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Notes About Tensor Fields of Type (1,1) on the Cross-Section in the Cotangent Bundle

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Abstract. The main purpose of this article is to improve the cross-section idea in the cotangent bundle T^*M^n which is given in [5], [7]. We investigate horizontal and diagonal lifts of tensor fields of type (1,1) on a cross-section in the cotangent bundle T^*M^n and we find some relation for them.

Keywords. Cross-section, cotangent bundle, vertical and horizontal lift, diagonal lift, almost complex structure, Nijenhuis tensor, Kählerian structure.

Mathematics Subject Classification (2010): 53C07, 53C15, 53B05

1 Introduction

The cross-section idea is very useful tool for physics and mathematic. On the differential geometry, vector fields and 1-forms can be regarded as cross-sections of the tangent bundle and the cotangent bundle, respectively. In [5] Yano study the behaviour of the lifts of tensor fields and connections on the cross-section in the cotangent bundle T^*M^n. In [4] Salimov et al show that the complete lift of almost complex structure when restricted to the cross-section determined by an almost analytic 1-form ω on M^n, is an almost complex structure. The lifts of tensor fields and connections are studied along the cross-section in the tangent, cotangent and tensor bundles [1], [2], [3], [6], [7]. In this study we investigate horizontal and diagonal lift of tensor fields of type (1,1) on the cross-section in the cotangent bundle and obtain some relation for them.

Let T^*M^n be the cotangent bundle of (M^n, g) Riemannian manifold and π the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1, ..., n$ on M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i$ $(i = 1, ..., n)$, where $x^{\bar{i}} = p_i$ are the components of the covector p in each cotangent space $T^*_xM^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, $i = 1, ..., n$. We denote by $\Im^r_s(T^*M^n)$ the set of all tensor fields of type (r, s) on T^*M^n.

Now, consider the complete and horizontal lifts ${}^CX, {}^HX \in \Im^1_0(T^*M^n)$ of vector field $X \in \Im^1_0(M^n)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M^n)$ of covector field (1-form) $\omega \in \Im^0_1(M^n)$

$$ {}^CX = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.1}$$

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$$ {}^{H}X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.2} $$

$$ {}^{V}\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.3} $$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, where Γ_{ij}^h are the components of the Levi-Civita connection ∇_g on M^n and X^i and ω_i are local components of X and ω (see [7] for more details).

Let $\theta \in \Im_1^0(M^n)$ be a 1-form whose local expression in $U \subset M^n$ is $\theta = \theta_i dx^i$. Then the correspondence $x \to \theta_x$, θ_x being the value of θ at $x \in M^n$, determines a mapping $\beta_\theta : M^n \to T^*M^n$, such that $\pi \circ \beta_\theta = I_{M^n}$ and the n-dimensional submanifold $\beta_\theta(M^n)$ of T^*M^n is called the cross-section determined by θ. The cross-section $\beta_\theta(M^n)$ is locally expressed by

$$ \begin{cases} x^h = x^h, \\ p_h = \theta_h(x^1, ..., x^n), \end{cases} \tag{1.4} $$

with respect to the coordinates (x^h, p_h) in T^*M^n.

Differentiating (1.4) by x^i, we find that the tangent vector $B_{(i)}$ to $\beta_\theta(M^n)$ have component

$$ B_{(i)} = ({B_i}^A) = (\frac{\partial x^A}{\partial x^i}) = \begin{pmatrix} \delta_i^h \\ \partial_i \theta_h \end{pmatrix}, \tag{1.5} $$

with respect to the natural frame $\{\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^{\bar{h}}}\}$ in T^*M^n.

Thus, we have

$$ BX = ({B_i}^A X^i) = \begin{pmatrix} X^h \\ X^i \partial_i \theta_h \end{pmatrix}, $$

for any $X \in \Im_0^1(M^n)$ which is defined along $\beta_\theta(M^n)$ in T^*M^n.

On the other hand, the fibre is locally represented by

$$ x^h = const., \quad p_h = p_h. \tag{1.6} $$

Thus, on differentiating (1.6) with respect to p_i, we find the tangent vector $C_{(i)}$ to the fibre have components

$$ C_{(i)} = ({C_i}^A) = \begin{pmatrix} 0 \\ \delta_h^i \end{pmatrix}, \tag{1.7} $$

with respect to the natural frame $\{\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^{\bar{h}}}\}$ in T^*M^n.

Thus, we denote by $C\omega$ the vector field with local components for any $\omega \in \Im_1^0(M^n)$

$$ C\omega = ({C_i}^A \omega_i) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} $$

which is tangent to the fibre along $\beta_\theta(M^n)$ in T^*M^n.

2n local vector fields $B_{(i)}$ and $C_{(i)}$ are linearly independent and form a frame along the cross-section $\beta_\theta(M^n)$. We call this the adapted (B,C)-frame along the cross-section [7].

We now, from equations (1.1), (1.2), (1.3), (1.5) and (1.7) see that ${}^{C}X$, ${}^{H}X$ and ${}^{V}\omega$ have respectively components

$$ {}^{C}X = \begin{pmatrix} X_h \\ -L_X \theta_h \end{pmatrix}, $$

$$ {}^{H}X = \begin{pmatrix} X_h \\ -X^m \nabla_m \theta_h \end{pmatrix}, \tag{1.8} $$

$$ {}^{V}\omega = \begin{pmatrix} 0 \\ \omega_h \end{pmatrix}, \tag{1.9} $$

with respect to the adapted (B,C)-frame on the cross-section of the cotangent bundle.

2 Horizontal lift of a symmetric affine connection on a cross-section in the cotangent bundle

The horizontal lift ${}^{H}\nabla$ of ∇ to T^*M^n has components ${}^{H}\Gamma^{K}_{JI}$ given by

$$ \begin{aligned} &{}^{H}\Gamma^{k}_{ji} = \Gamma^{k}_{ji}, \qquad &&{}^{H}\Gamma^{k}_{j\bar{i}} = {}^{H}\Gamma^{k}_{\bar{j}i} = {}^{H}\Gamma^{k}_{\bar{j}\bar{i}} = {}^{H}\Gamma^{\bar{k}}_{\bar{j}\bar{i}} = 0, \\ &{}^{H}\Gamma^{\bar{k}}_{j\bar{i}} = -\Gamma^{i}_{jk}, \qquad &&{}^{H}\Gamma^{\bar{k}}_{\bar{j}i} = -\Gamma^{j}_{ki}, \\ &{}^{H}\Gamma^{\bar{k}}_{ji} = p_a(-\partial_j \Gamma^{a}_{ik} + \Gamma^{a}_{kt}\Gamma^{t}_{ji} + \Gamma^{a}_{it}\Gamma^{t}_{kj}), \end{aligned} \tag{2.1} $$

with respect to the induced coordinates where Γ^{k}_{ji} are components of ∇ in U [7].

The affine connection ${}^{H}\tilde{\nabla}$ induced on $\beta_\theta(M^n)$ from the horizontal lift ${}^{H}\nabla$ of an affine connection ∇ in M^n has components of the form

$$ {}^{H}\tilde{\Gamma}^{k}_{ji} = (\partial_j B_i{}^{A} + {}^{H}\Gamma^{A}_{CB} B_j{}^{C} B_i{}^{B}) B^{k}{}_{A} \tag{2.2} $$

where $B^{k}{}_{A}$ are defined by

$$ (B^{k}{}_{A}, C^{k}{}_{A}) = (B_i{}^{A}, C_i{}^{A})^{-1} $$

and hence

$$ (B^{k}{}_{A}) = (\delta^{k}_{i}, 0), \quad (C^{k}{}_{A}) = (-\partial_i \theta_k, \delta^{i}_{k}). \tag{2.3} $$

Using (1.5), (1.7), (2.1) and (2.3) in (2.2), we find

$$ {}^{H}\tilde{\Gamma}^{k}_{ji} = \Gamma^{k}_{ji} $$

where Γ^{k}_{ji} are components of ∇ in M^n.

From (2.2), we have

$$ \partial_j B_i{}^{A} + {}^{H}\Gamma^{A}_{CB} B_j{}^{C} B_i{}^{B} - \Gamma^{k}_{ji} B_k{}^{A} = H^{\bar{s}}_{ji} C_s{}^{A} \tag{2.4} $$

i.e., the left hand side is a linear combination of $C_s{}^{A}$. To find the coefficient $H^{\bar{h}}_{ji}$, we put $A = \bar{h}$ in (2.4) and hence obtain

$$ H^{\bar{h}}_{ji} = \nabla_j \nabla_i \theta_h. $$

The coefficient $H^{\bar{h}}_{ji}$ in (2.4) define the second fundamental tensor of the submanifold $\beta_\theta(M^n)$ with respect to the normals $C_{(i)}$. When $H^{\bar{h}}_{ji} = 0$, $\beta_\theta(M^n)$ is said to be totally geodesic in T^*M^n [7].

Theorem 2.1 *The cross-section $\beta_\theta(M^n)$ in T^*M^n is totally geodesic in T^*M^n with the horizontal lift ${}^{H}\nabla$ of ∇ if and only if θ satisfy $\nabla_j \nabla_i \theta_k = 0$, θ_k being local components of θ.*

3 Horizontal and Diagonal Lifts of Tensor Fields of Type (1,1) on a Cross-Section

Let $\varphi \in \Im_1^1(M^n)$. Then the diagonal lift ${}^D\varphi$ of φ to T^*M^n has local components of the form

$$ {}^D\varphi : ({}^D\varphi_B^A) = \begin{pmatrix} \varphi_i^h & 0 \\ \Gamma_{is}\varphi_h^s + \Gamma_{hs}\varphi_i^s & -\varphi_h^i \end{pmatrix} \tag{3.1} $$

with respect to the induced coordinates (x^h, p_h) [7].

From (1.5), (1.7), (2.3) and (3.1), ${}^D\tilde{\varphi}$ has components ${}^D\tilde{\varphi}_\beta^\alpha$ given by

$$ \begin{aligned} {}^D\tilde{\varphi}_l^k &= {}^D\varphi_B^A B_l{}^B B^k{}_A = \varphi_l^k, \\ {}^D\tilde{\varphi}_{\tilde{l}}^k &= {}^D\varphi_B^A C_l{}^B B^k{}_A = 0, \\ {}^D\tilde{\varphi}_l^{\tilde{k}} &= {}^D\varphi_B^A B_l{}^B C^k{}_A = -(\varphi_l^h \nabla_h \theta_k + \varphi_k^h \nabla_l \theta_h), \\ {}^D\tilde{\varphi}_{\tilde{l}}^{\tilde{k}} &= {}^D\varphi_B^A C_l{}^B C^k{}_A = -\varphi_k^l, \end{aligned} $$

i.e. ${}^D\tilde{\varphi}$ has components

$$ {}^D\tilde{\varphi} : {}^D\tilde{\varphi}_B^A = \begin{pmatrix} \varphi_l^k & 0 \\ -(\varphi_l^h \nabla_h \theta_k + \varphi_k^h \nabla_l \theta_h) & -\varphi_k^l \end{pmatrix} \tag{3.2} $$

with respect to the adapted (B,C)-frame on the cross-section $\beta_\theta(M^n)$ in T^*M^n.

Using (1.8), (1.9) and (3.2) we have

$$ {}^D\tilde{\varphi}({}^H X) = {}^H(\varphi X), \qquad X \in \Im_0^1(M^n), \tag{3.3} $$

$$ {}^D\tilde{\varphi}({}^V \omega) = -{}^V(\omega \circ \varphi), \qquad \omega \in \Im_1^0(M^n), \tag{3.4} $$

which characterize ${}^D\tilde{\varphi}$, where $\omega \circ \varphi \in \Im_1^0(M^n)$.

The horizontal lift ${}^H\varphi$ of φ has components

$$ {}^H\varphi : ({}^H\varphi_B^A) = \begin{pmatrix} \varphi_i^h & 0 \\ -\Gamma_{is}\varphi_h^s + \Gamma_{hs}\varphi_i^s & \varphi_h^i \end{pmatrix} $$

with respect to the induced coordinates in T^*M^n.

Then using same calculation, ${}^H\tilde{\varphi}$ has components

$$ {}^H\tilde{\varphi} : {}^H\tilde{\varphi}_B^A = \begin{pmatrix} \varphi_l^k & 0 \\ -\varphi_l^h \nabla_h \theta_k + \varphi_k^h \nabla_l \theta_h & \varphi_k^l \end{pmatrix} \tag{3.5} $$

with respect to the adapted (B,C)-frame on the cross-section $\beta_\theta(M^n)$ in T^*M^n.

From (1.8), (1.9) and (3.5) we find

$$ {}^H\tilde{\varphi}({}^H X) = {}^H(\varphi X), \qquad X \in \Im_0^1(M^n), \tag{3.6} $$

$$ {}^H\tilde{\varphi}({}^V \omega) = {}^V(\omega \circ \varphi), \qquad \omega \in \Im_1^0(M^n), \tag{3.7} $$

which characterize ${}^H\tilde{\varphi}$, where $\omega \circ \varphi \in \Im_1^0(M^n)$.

Proposition 3.1 *[7] Let $\tilde{S}$ and $\tilde{T}$ be tensor fields in T^*M^n of type (0,s) or (1,s), where $s > 0$, such that*

$$\tilde{S}(\tilde{X}_s, ..., \tilde{X}_1) = \tilde{T}(\tilde{X}_s, ..., \tilde{X}_1)$$

for all vector fields $\tilde{X}_1, ..., \tilde{X}_s$ which are of the form ${}^V\omega$ or ${}^HZ$ where $\omega \in \Im_1^0(M^n)$ and $Z \in \Im_0^1(M^n)$. Then $\tilde{S} = \tilde{T}$.

Theorem 3.1 *If $\varphi, \phi \in \Im_1^1(M^n)$, then with respect symmetric affine connection ∇ in M^n, we have*

$${}^D\tilde{\varphi}{}^D\tilde{\phi} + {}^D\tilde{\phi}{}^D\tilde{\varphi} = {}^H(\varphi\phi + \phi\varphi)$$

$${}^D\tilde{\varphi}{}^H\tilde{\phi} + {}^D\tilde{\phi}{}^H\tilde{\varphi} = {}^H\tilde{\varphi}{}^D\tilde{\phi} + {}^H\tilde{\phi}{}^D\tilde{\varphi} = {}^D(\varphi\phi + \phi\varphi) \tag{3.8}$$

$${}^H\tilde{\varphi}{}^H\tilde{\phi} + {}^H\tilde{\phi}{}^H\tilde{\varphi} = {}^H(\varphi\phi + \phi\varphi) \tag{3.9}$$

with respect to the adapted (B,C)-frame of $\beta_\theta(M^n)$ in T^*M^n.

Proof. If $\varphi, \phi \in \Im_1^1(M^n)$, then we have by (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) and Proposition1

$$\begin{aligned}({}^D\tilde{\varphi}{}^D\tilde{\phi} + {}^D\tilde{\phi}{}^D\tilde{\varphi})({}^HX) &= {}^D\tilde{\varphi}{}^H(\phi X) + {}^D\tilde{\phi}{}^H(\varphi X)\\ &= {}^H((\varphi\phi + \phi\varphi)X) = {}^H(\varphi\phi + \phi\varphi){}^HX,\\ ({}^D\tilde{\varphi}{}^D\tilde{\phi} + {}^D\tilde{\phi}{}^D\tilde{\varphi})({}^V\omega) &= -{}^D\tilde{\varphi}{}^V(\omega \circ \phi) - {}^D\tilde{\phi}{}^V(\omega \circ \varphi)\\ &= {}^V(\omega \circ \varphi\phi) + {}^V(\omega \circ \phi\varphi) = {}^V(\omega \circ (\varphi\phi + \phi\varphi))\\ &= {}^H(\varphi\phi + \phi\varphi){}^V\omega\end{aligned}$$

which imply

$${}^D\tilde{\varphi}{}^D\tilde{\phi} + {}^D\tilde{\phi}{}^D\tilde{\varphi} = {}^H(\varphi\phi + \phi\varphi).$$

Same calculation we can prove (3.8) and (3.9).

Then putting $\tilde{\varphi} = \tilde{\phi}$ in Theorem 2, we have

$${}^D\tilde{\varphi}{}^D\tilde{\varphi} = {}^H(\varphi\varphi), \qquad ({}^D\tilde{\varphi})^2 = {}^H(\varphi^2).$$

Since ${}^H(id_{M^n}) = id_{\Im_1^1(M^n)}$, from (3.8), we have

Theorem 3.2 *If φ is almost complex structure in M^n, then the diagonal lift ${}^D\tilde{\varphi}$ of φ along the cross-section $\beta_\theta(M^n)$ is an almost complex structure in T^*M^n.*

Then we see, for $\tilde{\varphi} = \tilde{\phi}$ in Theorem 2,

$$({}^H\tilde{\varphi})^2 = {}^H(\varphi^2).$$

So we have

Theorem 3.3 *Let φ be an almost complex structure in M^n then the horizontal lift ${}^H\tilde{\varphi}$ of φ along the cross-section $\beta_\theta(M^n)$ is an almost complex structure in T^*M^n.*

It is well know that [7, p.238, p.277]

$$\begin{array}{ll} i)^V\omega^V f = 0, & ii)^H X^V f =^V (Xf), \\ iii)[^H X,^V \omega] =^V (\nabla_X \omega), & iv)[^V\omega,^V \theta] = 0, \\ v)[^H X,^H Y] =^H [X,Y] + \gamma R(X,Y) =^H [X,Y] +^V (pR(X,Y)) & \end{array} \tag{3.10}$$

for any $X, Y \in \Im^1_0(M^n), \omega, \theta \in \Im^0_1(M^n)$, where $pR(X,Y) = (p_i(R(X,Y)^i_j))$.

The Nijenhuis tensor N_φ of $\varphi \in \Im^1_1(M^n)$ defined by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y].$$

where $X, Y \in \Im^1_0(M^n)$.

We now consider Nijenhuis tensor of ${}^D\tilde{\varphi}$. Taking account of (3.3), (3.4) and (3.10), we have the following formulas:

$$\begin{aligned} &N_{^D\tilde{\varphi}}(^V\omega,^V\theta) = 0, \\ &N_{^D\tilde{\varphi}}(^H X,^V\omega) =^V (-(\nabla_{\varphi X}\varphi)\omega - \varphi(\nabla_X\varphi)\omega), \\ &N_{^D\tilde{\varphi}}(^H X,^H Y) =^H (N_\varphi(X,Y)) \\ &\quad +^V (pR(\varphi X,\varphi Y) + \varphi(pR(\varphi X,Y)) + \varphi(pR(X,\varphi Y)) + \varphi^2(pR(X,Y))) \end{aligned}$$

for any $X, Y \in \Im^1_0(M^n), \omega, \theta \in \Im^0_1(M^n)$.

Remark. Let (φ, g) be a Kählerian structure in M^n and ∇ be the Riemannian connection determined by the metric g. Then we see that
i) φ is an almost complex structure in M^n, i.e. $\varphi^2 = -I$;
ii) $\nabla\varphi = 0$;
iii) The curvature tensor R of ∇ satisfies $R(\varphi X, \varphi Y) = R(X,Y)$ for any $X, Y \in \Im^1_0(M^n)$ [7].

From (iii), we write $R(\varphi X, Y) = -R(X, \varphi Y)$ and

$$R(\varphi X, \varphi Y) + R(\varphi X, Y)\varphi + R(X, \varphi Y)\varphi + R(X,Y)\varphi^2 = 0$$

since $\varphi^2 = -I$.

Thus from (31) and (ii), we have

$$\begin{aligned} &\tilde{N}_{^D\tilde{\varphi}}(^V\omega,^V\theta) = 0, \\ &\tilde{N}_{^D\tilde{\varphi}}(^H X,^V\omega) = 0, \\ &\tilde{N}_{^D\tilde{\varphi}}(^H X,^H Y) = 0 \end{aligned}$$

for any $X, Y \in \Im^1_0(M^n), \omega, \theta \in \Im^0_1(M^n)$.

By Proposition 1, $\tilde{N}_{^D\tilde{\varphi}}$ is zero, since N is skew-symmetric so ${}^D\tilde{\varphi}$ is necessarly integrable. Hence we have

Theorem 3.4 *Let (φ, g) be a Kählerian structure in M^n and ∇ be the Riemannian connection determined by the metric g. Then the diagonal lift ${}^D\tilde{\varphi}$ of φ to T^*M^n along $\beta_\theta(M^n)$ is an complex structure in T^*M^n.*

Now we use same calculation for horizontal lift ${}^H\tilde{\varphi}$ of φ. From (3.6), (3.7) and (3.10) we have

$$\begin{aligned} &N_{^H\tilde{\varphi}}(^V\omega,^V\theta) = 0, \\ &N_{^H\tilde{\varphi}}(^H X,^V\omega) =^V ((\nabla_{\varphi X}\varphi)\omega - \varphi(\nabla_X\varphi)\omega), \\ &N_{^H\tilde{\varphi}}(^H X,^H Y) =^H (N_\varphi(X,Y)) \\ &\quad +^V (pR(\varphi X,\varphi Y) - \varphi(pR(\varphi X,Y)) + \varphi(pR(X,\varphi Y)) + \varphi^2(pR(X,Y))) \end{aligned}$$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$.

Let (φ, g) be a Kählerian structure in M^n and ∇ be the Riemannian connection determined by the metric g. Hence, from (32) and Remark (ii), we have

$$\tilde{N}_{^H\tilde{\varphi}}(^V\omega, ^V\theta) = 0,$$
$$\tilde{N}_{^H\tilde{\varphi}}(^HX, ^V\omega) = 0,$$
$$\tilde{N}_{^H\tilde{\varphi}}(^HX, ^HY) = 0$$

for any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. So $^H\tilde{\varphi}$ is integrable.

Theorem 3.5 *Let (φ, g) be a Kählerian structure in M^n and ∇ be the Riemannian connection determined by the metric g. Then the horizontal lift $^H\tilde{\varphi}$ of φ to T^*M^n along $\beta_\theta(M^n)$ is an complex structure in T^*M^n.*

Let now we consider the curvature tensor ^HR of the horizontal lift $^H\nabla$ of symmetric affine connection ∇ in M^n to T^*M^n. Then ^HR is determined by

$$^HR_{kji}{}^h = R_{kji}{}^h, \qquad ^HR_{kj\bar{i}}{}^{\bar{h}} = -R_{kjh}{}^i,$$
$$^HR_{kji}{}^{\bar{h}} = p_a(\Gamma^a_{ht}R_{kji}{}^t + \Gamma^a_{it}R_{kjh}{}^t)$$

the remaining components being zero, with respect to the induced coordinates [7, p.288].

Then $^H\tilde{R}$ has components $^H\tilde{R}_{\alpha\beta\gamma}{}^\theta$ given by

$$^H\tilde{R}_{ijk}{}^l = {}^H R_{ABC}{}^D B_i{}^A B_j{}^B B_k{}^C B^l{}_D = R_{ijk}{}^l,$$
$$^HR_{ijk}{}^{\bar{l}} = {}^H R_{ABC}{}^D B_i{}^A B_j{}^B B_k{}^C C^l{}_D = -R_{ijk}{}^h\nabla_l\theta_h - R_{ijl}{}^h\nabla_k\theta_h,$$
$$^HR_{ij\bar{k}}{}^{\bar{l}} = {}^H R_{ABC}{}^D B_i{}^A B_j{}^B C_k{}^C C^l{}_D = -R_{ijl}{}^k$$

and the others is zero with respect to the adapted (B,C)-frame on the cross-section $\beta_\theta(M^n)$ in T^*M^n.

Theorem 3.6 *$^H\tilde{R}_{\alpha\beta\gamma}{}^\theta$ is tangent to the cross-section $\beta_\theta(M^n)$ in T^*M^n if and only if $R = 0$ i.e. M^n is flat.*

References

1. Gezer, A., Salimov, A.: *Diagonal lifts of tensor fields of type (1,1) on cross-sections in tensor bundles and its applications*, J. Korean Math. Soc. **45** (2), 367–376 (2008).
2. Magden, A., Salimov, A.A.: *Complete lifts of tensor fields on a pure cross-section in the tensor bundle*, J. Geom. **93** (1-2), 128–138 (2009).
3. Salimov, A.: Tensor Operators and Their Applications, *Nova Science Publishers, New York* (2013).
4. Salimov, A., Gezer, A., Aslanci, S.: *On almost complex structures in the cotangent bundle*, Turkish J. Math. **35** (3), 487–492 (2011).
5. Yano, K.: *Tensor fields and connections on cross-sections in the cotangent bundle*, Tohoku Math. J. **19** (2), 32–48 (1967).
6. Yano, K.: *Tensor fields and connections on cross-sections in the tangent bundle of a differentiable manifold*, Proc. Roy. Soc. Edinburgh Sect. A **67**, 277–288 (1968).
7. Yano, K., Ishihara, S.: Tangent and cotangent bundles, *Marcel Dekker Inc., N.Y.* (1973).