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On Calderon's maximal function

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Abstract. This paper is devoted to the study of Calderon's maximal function. We prove the estimate of the maximal functions in terms of some metric characteristics.

Keywords. Maximal function, Smoothness of function, Calderon's maximal function.

Mathematics Subject Classification (2010): 42B25, 47A63

1 Introduction

Properties of different maximal functions, including maximal functions measuring smoothness are applied to the study of differential and smoothness properties of functions, structural properties of singular integrals and potentials. These properties also have important applications in the study of problems of almost everywhere convergence of different approximating aggregates in the study of the applications of various integral operators in functional spaces, including spaces associated with functions' smoothness properties ([3], [4], [7]). In the present paper, we investigate some maximal functions measuring smoothness of functions.

2 Maximal functions and their properties

Let R^n be n-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$. By $L^p_{loc}(R^n)$ $(1 \leq p < \infty)$ we denote the class of all locally summable p-th power functions and by $L^\infty_{loc}(R^n)$ the class of all locally essential bounded functions defined on R^n. Denote by P_k the set of all polynomials in R^n of degree less than or equal to $k \in N \cup \{0\}$, where N is the set of all positive integers.

Let $\varPhi$ be the class of all positive monotonically increasing functions on $(0, +\infty)$ and $\varPhi_k$ be $(k \in N)$ the class of all functions $\varphi \in \varPhi$ such that $\frac{\varphi(t)}{t^k}$ is almost decreasing [1].

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[1] A non-negative function $h(t)$, $t \in (0, +\infty)$, is called almost decreasing if $\exists c > 0$ $\forall t_1, t_2 \in (0, +\infty)$: $(t_1 < t_2 \Rightarrow h(t_1) \geq c \cdot h(t_2))$

Let $B(a,r)$ be a closed ball in R^n with radius $r>0$ and with center at the point $a\in R^n$, that is $B(a,r):=\{x\in R^n:\,|x-a|\le r\}$, $1\le p\le\infty$, $\varphi\in\Phi$, $f\in L^p_{loc}(R^n)$, $k\in N$ and

$$N^{(\varphi)}_{k,p}f(x):=\sup_{r>0}\frac{1}{\varphi(r)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-P_x(t)|^p\,dt\right)^{\frac{1}{p}},1\le p<\infty,$$

$$N^{(\varphi)}_{k,\infty}f(x):=\sup_{r>0}\frac{1}{\varphi(r)}\|f-P_x\|_{L^\infty(B(x,r))},$$

where $|B(x,r)|$ is the volume of the ball $B(x,r)$ and it is assumed that exists the polynomial $P_x\in P_{k-1}$ such that the supremum is finite. Otherwise, i.e. if such a polynomial does not exist then we will assume that $N^{(\varphi)}_{k,p}f(x)=+\infty$.

In the case $\varphi(t)=t^\alpha$, $\alpha>0$, the value $N^{(\varphi)}_{k,p}f(x)$ is called as Calderon's maximal function (see [1], [2], [3]).

We also introduce the following maximal function:

$$N^{\varphi}_{k,p}f(x):=\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{1}{\varphi(r)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-\pi(t)|^p\,dt\right)^{\frac{1}{p}},1\le p<\infty,$$

$$N^{\varphi}_{k,\infty}f(x):=\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{1}{\varphi(r)}\|f-\pi\|_{L^\infty(B(x,r))}.$$

For the case $\varphi(t)=t^\alpha$, $\alpha>0$, see [10].

We apply orthogonalization process with respect to the scalar product

$$(f,g):=\frac{1}{|B(0,1)|}\int_{B(0,1)}f(t)\,g(t)\,dt$$

to the system of power functions $\{x^\nu\}$, $|\nu|\le k$, arranged in partially lexicographical order (see [6]), where $k\in N$, $x=(x_1,x_2,\ldots,x_n)\in R^n$, $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$, $\nu_1,\nu_2,\ldots,\nu_n$ are non-negative integers, $x^\nu=x_1^{\nu_1}\cdot x_2^{\nu_2}\cdot\ldots\cdot x_n^{\nu_n}$, $|\nu|=\nu_1+\nu_2+\ldots+\nu_n$.

Let $f\in L^1_{loc}(R^n)$, $k\in N\cup\{0\}$. Put (see. [3], [5])

$$P_{k,B(a,r)}f(x):=\sum_{|\nu|\le k}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}f(t)\,\varphi_\nu\left(\frac{t-a}{r}\right)dt\right)\cdot\varphi_\nu\left(\frac{x-a}{r}\right).$$

It is easy to see that $P_{k,B(a,r)}f(x)$ it is a polynomial of degree at most k. For the function $f\in L^p_{loc}(R^n)$ $(1\le p\le\infty)$ denote

$$\Omega_k(f,\,B(a,r))_p:=\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}\left|f(t)-P_{k-1,B(a,r)}f(t)\right|^p dt\right)^{\frac{1}{p}}(1\le p<\infty),$$

$$\Omega_k(f,B(a,r))_\infty:=ess\sup\left\{\left|f(t)-P_{k-1,B(a,r)}f(t)\right|:t\in B(a,r)\right\}.$$

$\Omega_k(f,\,B(a,\,r))_p$ $(1\le p\le\infty)$ is called the mean oscillation of the k-th order of the function f on the ball $B(a,\,r)$ in the metric L^p. We introduce the following local metric characteristics of $f\in L^p_{loc}(R^n)$:

$$m^k_f(x_0;\delta)_p:=\sup\left\{\Omega_k\left(f,B(x_0;r)_p\right):r\le\delta\right\}(\delta>0),$$

where $x_0 \in R^n$ is a fixed point.

Let $x \in R^n$, $k \in N$ and for any ν with condition $|\nu| \le k-1$ there is a finite limit

$$\lim_{r\to 0} D^{\nu} P_{k-1,B(x,r)} f(x) =: D_{\nu} f(x), \tag{2.1}$$

where $D^{\nu} g := \frac{\partial^{|\nu|} g}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \ldots \partial x_n^{\nu_n}}$.

Let

$$P_{k-1,\,x} f(t) := \sum_{|\nu|\le k-1} D^{\nu} f(x) \cdot \frac{(t-x)^{\nu}}{\nu!}, \tag{2.2}$$

$$n_f^k (x;\delta)_p := \sup\left\{ |B(x,r)|^{-\frac{1}{p}} \cdot \|f - P_{k-1,x} f\|_{L^p(B(x,r))} : r \le \delta \right\}, (x \in R^n, \delta > 0),$$

where $\nu! = \nu_1!\, \nu_2! \ldots \nu_n!$.

Let $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, $\varphi \in \Phi$, $k \in N$. Designate (see [6])

$$N_{k,\varphi,p} f(x) := \sup_{r>0} \frac{1}{\varphi(r)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - P_{k-1,x} f(t)|^p \, dt \right)^{\frac{1}{p}}, x \in R^n,$$

with appropriate modification in the case of $p = \infty$. If at least one of the limits (2.1) does not exist, we will assume that $N_{k,\,\varphi,\,p} f(x) = +\infty$.

Proposition 2.1 *Let $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, $k \in N$ and $\varphi \in \Phi$. Then the following inequality holds*

$$N^{\varphi}_{k,\,p} f(x) \le N^{(\varphi)}_{k,p} f(x) \le 2 N^{\varphi}_{k,p} f(x), x \in R^n. \tag{2.3}$$

Proof. The following inequality

$$N^{\varphi}_{k,p} f(x) \le N^{(\varphi)}_{k,p} f(x), x \in R^n, \tag{2.4}$$

is obvious.

If $N^{\varphi}_{k,p} f(x) = +\infty$, then the second part of (2.3) is also obvious. Therefore, let $N^{\varphi}_{k,p} f(x) < +\infty$. Then we obtain that for any positive number ε there is a polynomial $\pi = \pi_{\varepsilon} \in P_{k-1}$ such that

$$\sup_{r>0} \frac{1}{\varphi(r)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - \pi(t)|^p \, dt \right)^{\frac{1}{p}} \le N^{\varphi}_{k,p} f(x) + \varepsilon,$$

with appropriate modification in the case $p = \infty$.

In particular, for $\varepsilon = N^{\varphi}_{k,\,p} f(x)$ we have

$$\sup_{r>0} \frac{1}{\varphi(r)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - \pi(t)|^p \, dt \right)^{\frac{1}{p}} \le 2 N^{\varphi}_{k,p} f(x).$$

It follows that

$$N^{(\varphi)}_{k,p} f(x) \le 2 N^{\varphi}_{k,p} f(x), x \in R^n. \tag{2.5}$$

From (2.4) and (2.5) the required inequality (2.3) holds true.

Proposition 2.2 *Let $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, $k \in N$, $\varphi \in \Phi$ and*

$$\delta^{k-1} \int_0^{\delta} \frac{\varphi(t)}{t^k} dt = O(\varphi(\delta)), \delta > 0. \tag{2.6}$$

Then there is a constant $c > 0$ such that

$$N^{\varphi}_{k,p} f(x) \le N_{k,\varphi,p} f(x) \le c \cdot N^{\varphi}_{k,p} f(x), x \in R^n, \tag{2.7}$$

where the constant c is independent of f and x.

Proof. The correctness of inequality

$$N^{\varphi}_{k,p} f(x) \le N_{k,\varphi,p} f(x), x \in R^n, \tag{2.8}$$

(even without the condition (2.6)) is obvious.

If $N^{\varphi}_{k,p} f(x) = +\infty$, then the second part of (2.7) is also obvious. Now let $N^{\varphi}_{k,p} f(x) < +\infty$. Then there is a polynomial $\pi \in P_{k-1}$ such that

$$N^{\varphi}_{k,p} f(x) \le \sup_{r>0} \frac{1}{\varphi(r)} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - \pi(t)|^p dt \right)^{\frac{1}{p}} \le 2N^{\varphi}_{k,p} f(x),$$

with appropriate modification in the case $p = \infty$.

It follows that

$$\forall r > 0 : \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - \pi(t)|^p dt \right)^{\frac{1}{p}} \le 2N^{\varphi}_{k,p} f(x) \cdot \varphi(r).$$

Hence, we obtain that

$$m^k_f(x;r)_p \le c \cdot N^{\varphi}_{k,p} f(x) \cdot \varphi(r), r > 0, \tag{2.9}$$

where the constant $c > 0$ does not depend on x, r, f and φ.

Since

$$\int_0^1 \frac{m^k_f(x;t)_p}{t^k} dt \le c \cdot N^{\varphi}_{k,p} f(x) \cdot \int_0^1 \frac{\varphi(t)}{t^k} dt < +\infty,$$

by Theorem 2.3 from [9] the characteristics $n^k_f(x;\, r)_p$is reasonable and the following inequality is true

$$n^k_f(x;r)_p \le c \left(m^k_f(x;r)_p + r^{k-1} \int_0^r \frac{m^k_f(x;t)_p}{t^k} dt \right), \tag{2.10}$$

where the constant $c > 0$ does not depend on r, f and x.

Further, applying Proposition 4 from [6], and inequality (2.9) and (2.10) we have

$$N_{k,\varphi,p} f(x) = \sup_{r>0} \frac{n^k_f(x;r)_p}{\varphi(r)} \le$$

$$\le c \cdot \sup_{r>0} \frac{1}{\varphi(r)} \left(m^k_f(x;r)_p + r^{k-1} \int_0^r \frac{m^k_f(x;t)_p}{t^k} dt \right) \le$$

$$\leq c\cdot\sup_{r>0}\frac{m_f^k(x;\,r)_p}{\varphi(r)}+c\cdot\sup_{r>0}\frac{1}{\varphi(r)}\cdot r^{k-1}\int_0^r\frac{m_f^k(x;t)_p}{t^k}dt\leq$$

$$\leq c_1\cdot N_{k,p}^{\varphi}f(x)+c\cdot\sup_{r>0}\frac{1}{\varphi(r)}\cdot r^{k-1}\int_0^r\frac{m_f^k(x;t)_p}{\varphi(t)}\cdot\frac{\varphi(t)}{t^k}dt\leq$$

$$\leq c_1\cdot N_{k,p}^{\varphi}f(x)+c_1\cdot N_{k,p}^{\varphi}f(x)\cdot\sup_{r>0}\frac{1}{\varphi(r)}\cdot r^{k-1}\int_0^r\frac{\varphi(t)}{t^k}dt\leq$$

$$\leq c_2\cdot N_{k,p}^{\varphi}f(x),$$

i.e. $N_{k,\varphi,p}f(x)\leq c_2\cdot N_{k,p}^{\varphi}f(x)$, $x\in R^n$. This, together with (2.8) gives the desired relation (2.7). The proposition is proved.

3 Estimates of maximal functions in terms of metric characteristics

We introduce a new local metric characteristics of a function $f\in L_{loc}^p(R^n)$:

$$n_f^{k,\pi}(x;\delta)_p:=\sup_{0<r\leq\delta}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-\pi(t)|^p\,dt\right)^{\frac{1}{p}},1\leq p<\infty,$$

$$n_f^{k,\pi}(x;\delta)_\infty:=\|f-\pi\|_{L^\infty(B(x,\delta))}\,,p=\infty.$$

where $\pi\in P_{k-1}$, $k\in N$, $x\in R^n$, $\delta>0$.

In this notation $n_f^k(x;\delta)_p\equiv n_f^{k,\pi}(x;\delta)_p$, where $\pi(t)\equiv P_{k-1,x}f(t)$ is a polynomial defined by equality (2.2).

Proposition 3.1 *Let $f\in L_{loc}^p(R^n)$, $1\leq p\leq\infty$, $k\in N$, $\varphi\in\Phi$. Then the following inequalities hold true*

$$1)\frac{1}{2}\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{n_f^{k,\pi}(x,r)_p}{\varphi(r)}\leq N_{k,p}^{\varphi}f(x)\leq\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{n_f^{k,\pi}(x,r)_p}{\varphi(r)},x\in R^n;$$

$$2)\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{n_f^{k,\pi}(x,r)_p}{\varphi(r)}\leq N_{k,p}^{(\varphi)}f(x)\leq 2\cdot\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{n_f^{k,\pi}(x,r)_p}{\varphi(r)},x\in R^n.$$

Proof. Let $1\leq p<\infty$. From the definition of $N_{k,p}^{\varphi}f(x)$ we obtain that for any polynomial $\pi\in P_{k-1}$,

$$N_{k,p}^{\varphi}f(x)\leq\sup_{r>0}\frac{1}{\varphi(r)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-\pi(t)|^p\,dt\right)^{\frac{1}{p}}\leq$$

$$\leq\sup_{r>0}\frac{1}{\varphi(r)}\cdot n_f^{k,\pi}(x,r)_p\,,$$

i.e., $\forall\pi\in P_{k-1}$: $N_{k,p}^{\varphi}f(x)\leq\sup\limits_{r>0}\frac{n_f^{k,\pi}(x,r)_p}{\varphi(r)}$. It follows that

$$N_{k,\,p}^{\varphi}f(x)\leq\inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{n_f^{k,\,\pi}(x,\,r)_p}{\varphi(r)},x\in R^n. \tag{3.1}$$

Let $N_{k,p}^{(\varphi)}f(x) < +\infty$ and $P_x \in P_{k-1}$ is the polynomial appearing in the definition of $N_{k,p}^{(\varphi)}f(x)$. Then we have

$$N_{k,p}^{(\varphi)}f(x) \geq \frac{1}{\varphi(r)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-P_x(t)|^p\,dt\right)^{\frac{1}{p}}, r>0.$$

From here

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(t)-P_x(t)|^p\,dt\right)^{\frac{1}{p}} \leq N_{k,p}^{(\varphi)}f(x)\cdot\varphi(r), r>0, x\in R^n,$$

$$n_f^{k,P_x}(x,r)_p \leq N_{k,p}^{(\varphi)}f(x)\cdot\varphi(r), r>0, x\in R^n,$$

$$N_{k,p}^{(\varphi)}f(x) \geq \frac{n_f^{k,P_x}(x,r)_p}{\varphi(r)}, r>0, x\in R^n.$$

In this way,

$$N_{k,p}^{(\varphi)}f(x) \geq \sup_{r>0}\frac{n_f^{k,P_x}(x,r)_p}{\varphi(r)}, x\in R^n, \tag{3.2}$$

and therefore

$$N_{k,p}^{(\varphi)}f(x) \geq \inf_{\pi\in P_{k-1}}\sup_{r>0}\frac{n_f^{k,\pi}(x,r)_p}{\varphi(r)}, x\in R^n. \tag{3.3}$$

If $N_{k,p}^{(\varphi)}f(x) = +\infty$, then the correctness of the inequality (3.3) is obvious. The case $p=\infty$ can be considered in a similar way. From Proposition 1, the inequalities (3.1) and (3.3) we obtain the required result. The proposition is proved.

Proposition 3.2 *Let $f\in L_{loc}^p(R^n)$, $1\leq p\leq\infty$, $k\in N$, $\varphi\in\Phi$. Then if $N_{k,p}^{(\varphi)}f(x) < +\infty$, then the following equality is true*

$$N_{k,p}^{(\varphi)}f(x) = \sup_{r>0}\frac{n_f^{k,P_x}(x,r)_p}{\varphi(r)}, x\in R^n, \tag{3.4}$$

where $P_x\in P_{k-1}$ is the polynomial appearing in the definition of $N_{k,p}^{(\varphi)}f(x)$.

Proof. From the definition of the value of $N_{k,p}^{(\varphi)}f(x)$ we obtain that

$$N_{k,p}^{(\varphi)}f(x) \leq \sup_{r>0}\frac{n_f^{k,P_x}(x,r)_p}{\varphi(r)}, x\in R^n.$$

This inequality together with (3.2) gives the required equality (3.4).

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