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Characterizations of parabolic fractional integral operators on generalized parabolic Morrey spaces

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Abstract. In this paper we shall give a characterization for the Spanne-Guliyev and Adams-Guliyev type boundedness of the parabolic fractional integral operator $I_{\alpha,P}$ in the generalized parabolic Morrey spaces $M_{p,\varphi,P}(R^n)$. We also give criteria for the weak versions of the Spanne-Guliyev and Adams-Guliyev type boundedness of $I_{\alpha,P}$ on the generalized parabolic Morrey spaces.

Keywords. Parabolic fractional integral operator; parabolic generalized parabolic Morrey space.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction

Let P be a real $n \times n$ matrix, whose all eigenvalues have a positive real part. Let $A_t = t^P \;\; (t > 0)$, and the set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$\begin{aligned}
&(a)\;\; \rho(A_t x) = t\rho(x),\;\; t > 0,\;\; \text{for every}\;\; x \in R^n;\\
&(b)\;\; \rho(0) = 0,\;\; \rho(x-y) = \rho(y-x) \geq 0\\
&\qquad \text{and}\;\; \rho(x-y) \leq k(\rho(x-z) + \rho(y-z));\\
&(c)\;\; dx = \rho^{\gamma-1} d\sigma(w) d\rho,\;\; \text{where}\;\; \rho = \rho(x), w = A_{\rho^{-1}}x\\
&\qquad \text{and}\; d\sigma(w) \text{is a}\; C^\infty \text{measure on the ellipsoid}\; \{w : \rho(w) = 1\}.
\end{aligned}$$

Then, $\{R^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $E(x,r) = \{y \in R^n : \rho(x-y) < r\}$, with the Lebesgue measure $|E(x,r)| = v_n r^\gamma$, where v_n is the volume of the unit ellipsoid in R^n. Let also ${}^{\complement}E(x,r) = R^n \setminus E(x,r)$ be the complement of $E(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $E_I(x,r) = B(x,r)$.

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Let $f \in L_1^{\rm loc}(R^n)$. The maximal operator M and the fractional integral operator $I_{\alpha,P}$ are defined by

$$M_P f(x) = \sup_{r>0} |E(x,r)|^{-1} \int_{E(x,r)} |f(y)| dy,$$

$$I_{\alpha,P} f(x) = \int_{R^n} \frac{f(y)dy}{\rho(x-y)^{Q-\alpha}}, \qquad 0 < \alpha < Q,$$

where Q is the homogeneous dimension of $\{R^n, \rho, dx\}$ and $|E(x,t)|$ is the Lebesgue measure of the parabolic ball $E(x,t)$.

The operators M_P and $I_{\alpha,P}$ play an important role in real and harmonic analysis and applications (see, for example [22]).

In the present work, we shall give a characterization for the Spanne-Guliyev and Adams-Guliyev type boundedness of the operator $I_{\alpha,P}$ on the generalized parabolic Morrey spaces. Also we give criteria for the weak versions of Spanne-Guliyev and Adams-Guliyev type boundedness of the operator $I_{\alpha,P}$ on the generalized parabolic Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized Morrey spaces

In the study of local properties of solutions to partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(R^n)$ play an important role, see [6]. They were introduced by C. Morrey in 1938 [17]. The parabolic Morrey space is defined as follows: for $1 \le p \le \infty$, $0 \le \lambda \le Q$, a function $f \in L_{p,\lambda,P}(R^n)$ if $f \in L_p^{\rm loc}(R^n)$ and

$$\|f\|_{L_{p,\lambda,P}} := \sup_{x\in R^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(E(x,r))} < \infty,$$

(If $\lambda = 0$, then $L_{p,0,P}(R^n) = L_p(R^n)$; if $\lambda = Q$, then $L_{p,Q,P}(R^n) = L_\infty(R^n)$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,P}(R^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on R^n.)

We also denote by $WL_{p,\lambda,P}(R^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(R^n)$ for which

$$\|f\|_{WL_{p,\lambda,P}} \equiv \|f\|_{WL_{p,\lambda,P}(R^n)} = \sup_{x\in R^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(E(x,r))} < \infty,$$

where $WL_p(E(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(E(x,r))} = \sup_{t>0} t \left|\{y \in E(x,r) : \, |f(y)| > t\}\right|^{1/p}. \tag{2.1}$$

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then $I_{\alpha,P}$ is bounded from $L_p(R^n)$ to $L_q(R^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, $I_{\alpha,P}$ is bounded from $L_1(R^n)$ to $WL_q(R^n)$ if and only if $\alpha = Q\left(1 - \frac{1}{q}\right)$. S. Spanne (published by J. Peetre [20]) and D.R. Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results, can be summarized as follows.

Theorem 2.1 *(Spanne) Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$. *Set* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ *and* $\frac{\lambda}{p} = \frac{\mu}{q}$. *Then there exists a constant* $C > 0$ *independent of* f *such*

$$\|I_{\alpha,P} f\|_{L_{q,\mu,P}} \leq C \|f\|_{L_{p,\lambda,P}}$$

for every $f \in L_{p,\lambda,P}(R^n)$.

Theorem 2.2 *(Adams) Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. *Then there exists a constant* $C > 0$ *independent of* f *such*

$$\|I_{\alpha,P} f\|_{L_{q,\lambda,P}} \leq C \|f\|_{L_{p,\lambda,P}}$$

for every $f \in L_{p,\lambda,P}(R^n)$.

We find it convenient to define the generalized parabolic Morrey spaces in the form following.

Definition 2.1 *Let* $1 \leq p < \infty$ *and* $\varphi(x,r)$ *be a positive measurable function on* $R^n \times (0,\infty)$. *The generalized parabolic Morrey space* $M_{p,\varphi,P}(R^n)$ *is defined for all functions* $f \in L_p^{loc}(R^n)$ *by the finite norm*

$$\|f\|_{M_{p,\varphi,P}} = \sup_{x \in R^n, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(x,r)} \|f\|_{L_p(E(x,r))}.$$

Also the weak generalized parabolic Morrey space $WM_{p,\varphi,P}(R^n)$ *is defined for all functions* $f \in L_p^{loc}(R^n)$ *by the finite norm*

$$\|f\|_{WM_{p,\varphi,P}} = \sup_{x \in R^n, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(x,r)} \|f\|_{WL_p(E(x,r))}.$$

According to this definition, we recover the space $L_{p,\lambda,P}(R^n)$ under the choice $\varphi(x,r) = r^{\frac{\lambda-Q}{p}}$:

$$L_{p,\lambda,P}(R^n) = M_{p,\varphi,P}(R^n)\Big|_{\varphi(x,r)=r^{\frac{\lambda-Q}{p}}}.$$

Lemma 2.1 *Let* $\varphi(x,r)$ *be a positive measurable function on* $R^n \times (0,\infty)$.

(i) If

$$\sup_{t<r<\infty} \frac{r^{-\frac{Q}{p}}}{\varphi(x,r)} = \infty \quad \text{for some } t > 0 \text{ and for all } x \in R^n, \tag{2.2}$$

then $M_{p,\varphi,P}(R^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau > 0 \text{ and for all } x \in R^n, \tag{2.3}$$

then $M_{p,\varphi,P}(R^n) = \Theta$.

Proof. (i) Let (2.2) be satisfied and f be not equivalent to zero. Then $\sup_{x\in R^n} \|f\|_{L_p(E(x,t))} > 0$, hence

$$\|f\|_{M_{p,\varphi,P}} \geq \sup_{x\in R^n} \sup_{t<r<\infty} \varphi(x,r)^{-1} r^{-\frac{Q}{p}} \|f\|_{L_p(E(x,r))}$$
$$\geq \sup_{x\in R^n} \|f\|_{L_p(E(x,t))} \sup_{t<r<\infty} \varphi(x,r)^{-1} r^{-\frac{Q}{p}}.$$

Therefore $\|f\|_{M_{p,\varphi,P}} = \infty$.
(ii) Let $f \in M_{p,\varphi,P}(R^n)$ and (2.3) be satisfied. Then there are two possibilities:
Case 1: $\sup_{0<r<t} \varphi(x,r)^{-1} = \infty$ for all $t > 0$.
Case 2: $\sup_{0<r<t} \varphi(x,r)^{-1} < \infty$ for some $s \in (0,\tau)$.

For Case 1, by Lebesgue differentiation theorem, for almost all $x \in R^n$,

$$\lim_{r\to 0+} \frac{1}{|E(x,r)|} \|f\|_{L_1(E(x,r))} = |f(x)|. \tag{2.4}$$

We claim that $f(x) = 0$ for all those x. Indeed, fix x and assume $|f(x)| > 0$. Then by Lemma 2.2 and (2.4) there exists $t_0 > 0$ such that

$$r^{-\frac{Q}{p}} \|f\|_{L_p(E(x,r))} \geq 2^{-1} c_2^{\frac{1}{p}} |f(x)|$$

for all $0 < r \leq t_0$. Consequently,

$$\|f\|_{M_{p,\varphi,P}} \geq \sup_{0<r<t_0} \varphi(x,r)^{-1} r^{-\frac{Q}{p}} \|f\|_{L_p(E(x,r))} \geq 2^{-1} c_2^{\frac{1}{p}} |f(x)| \sup_{0<r<t_0} \varphi(x,r)^{-1}.$$

Hence $\|f\|_{M_{p,\varphi,P}} = \infty$, so $f \notin M_{p,\varphi,P}(R^n)$ and we arrive at contradiction.

Note that Case 2 implies that $\sup_{s<r<\tau} \varphi(x,r)^{-1} = \infty$, hence

$$\sup_{s<r<\infty} \varphi(x,r)^{-1} r^{-\frac{Q}{p}} \geq \sup_{s<r<\tau} \varphi(x,r)^{-1} r^{-\frac{Q}{p}} \geq \tau^{-\frac{Q}{p}} \sup_{s<r<\tau} \varphi(x,r)^{-1} = \infty,$$

which is the case in (i).

Remark 2.1 We denote by Ω_p the sets of all positive measurable functions φ on $R^n \times (0,\infty)$ such that for all $t > 0$,

$$\sup_{x\in R^n} \left\| \frac{r^{-\frac{Q}{p}}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in R^n} \left\| \varphi(x,r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_p$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \leq C\varphi(s) \qquad (\text{resp. } \varphi(r) \geq C\varphi(s)) \quad \text{for } r \leq s.$$

Let $1 \le p < \infty$. Denote by $\mathcal{G}_p$ the the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{\frac{Q}{p}}\varphi(t) \in (0,\infty)$ is almost increasing.

Seemingly the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{n/p} \le \rho(T)T^{n/p}$ for all $0 < t \le T < \infty$ and that $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = \mathcal{M}_{p,\rho}(\mathbb{R}^n)$.

By elementary calculations we have the following, which shows particularly that the space $M_{p,\varphi,P}(R^n)$ is not trivial, see for example, [4].

Lemma 2.2 *Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $E_0 = E(x_0, r_0)$ and $\chi_{_{E_0}}$ is the characteristic function of the ball E_0, then $\chi_{_{E_0}} \in M_{p,\varphi,P}(R^n)$. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{_{E_0}}\|_{WM_{p,\varphi,P}} \le \|\chi_{_{E_0}}\|_{M_{p,\varphi,P}} \le \frac{C}{\varphi(r_0)}.$$

Proof. Let $\varphi \in \mathcal{G}_p$, $1 \le p < \infty$, $E_0 = E(x_0, r_0)$ denote an arbitrary ball in R^n. It is easy to see that

$$\|\chi_{_{E_0}}\|_{WM_{p,\varphi,P}} = \sup_{x\in R^n, r>0} \frac{1}{\varphi(r)}\Big(\frac{|E(x,r)\cap E_0|}{|E(x,r)|}\Big)^{1/p} \ge \frac{1}{\varphi(r_0)}\Big(\frac{|E_0\cap E_0|}{|E_0|}\Big)^{1/p} = \frac{1}{\varphi(r_0)}.$$

Now, if $r \le r_0$, then $\varphi(r_0) \le C\varphi(r)$ and

$$\frac{1}{\varphi(r)}\Big(\frac{|E(x,r)\cap E_0|}{|E(x,r)|}\Big)^{1/p} \le \frac{1}{\varphi(r)} \le \frac{C}{\varphi(r_0)}$$

for all $x \in R^n$.

On the other hand, if $r_0 \le r$, we have $\varphi(r_0)r_0^{Q/p} \le C\varphi(r)r^{Q/p}$ for all $x \in R^n$ and

$$\frac{1}{\varphi(r)}\Big(\frac{|E(x,r)\cap E_0|}{|E(x,r)|}\Big)^{1/p} = \frac{|E(x,r)\cap E_0|^{1/p}}{c_2^{1/p}\varphi(r)r^{Q/p}} \le \frac{|E_0|^{1/p}}{c_2^{1/p}\varphi(r)r^{Q/p}} = \frac{r_0^{Q/p}}{\varphi(r)r^{Q/p}} \le \frac{C}{\varphi(r_0)}$$

for all $x \in R^n$. This completes the proof.

3 Parabolic fractional integral operator in the spaces $M_{p,\varphi,P}(R^n)$

31 Spanne-Guliyev type result

The following theorem was proved in [14].

Theorem 3.1 *[14] Let $1 \le p < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ and the pair (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty \frac{\operatorname{ess\,sup}\limits_{r<s<\infty} \varphi_1(x,s)s^{\frac{Q}{p}}}{r^{\frac{Q}{q}}}\frac{dr}{r} \le C\,\varphi_2(x,t), \tag{3.1}$$

where C does not depend on x and r. Then for $p > 1$ the operator $I_{\alpha,P}$ is bounded from $M_{p,\varphi_1,P}(R^n)$ to $M_{q,\varphi_2,P}(R^n)$ and for $p = 1$ $I_{\alpha,P}$ is bounded from $M_{1,\varphi_1}(R^n)$ to $WM_{q,\varphi_2}(R^n)$.

For proving our main results, we need the following estimate.

Lemma 3.1 *If* $E_0 := E(x_0, r_0)$*, then* $r_0^\alpha \le c_2 \big(2c_0\big)^{Q-\alpha} I_{\alpha,P}\chi_{E_0}(x)$ *for every* $x \in E_0$.

Proof. If $x, y \in E_0$, then $\rho(x-y) \le c_0\big(\rho(x^{-1}x_0)+\rho(x_0^{-1}y)\big) < 2c_0r_0$. Since $0 < \alpha < Q$, we get $r_0^{\alpha-Q} \le \big(2c_0\big)^{Q-\alpha}\rho(x-y)^{\alpha-Q}$. Therefore

$$I_{\alpha,P}\chi_{E_0}(x) = \int_{R^n} \chi_{E_0}(y)\rho(x-y)^{\alpha-Q}dy = \int_{E_0} \rho(x-y)^{\alpha-Q}dy \ge c_2\big(2c_0\big)^{Q-\alpha} r_0^\alpha.$$

The following theorem is one of our main results.

Theorem 3.2 *Let* $0 < \alpha < Q$*,* $p, q \in [1, \infty)$*,* $\varphi_1 \in \Omega_p$ *and* $\varphi_2 \in \Omega_q$*.*

1. *If* $1 < p < \frac{Q}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$*, then condition* (3.1) *is sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi_1,P}(R^n)$ *to* $M_{q,\varphi_2,P}(R^n)$*.*

2. *If the function* $\varphi_1 \in \mathcal{G}_p$*, then the condition*

$$t^\alpha \varphi_1(t) \le C\varphi_2(t), \tag{3.2}$$

for all $t > 0$*, where* $C > 0$ *does not depend on t, is necessary for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi_1,P}(R^n)$ *to* $M_{q,\varphi_2,P}(R^n)$*.*

3. *Let* $1 < p < \frac{Q}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$*. If* $\varphi_1 \in \mathcal{G}_p$ *satisfies the regularity condition*

$$\int_t^\infty r^{\alpha-1}\,\varphi_1(r)dr \le Ct^\alpha\varphi_1(t), \tag{3.3}$$

for all $t > 0$*, where* $C > 0$ *does not depend on t, then condition* (3.2) *is necessary and sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi_1,P}(R^n)$ *to* $M_{q,\varphi_2,P}(R^n)$*.*

Proof. The first part of the theorem was proved in Theorem 3.1.

We shall now prove the second part. Let $E_0 = E(x_0, t_0)$ and $x \in E_0$. By Lemma 3.1 we have $t_0^\alpha \le CI_{\alpha,P}\chi_{E_0}(x)$. Therefore, by Lemma 2.2 and Lemma 3.1

$$\begin{aligned} t_0^\alpha &\lesssim |E_0|^{-\frac{1}{p}} \|I_{\alpha,P}\chi_{E_0}\|_{L_q(E_0)} \lesssim \varphi_2(t_0)\|I_{\alpha,P}\chi_{E_0}\|_{M_{q,\varphi_2,P}} \\ &\lesssim \varphi_2(t_0)\|\chi_{E_0}\|_{M_{p,\varphi_1,P}} \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)} \qquad \text{or} \end{aligned}$$

$$t_0^\alpha \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)} \quad \text{for all} \quad t_0 > 0 \Longleftrightarrow t_0^\alpha\varphi_1(t_0) \lesssim \varphi_2(t_0).$$

Since this is true for every $x \in R^n$ and $t_0 > 0$, we are done. The third statement of the theorem follows from first and second parts of the theorem.

The following weak version of Theorem 3.2 is proved in the same way.

Theorem 3.3 *Let* $0 < \alpha < Q$*,* $p, q \in [1, \infty)$*,* $\varphi_1 \in \Omega_p$ *and* $\varphi_2 \in \Omega_q$*.*

1. *If* $1 \le p < \frac{Q}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$*, then condition* (3.1) *is sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi_1,P}(R^n)$ *to* $WM_{q,\varphi_2}(R^n)$*.*

2. *If the function* $\varphi_1 \in \mathcal{G}_p$*, then condition* (3.2) *is necessary for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi_1,P}(R^n)$ *to* $WM_{q,\varphi_2}(R^n)$*.*

3. *Let* $1 \le p < \frac{Q}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$*. If* $\varphi_1 \in \mathcal{G}_p$ *satisfies the regularity condition* (3.3)*, then condition* (3.2) *is necessary and sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi_1,P}(R^n)$ *to* $WM_{q,\varphi_2}(R^n)$*.*

Remark 3.1 If we take $\varphi_1(t) = t^{\frac{\lambda-Q}{p}}$ and $\varphi_2(t) = t^{\frac{\mu-Q}{q}}$ in Theorem 3.2, then conditions (3.3) and (3.2) are equivalent to $0 < \lambda < Q - \alpha p$ and $\frac{\lambda}{p} = \frac{\mu}{q}$, respectively. Therefore, we get the following strong and weak version of the Spanne-Guliyev result for parabolic Morrey spaces.

Corollary 3.1 *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then the operator $I_{\alpha,P}$ is bounded from $L_{p,\lambda,P}(R^n)$ to $L_{q,\mu,P}(R^n)$ if and only if $\frac{\lambda}{p} = \frac{\mu}{q}$.*

Corollary 3.2 *Let $0 < \alpha < Q$, $1 \le p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then the operator $I_{\alpha,P}$ is bounded from $L_{p,\lambda,P}(R^n)$ to $WL_{q,\mu,P}(R^n)$ if and only if $\frac{\lambda}{p} = \frac{\mu}{q}$.*

32 Adams-Guliyev type results

The following theorem was proved in [14].

Theorem 3.4 *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s)\, s^{\frac{Q}{p}} \le C\, \varphi_2(x,r), \tag{3.4}$$

where C does not depend on x and r. Then for $p > 1$, the operator M_P is bounded from $M_{p,\varphi_1,P}(R^n)$ to $M_{p,\varphi_2}(R^n)$ and for $p = 1$, the operator M_P is bounded from $M_{1,\varphi_1}(R^n)$ to $WM_{1,\varphi_2}(R^n)$.

The following is the Adams-Guliyev type result for the parabolic fractional integral.

Theorem 3.5 *[13] Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and let $\varphi \in \Omega_p$ satisfy the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi(x,s)^{\frac{1}{p}}\, s^{\frac{Q}{p}} \le C\, \varphi(x,r)^{\frac{1}{p}}, \tag{3.5}$$

and

$$\int_r^\infty t^{\alpha-1} \varphi(x,t)^{\frac{1}{p}} dt \le C r^{-\frac{\alpha p}{q-p}}, \tag{3.6}$$

where C does not depend on $x \in R^n$ and $r > 0$. Then for $p > 1$, the operator $I_{\alpha,P}$ is bounded from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$ and for $p = 1$, the operator $I_{\alpha,P}$ is bounded from $M_{1,\varphi,P}(R^n)$ to $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

The following theorem is one of our main results.

Theorem 3.6 *Let $0 < \alpha < Q$, $1 < p < q < \infty$ and $\varphi \in \Omega_p$.*

1. *If $\varphi(x,t)$ satisfies condition* (3.5)*, then the condition* (3.6) *is sufficient for boundedness of $I_{\alpha,P}$ from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$.*
2. *If $\varphi \in \mathcal{G}_p$, then the condition*

$$r^\alpha \varphi(r)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{3.7}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary for boundedness of $I_{\alpha,P}$ from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

3. *If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition*

$$\int_r^\infty t^{\alpha-1}\varphi(t)^{\frac{1}{p}}dt \le Cr^\alpha \varphi(r)^{\frac{1}{p}}, \tag{3.8}$$

for all $r > 0$, where $C > 0$ does not depend on r, then condition (3.7) *is necessary and sufficient for boundedness of $I_{\alpha,P}$ from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$.*

Proof. The first part of the theorem is a corollary of Theorem 3.5.

We shall now prove the second part. Let $E_0 = E(x_0, t_0)$ and $x \in E_0$. By Lemma 3.1 we have $t_0^\alpha \le CI_{\alpha,P}\chi_{E_0}(x)$. Therefore, by Lemma 2.2 and Lemma 3.1 we have

$$\begin{aligned} t_0^\alpha &\lesssim |E_0|^{-\frac{1}{q}} \|I_{\alpha,P}\chi_{E_0}\|_{L_q(E_0)} \lesssim \varphi(t_0)^{\frac{1}{q}} \|I_{\alpha,P}\chi_{E_0}\|_{M_{q,\varphi^{\frac{1}{q}},P}} \\ &\lesssim \varphi(t_0)^{\frac{1}{q}} \|\chi_{E_0}\|_{M_{p,\varphi^{\frac{1}{p}},P}} \lesssim \varphi(t_0)^{\frac{1}{q}-\frac{1}{p}} \end{aligned}$$

or

$$t_0^\alpha \varphi(t_0)^{\frac{1}{p}-\frac{1}{q}} \lesssim 1 \text{ for all } t_0 > 0 \Longleftrightarrow t_0^\alpha \varphi(t_0)^{\frac{1}{p}} \lesssim t_0^{-\frac{\alpha p}{q-p}}.$$

Since this is true for every $x \in R^n$ and $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem.

The following weak version of Theorem 3.6 is proved in the same way.

Theorem 3.7 *Let $0 < \alpha < Q$, $1 \le p < q < \infty$ and $\varphi \in \Omega_p$.*

1. *If $\varphi(x,t)$ satisfies condition* (3.5)*, then condition* (3.6) *is sufficient for boundedness of $I_{\alpha,P}$ from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.*

2. *If $\varphi \in \mathcal{G}_p$, then condition* (3.7) *is necessary for boundedness of $I_{\alpha,P}$ from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.*

3. *If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition* (3.8)*, then condition* (3.7) *is necessary and sufficient for boundedness of $I_{\alpha,P}$ from $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ to $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.*

The following pointwise estimate plays a key role where we prove our main results.

Theorem 3.8 *Let $1 \le p < \infty$, $0 < \alpha < Q$ and $f \in L_p^{loc}(R^n)$. Then*

$$|I_{\alpha,P}f(x)| \le Ct^\alpha \, M_P f(x) + C\int_t^\infty r^{\alpha-\frac{Q}{p}-1} \|f\|_{L_p(E(x,r))}dr, \tag{3.9}$$

where C does not depend on f, x and t.

Proof. Write $f = f_1 + f_2$, where $f_1 = f\chi_{2c_0E}$, $f_2 = f\chi_{\complement_{(2c_0E)}}$ and $E = E(x,t)$. Then

$$I_{\alpha,P}f(x) = I_{\alpha,P}f_1(x) + I_{\alpha,P}f_2(x).$$

For $I_{\alpha,P}f_1(x)$, following Hedberg's trick (see for instance [22], p. 354), for all $z \in R^n$ we obtain $|I_{\alpha,P}f_1(z)| \le C_1 t^\alpha M_P f(z)$. For $I_{\alpha,P}f_2(z)$ with $z \in E$ we have

$$|I_{\alpha,P}f_2(z)| \le \int_{\complement_{(2c_0E)}} \rho(x-y)^{\alpha-Q}|f(y)|dy \le C\int_{2c_0t}^\infty r^{\alpha-\frac{Q}{p}-1}\|f\|_{L_p(E(x,r))}dr, \tag{3.10}$$

which proves (3.9).

The following is the Adams-Guliyev type result for the parabolic fractional integral (see [9]).

Theorem 3.9 (Adams-Guliyev type result). *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$ *and* $\varphi \in \Omega_p$ *satisfy condition* (3.5) *and*

$$t^{\alpha}\varphi(x,t) + \int_t^{\infty} r^{\alpha-1}\,\varphi(x,r)dr \le C\varphi(x,t)^{\frac{p}{q}}, \tag{3.11}$$

where C *does not depend on* $x \in R^n$ *and* $r > 0$. *Then for* $p > 1$, *the operator* $I_{\alpha,P}$ *is bounded from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$ *and for* $p = 1$, *the operator* $I_{\alpha,P}$ *is bounded from* $M_{1,\varphi,P}(R^n)$ *to* $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi,P}(R^n)$. By Theorem 3.8 inequality (3.9) is valid. Then from condition (3.5) and inequality (3.9) we get

$$\begin{aligned} |I_{\alpha,P}f(x)| &\lesssim t^{\alpha}\, M_P f(x) + \int_t^{\infty} r^{\alpha-\frac{Q}{p}-1}\|f\|_{L_p(E(x,r))}dr \\ &\le t^{\alpha}\, M_P f(x) + \|f\|_{M_{p,\varphi,P}} \int_t^{\infty} r^{\alpha-1}\varphi(x,r)dr. \end{aligned} \tag{3.12}$$

Thus, by (3.11) and (3.12) we obtain

$$\begin{aligned} &|I_{\alpha,P}f(x)| \lesssim \min\left\{\varphi(x,t)^{\frac{p}{q}-1}M_P f(x), \varphi(x,t)^{\beta}\|f\|_{M_{p,\varphi,P}}\right\} \\ &\lesssim \sup_{s>0}\min\left\{s^{\frac{p}{q}-1}M_P f(x), s^{\frac{p}{q}}\|f\|_{M_{p,\varphi,P}}\right\} = (M_P f(x))^{\frac{p}{q}}\,\|f\|_{M_{p,\varphi,P}}^{1-\frac{p}{q}}, \end{aligned} \tag{3.13}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 3.4 and (3.13), we get

$$\|I_{\alpha,P}f\|_{M_{q,\varphi^{\frac{1}{q}},P}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{1-\frac{p}{q}}\ \|M_P f\|_{M_{p,\varphi^{\frac{1}{p}},P}}^{\frac{p}{q}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}},P}}, \quad \text{if} \quad 1<p<q<\infty$$

and

$$\|I_{\alpha,P}f\|_{WM_{q,\varphi^{\frac{1}{q}},P}} \lesssim \|f\|_{M_{1,\varphi,P}}^{1-\frac{1}{q}}\,\|M_P f\|_{M_{1,\varphi,P}}^{\frac{1}{q}} \lesssim \|f\|_{M_{1,\varphi,P}}, \quad \text{if} \quad p=1<q<\infty.$$

The following theorem is one of our main results.

Theorem 3.10 *Let* $0 < \alpha < Q$, $1 < p < q < \infty$ *and* $\varphi \in \Omega_p$.

1. *If* $\varphi(x,t)$ *satisfies condition* (3.5), *then condition* (3.11) *is sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$.
2. *If* $\varphi \in \mathcal{G}_p$, *then the condition*

$$r^{\alpha}\varphi(r)^{\frac{1}{p}} \le C\varphi(r)^{\frac{1}{q}}, \tag{3.14}$$

for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is necessary for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

3. *If* $\varphi \in \mathcal{G}_p$ *satisfies the regularity condition* (3.8), *then condition* (3.14) *is necessary and sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $M_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

The following weak version of Theorem 3.10 is proved in the same way.

Theorem 3.11 *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$ *and* $\varphi \in \Omega_p$.

1. *If* $\varphi(x,t)$ *satisfies condition* (3.5), *then condition* (3.11) *is sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

2. *If* $\varphi \in \mathcal{G}_p$, *then condition* (3.14) *is necessary for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

3. *If* $\varphi \in \mathcal{G}_p$ *satisfies the regularity condition* (3.8), *then condition* (3.14) *is necessary and sufficient for boundedness of* $I_{\alpha,P}$ *from* $M_{p,\varphi^{\frac{1}{p}},P}(R^n)$ *to* $WM_{q,\varphi^{\frac{1}{q}},P}(R^n)$.

Remark 3.2 If we take $\varphi(t) = t^{\lambda-Q}$ in Theorem 3.6, then condition (3.8) is equivalent to $0 < \lambda < Q - \alpha p$ and condition (3.7) is equivalent to $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. Therefore, we get the following strong and weak version of the Adams-Guliyev result for parabolic Morrey spaces.

Corollary 3.3 *Let* $0 < \alpha < Q$, $1 < p < q < \infty$ *and* $0 < \lambda < Q - \alpha p$. *Then the operator* $I_{\alpha,P}$ *is bounded from* $L_{p,\lambda,P}(R^n)$ *to* $L_{q,\lambda,P}(R^n)$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$.

Corollary 3.4 *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$ *and* $0 < \lambda < Q - \alpha p$. *Then the operator* $I_{\alpha,P}$ *is bounded from* $L_{p,\lambda,P}(R^n)$ *to* $WL_{q,\lambda,P}(R^n)$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$.

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