\left[U_2(\xi)\,\psi_k^{(2)}(\xi)\right]_i\right||J_0(\xi-x,R,\mu_k,\nu)|\,d\xi\right]\mu_k^{-3}\alpha_k$$

$$\leq C\|U_2\|_{1i}\sum_{2\leq|\mu_k-\nu|\leq\nu/2}\alpha_k\mu_k^{-1}\ln\frac{\nu}{|\mu_k-\nu|}$$

$$\leq C\alpha(\nu)\,\nu^{-1}\sum_{2\leq|\mu_k-\nu|\leq\nu/2}\ln\frac{\nu}{|\mu_k-\nu|}$$

$$\leq C\nu^{-1}\alpha(\nu)\sum_{n=2}^{[\nu/2]}\ln\frac{\nu}{n}\left(\sum_{n\leq|\mu_k-\nu|\leq n+1}1\right)$$

$$\leq C\nu^{-1}\alpha(\nu)\sum_{n=2}^{[\nu/2]}\ln\nu/n\leq C\alpha(\nu)\,\nu^{-1}\ln\frac{\nu^{[\nu/2]}}{[\nu/2]!}.$$

By the Stirling formula $n!=n^ne^{-n}\sqrt{2\pi n}\left(1+\frac{\omega}{\sqrt{n}}\right)$, $|\omega|\leq 1$, hence it follows the estimation $\gamma_1^2(x)\leq C\alpha(\nu)=O(\alpha(\nu))$.

In the same way we get

$$\gamma_1^3(x) \le C \sum_{|\mu_k-\nu|\le 2} \alpha_k \mu_k^{-1} \ln\nu \le C\alpha(\nu)\,\nu^{-1}\ln\nu \sum_{|\mu_k-\nu|\le 2} 1 \le C\alpha(\nu),$$

$$\gamma_1^4(x) \le C \sum_{\mu_k\ge 3\nu/2} \alpha_k \mu_k^{-1} \frac{\nu}{\mu_k} \le C\nu \sum_{\mu_k\ge 3\nu/2} \alpha_k \mu_k^{-2}$$

$$\le C\nu\alpha(\nu) \sum_{\mu_k\ge 3\nu/2} \mu_k^{-2} \le C\alpha(\nu).$$

Estimate now the sum $\gamma_1^1(x)$. For that at first we note that

$$\gamma_1^1(x) = C \sum_{1\le\mu_k\le\nu/2} S_{R_0}$$

$$\times \left[\int_x^{x+R} \left| \left[U_2(\xi)\,\psi_k^{(2)}(\xi) \right]_i \right| |J_0(\xi - x, R, \mu_k, \nu)|\, d\xi \right] \mu_k^{-3}\alpha_k$$

$$\le C \sum_{1\le\mu_k\le\nu/2} S_{R_0}$$

$$\times \left[\int_x^{x+R} \sum_{j=1}^m |u_{2ij}(\xi)| \left| \varphi_{kj}^{(2)}(\xi) \right| |J_0(\xi - x, R, \mu_k, \nu)|\, d\xi \right] \mu_k^{-3}\alpha_k$$

$$\equiv \gamma_1^1\left(x, U_2^i\right)$$

where $U_2^i(x) = (u_{2i1}(x), u_{2i2}(x), ..., u_{2im}(x))$ i.e. the i-th row of the matrix $U_2(x)$.

Let $Q_i^n(x) = (q_{i1}^n(x), q_{i2}^n(x), ..., q_{im}^n(x))$ where $q_{ik}^n(x)$ is an algebraic polynomial of the best approximation of the function $u_{2ik}(x)$ in the metric $L_1(G)$ of degree n.

Obviously, $\gamma_1^1\left(x, U_2^i\right) \le \gamma_1^1\left(x, U_2^i - Q_i^n\right) + \gamma_1^1(x, Q_i^n)$.

In the case when the system $\{\psi_k(x)\}_{k=1}^\infty$ is uniformly bounded, having applied estimations (2.2), (2.7), (2.17), (2.18) and the Holder inequality, we get

$$\gamma_1^1\left(x, U_2^i - Q_i^n\right) \le C\nu^{-1} \left(\sum_{1\le\mu_k\le\nu/2} \alpha_k \right) \sum_{j=1}^m \left\| u_{2ij} - q_{ij}^n \right\|_1$$

$$\le C\nu^{-1}\ln\nu \sum_{j=1}^m \left\| u_{2ij} - q_{ij}^n \right\|_1 = C\alpha(\nu)\ln\nu \sum_{j=1}^m \left\| u_{2ij} - q_{ij}^n \right\|_1;$$

$$\gamma_1^1(x, Q_i^n) \le C\nu^{-1} \left(\sum_{1\le\mu_k\le\nu/2} \alpha_k \mu_k^{(1-\beta)/\beta} \right) \sum_{j=1}^m \left\| q_{ij}^n \right\|_\beta$$

$$\le C \frac{\nu^{-1}}{1-\beta^{-1}} \sum_{j=1}^m \left\| q_{ij}^n \right\|_\beta = C \frac{\alpha(\nu)}{1-\beta^{-1}} \sum_{j=1}^m \left\| q_{ij}^n \right\|_\beta,$$

where $1 < \beta < \infty$. In these inequalities of the we take the known estimations

$$\left\| u_{2ij} - q_{ij}^n \right\|_1 \le const\,\omega_1\left(u_{2ij}, n^{-1}\right), \text{ (see [14])}$$

$$\left\|q_{ij}^n\right\|_\beta \leq constn^{2\left(1-\beta^{-1}\right)}\left\|q_{ij}^n\right\|_1, \text{ (see [15])}$$

into account, we find

$$\gamma_1^1\left(x, U_2^i - Q_i^n\right) \leq C\alpha\left(\nu\right)\ln\nu\Omega_{1i}\left(U_2, n^{-1}\right); \tag{2.20}$$

$$\gamma_1^1\left(x, Q_i^n\right) \leq C\alpha\left(\nu\right)\frac{n^{2\left(1-\beta^{-1}\right)}}{1-\beta^{-1}}\left\|U_2\right\|_{1i}. \tag{2.21}$$

Since $\min_{1<\beta<\infty}\frac{n^{2\left(1-\beta^{-1}\right)}}{1-\beta^{-1}} = 2e\ln n$, from inequalities (2.20), (2.21) it follows that $\gamma_1^1\left(x, U_2^i\right) \leq C\alpha\left(\nu\right)\left(1+T_i\left(\nu\right)\right)$ for $r=1$.

In the case $r>1$, applying estimations (2.7), (2.17), (2.18) and the Holder inequality, we find

$$\gamma_1^1\left(x, U_2^i\right) \leq C\nu^{-1}\left(\sum_{j=1}^m\left\|u_{2ij}\right\|_r\right)\sum_{1\leq\mu_k\leq\nu/2}\mu_k^{-(2-1/r)}$$
$$\leq \frac{C}{1-r^{-1}}\left\|U_2\right\|_{1i}\alpha\left(\nu\right).$$

If the system $\{\psi_k\left(x\right)\}_{k=1}^\infty$ is not uniformly bounded, then taking $\alpha_k = \mu_k^{-\frac{1}{2}}$ into account and using estimations (2.2), (2.17), (2.18), we have

$$\gamma_1^1\left(x, U_2^i\right) \leq C\nu^{-1}\left(\sum_{j=1}^m\left\|u_{2ij}\right\|_r\right)\sum_{1\leq\mu_k\leq\nu/2}\mu_k^{-\frac{1}{2}}$$
$$\leq C\nu^{-\frac{1}{2}}\left\|U_2\right\|_{1i} = C\alpha\left(\nu\right)\left\|U_2\right\|_{1i}.$$

Thus, for the series $S_5\left(x\right)$ the following estimation is fulfilled

$$\left|S_5\left(x\right)\right| \leq C\left(K, U_2\right)\begin{cases}\alpha\left(\nu\right) & \text{for } r>1\\ \alpha\left(\nu\right)\left(T_i\left(\nu\right)+1\right) & \text{for } r=1.\end{cases} \tag{2.22}$$

The series $S_6\left(x\right)$ is estimated just in the same way as the series $S_5\left(x\right)$, and for it estimation (2.2) is valid. The series $S_7\left(x\right)$ and $S_8\left(x\right)$ are estimated in the same way. Therewith estimations (2.4) and (2.8) are used. Estimation (2.22) is fulfilled for these series as well.

Thus, for absolute value of the integral

$$\int_{x_1}^{x_2}\left[\widetilde{W}^i\left(x, y, \nu, R_0\right) - \theta^i\left(x, y, \nu\right)\right]dy$$

it holds estimation (2.13). On the other hand,

$$\left|\int_{x_1}^{x_2}\left[\widetilde{W}^i\left(x, y, \nu, R_0\right) - D^i\left(x-y, \nu\right)\right]dy\right|$$
$$= \left|\int_{x_1}^{x_2}\left\{S_{R_0}\left[V\left(x, y, \nu, R\right)\right] - D\left(x-y, \nu\right)\right\}dy\right|.$$

Since the right hand side of this equality corresponds to the case $m = 1$ with the coefficients $U_l(x) = 0$, $l = \overline{2,4}$ (trigonometric system is a system of eigen functions of the operator $lu = u^{(4)}$, $u^{(j)}(0) = u^{(j)}(1)$, $j = \overline{0,3}$) then by the above reasonings it follows

$$\left|\int_{x_1}^{x_2} \left\{S_{R_0}\left[V(x,y,\nu,R) - D(x-y,\nu)\right]\right\} dy\right| \le C(K)\,\nu^{-1}.$$

Applying in the left hand side of (2.13) the triangle inequality, we get

$$\left|\int_{x_1}^{x_2} \left[\theta^i(x,y,\nu) - D^i(x-y,\nu)\right] dy\right|$$

$$\le \left|\int_{x_1}^{x_2} \left[\widetilde{W}^i(x,y,\nu,R_0) - \theta^i(x,y,\nu)\right] dy\right|$$

$$+ \left|\int_{x_1}^{x_2} \left[\widetilde{W}^i(x,y,\nu,R_0) - D^i(x-y,\nu)\right] dy\right|$$

$$\le \left|\int_{x_1}^{x_2} \left[\widetilde{W}^i(x,y,\nu,R_0) - \theta^i(x,y,\nu)\right] dy\right|$$

$$+ \left|\int_{x_1}^{x_2} \left\{S_{R_0}\left[V(x,y,\nu,R) - D(x-y,\nu)\right]\right\} dy\right|$$

$$\le C(K)\begin{cases} \alpha(\nu) & \text{for} \quad r > 1 \\ \alpha(\nu)\left(1 + T_j(\nu)\right) & r = 1. \end{cases}$$

Lemma 2.3 is proved.

Proof of theorems 1.1 and 1.2. Let $f(x)$ be an arbitrary vector-function from the class $W_1^1(G)$. Represent the function $f(x)$ in the form $f(x) = \varphi(x) + \psi(x)$, where $\varphi(0) = (0,0,...,0)^T$, $\varphi(1) = f(1)$ and each component $\varphi_i(x)$ of the vector-function $\varphi(x)$ is a linear function. If suffices to establish estimations (1.1) and (1.2) for each vector-function $\varphi(x)$ and $\psi(x)$. Prove that these estimations for the vector-function $\varphi(x)$ (these estimations are proved in the same way for $\psi(x)$). Transform $\Delta_\nu^i(x,\varphi)$ in the following way

$$\Delta_\nu^i(x,\varphi) = \sigma_\nu^i(x,\varphi) - S_\nu(x,\varphi_i) = \left(\varphi, \overline{\theta^i(x,y,\nu) - D^i(x-y,\nu)}\right)$$
$$= \left(\theta^i(x,y,\nu) - D^i(x-y,\nu), \overline{\varphi}\right).$$

Here, carrying out integration by parts and taking into account $\varphi(1) = \int\limits_0^1 \varphi/(t)\,dt$, we find

$$\Delta_\nu^i(x,\varphi) = \left\langle \int_0^1 \left[\theta^i(x,\xi,\nu) - D^i(x-\xi,\nu)\right] d\xi, \int_0^1 \overline{\varphi'(t)} dt \right\rangle$$

$$-\left(\int_0^y \left[\theta^i(x,\xi,\nu) - D^i(x-\xi,\nu)\right] d\xi, \overline{\varphi'(y)}\right).$$

Having applied in this relation lemma 2.3, we get the estimation

$$\max_{x\in K}\left|\Delta_\nu^i(x,\varphi)\right| \le C(K,U)\left\|\varphi'\right\|_{1,m}\left\{\begin{array}{cc} \alpha(\nu) & \text{for } r>1 \\ \alpha(\nu)(1+T_i(\nu)) & \text{for } r=1 \end{array}\right. .$$

The same estimation will be valid for the vector-function $\psi(x)$. Consequently, for an arbitrary vector-function $f(x) \in W_{1,m}^1(G)$ the following estimation is valid:

$$\left\|\Delta_\nu^i(\cdot,\varphi)\right\|_{C(K)} \le C(K,U)\left\|f'\right\|_{1,m}\left\{\begin{array}{cc} \alpha(\nu) & \text{for } r>1 \\ \alpha(\nu)(1+T_i(\nu)) & \text{for } r=1 \end{array}\right. . \tag{2.23}$$

Theorems 1.1 and 1.2 are proved.

Proof of theorem 1.3. Since $f(x) \in W_{1,m}^1(G)$ and has a compact support, then for any $\varepsilon > 0$ exists $g(x) \in C_{0,m}^\infty(G)$ such that

$$\left\|f' - g'\right\|_{1,m} < \varepsilon. \tag{2.24}$$

Taking into account inequalities (2.23), (2.24), we will estimate $\Delta_\nu^i(x,f)$ for $f(x) \in W_{1,m}^1(G)$ with a compact support. Obviously,

$$\left\|\Delta_\nu^i(\cdot,\varphi)\right\|_{C(K)} \le \left\|\Delta_\nu^i(\cdot,f-g)\right\|_{C(K)} + \left\|\sigma_\nu^i(\cdot,g) - g_i\right\|_{C(K)}$$
$$+ \left\|S_\nu(\cdot,g_i) - g_i\right\|_{C(K)}, \tag{2.25}$$

where $g(x) = (g_1(x), g_2(x), ..., g_m(x))^T$.

For any function $g_i(x) \in C_0^\infty(G)$ the following estimation is fulfilled:

$$\left\|S_\nu(\cdot,g_i) - g_i\right\|_{C(K)} = o\left(\nu^{-1}\right), \nu \to +\infty. \tag{2.26}$$

By inequality (2.23) it holds the estimation

$$\left\|\Delta_\nu^i(\cdot,f-g)\right\|_{C(K)} \le C(K)\left\|f'-g'\right\|_{1,m}\left\{\begin{array}{cc} \alpha(\nu) & \text{for } r>1 \\ \alpha(\nu)(1+T_i(\nu)) & \text{for } r=1 \end{array}\right. .$$

Taking here inequality (2.24)into account, we get

$$\left\|\Delta_\nu^i(\cdot,f-g)\right\|_{C(K)} \le C(K)\,\varepsilon\left\{\begin{array}{cc} \alpha(\nu) & \text{for } r>1 \\ \alpha(\nu)(1+T_i(\nu)) & \text{for } r=1 \end{array}\right. . \tag{2.27}$$

Estimate now the difference $\sigma_\nu^i(x,g) - g_i(x)$ for $x \in K$. To this end, we estimate the Fourier coefficients (g,ψ_k). For that we represent the Fourier coefficients of the vector-function $g(x)$ in the form $(\mu_k \ge 1)$:

$$(g,\psi_k) = \frac{1}{\mu_k^4}(g, L\psi_k) = \left(g,\psi_k^{(4)}\right)\mu_k^{-4} + \left(U_2 g,\psi_k^{(2)}\right)\mu_k^{-4}$$
$$+ \left(U_3 g,\psi_k'\right)\mu_k^{-4} + \left(U_4 g,\psi_k\right)\mu_k^{-4}.$$

Carrying out integration by parts in the first addend with regard to $g_i(x) \in C_0^\infty(G)$, $i = \overline{1,m}$ we get

$$(g,\psi_k) = \mu_k^{-4}\left(g^{(4)},\psi_k\right) + \left(U_3 g,\psi_k'\right)\mu_k^{-4} + \left(U_4 g,\psi_k\right)\mu_k^{-4}$$

$$+\left(U_2 g, \psi_k^{(2)}\right)\mu_k^{-4} = O\left(\mu_k^{-2}\alpha_k\right) + \left(U_2 g, \psi_k^{(2)}\right)\mu_k^{-4}. \qquad (2.28)$$

In the expression $\left(U_2 g, \psi_k^{(2)}\right)$ we apply formula (2.10) for $l = 2$.

$$\overline{\left(U_2 g, \psi_k^{(2)}\right)} = \mu_k^2 \left\{ \sum_{j=1}^{3} (-i\omega_j)^2 \sum_{s=1}^{m} X_{kjs}(0) \int_0^1 \overline{(U_2(t)\, g(t))_s} \exp(-i\omega_j \mu_k t)\, dt \right.$$

$$+ (-i\omega_4)^2 \sum_{s=1}^{m} B_{k4s}(0) \int_0^1 \overline{(U_2(t)\, g(t))_s} \exp(i\omega_4 \mu_k (1-t))\, dt$$

$$- \sum_{j=1}^{3} \omega_j^3 \sum_{s=1}^{m} (M(\xi, \psi_k))_s \int_\xi^1 \overline{(U_2(t)\, g(t))_s} \exp(-i\omega_j \mu_k (\xi - t))\, dt d\xi$$

$$\left. + \omega_4^3 \sum_{s=1}^{m} (M(\xi, \psi_k))_s \int_0^\xi \overline{(U_2(t)\, g(t))_s} \exp(i\omega_4 \mu_k (\xi - t))\, dt d\xi \right\},$$

where the index s indicates the s-th component of the appropriate vector. Hence, from estimations (2.11), (2.12) we find

$$\left| \left(U_2 g, \psi_k^{(2)}\right) \right|$$

$$\leq const \mu_k^2 \left\{ \|\psi_k\|_{\infty,m} \sum_{j=1}^{3} \sum_{s=1}^{m} \left| \int_0^1 \overline{(U_2(t)\, g(t))_s} \exp(-i\omega_j \mu_k t)\, dt \right| \right.$$

$$+ \|\psi_k\|_{\infty,m} \sum_{s=1}^{m} \left| \overline{(U_2(t)\, g(t))_s} \exp(i\omega_4 \mu_k (1-t))\, dt \right|$$

$$\left. + \mu_k^{-1} \|\psi_k\|_{\infty,m} \left(\sum_{l=2}^{4} \|U_l\|_1 \mu_k^{2-l} \right) \sum_{s=1}^{m} \|U_2 g\|_{1,m} \right\}.$$

Taking here the inequalities (see [10])

$$\left| \int_0^1 (U_2(t)\, g(t))_s \exp(-i\omega_j \mu_k t)\, dt \right|$$

$$\leq const \left\{ \omega_1\left((U_2(t)\, g(t))_s, \mu_k^{-1} \right) + \mu_k^{-1} \|(U_2 g)_s\|_1 \right\}, \quad j = \overline{1,3}, \quad s = \overline{1,m};$$

$$\left| \int_0^1 \overline{(U_2(t)\, g(t))_s} \exp(-i\omega_4 \mu_k (1-t))\, dt \right|$$

$$\leq const \left\{ \omega_1\left((U_2 g)_s, \mu_k^{-1} \right) + \mu_k^{-1} \|(U_2 g)_s\|_1 \right\}, \quad s = \overline{1,m}$$

into account, for $\mu_k \geq 4\pi$ we get

$$|(U_2 g, \psi_k)| \leq const \mu_k^2 \left\{ \omega_{1,m}\left(U_2 g, \mu_k^{-1}\right) + \mu_k^{-1} \|U_2 g\|_{1,m} \right\} \|\psi_k\|_{\infty,m}, \qquad (2.29)$$

where $\omega_{1,m}(f,\delta) = \max_{1\le s\le m}\omega_1(f_s,\delta)$, $\omega_1(\cdot,\delta)$ is a modulus of continuity in $L_1(0,1)$.
Consequently, from (2.28) and (2.29) we find

$$(g,\psi_k) = O\left(\mu_k^{-2}\alpha_k\right) + O\left(\mu_k^{-2}\right)\left\{\omega_{1,m}\left(U_2 g,\mu_k^{-1}\right) + \mu_k^{-1}\|U_2 g\|_{1,m}\right\}\|\psi_k\|_{\infty,m}$$

$$= O\left(\mu_k^{-2}\alpha_k\right) + O\left(\mu_k^{-1}\alpha_k\right)\left\{\omega_{1,m}\left(U_2 g,\mu_k^{-1}\right) + \mu_k^{-1}\|U_2 g\|_{1,m}\right\}$$

$$= O\left(\mu_k^{-2}\alpha_k\right) + O\left(\mu_k^{-1}\alpha_k\right)\delta_k, \quad \delta_k\to 0, \delta_k>0.$$

This estimation and estimation (2.18) allows to state that the Fourier series of the vector-function $g(x)$ in the system $\{\psi_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the compact K. Indeed, for $x\in K$

$$\sum_{k=1}^{\infty}|(g,\psi_k)||\psi_k(x)| \le \sum_{0\le\mu_k\le 4\pi}|(g,\psi_k)||\psi_k(x)| + \sum_{\mu_k>4\pi}|(g,\psi_k)||\psi_k(x)|$$

$$\le C\|g\|_{1,m}\sum_{1\le\mu_k\le 4\pi}1 + C\sum_{\mu_k>4\pi}\left\{\mu_k^{-2}\alpha_k + \delta_k\mu_k^{-1}\alpha_k\right\}\|\psi_k\|_{2,m} < \infty.$$

Therefore for $\mu_k > 4\pi$ and $x\in K$ by virtue of $\|\psi_k\|_{2,m} = 1$ we get

$$|\sigma_\nu(x,g) - g(x)| = \left|\sum_{\mu_k\le\nu}(g,\psi_k)\psi_k(x) - g(x)\right|$$

$$\le C\sum_{\mu_k>\nu}\left\{\mu_k^{-2}\alpha_k + \delta_k\mu_k^{-1}\alpha_k\right\} \le C\nu^{-1}\alpha(\nu) + C\left(\sup_{\mu_k>\nu}\delta_k\right)\alpha(\nu)$$

$$= o(\alpha(\nu)), \quad \nu\to+\infty.$$

Thus, the estimation

$$\sigma_\nu^i(x,g) - g_i(x) = o(\alpha(\nu)), \quad \nu\to\infty, \tag{2.30}$$

is fulfilled for any $i=\overline{1,m}$.

By the obtained estimations (2.26), (2.27), (2.30) and arbitrariness of $\varepsilon>0$ from (2.25) we get estimation (1.5). Theorem 1.3 is proved.

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