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On One Kind of Positive Operators in Lebesgue Space Of Harmonic Functions

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Abstract. In this work we consider some system of positive operators in Lebesgue space of harmonic functions and prove Korovkin type theorem in this space. Also we defined the statistical convergence, statistical positivity and proved statistical version of Korovkin type theorem in the same space.

Keywords. Lebesgue space of harmonic functions · Korovkin type theorem · Poisson kernel · statistical convergence · k-positivity

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1 Introduction

Korovkin-type theorems furnish simple and useful tools for ascertaining whether a given sequence of positive linear operators, acting on some function space is an approximation process or, equivalently, converges strongly to the identity operator [2].

Roughly speaking, these theorems exhibit a variety of test subsets of functions which guarantee that the approximation (or the convergence) property holds on the whole space provided it holds on them.

The custom of calling these kinds of results "Korovkin-type theorems" refers to P.P. Korovkin who in 1953 discovered such a property for the functions 1, x and x^2 in the space $C([0,1])$ of all continuous functions on the real interval $[0,1]$ as well as for the functions 1, $\cos$ and $\sin$ in the space of all continuous 2π-periodic functions on the real line ([14,15]).

After this discovery, several mathematicians have undertaken the program of extending Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on. Such developments delineated a theory which is nowadays referred to as Korovkin-type approximation theory. More precisely one can see the works [11–13,17–20].

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This theory has fruitful connections with real analysis, functional analysis, harmonic analysis, measure theory and probability theory, summability theory and partial differential equations. But the foremost applications are concerned with constructive approximation theory which uses it as a valuable tool.

The idea of statistical convergence was first proposed by A.Zigmund (Zygmund, 1979) [23] in his famous monograph where he talked about "almost convergence". The first definition of it was given by H. Fast (Fast, 1951) [6] and H. Steinhaus (Steinhaus, 1951) [22]. Later, this concept has been generalized in many directions [1,3–5,7,9,16,21].

In this work we consider some system of positive operators in Lebesgue space of harmonic functions, proved Korovkin type theorem in this space. We defined statistical convergence, statical positivity and proved statistical version of Korovkin type theorem in the same space. Also we mention about k-positivity, the sequence of k-positive linear operators and we will apply statistical approach to the results from the paper A. Gadjiev [7,8,10].

2 Preliminaries

In this work we will use the standard notation. N is the set of all positive integers; R is the set of all real numbers; C is the set of all complex numbers. We will denote the unit disc on C by ω.

Denote by $C[-\pi,\pi]$ the space of continuous functions on $[-\pi,\pi]$ with sup-norm:

$$C_{2\pi}[-\pi,\pi] \equiv \{f \in C[-\pi,\pi] : f(-\pi) = f(\pi)\}.$$

Let us define the positivity for complex valued functions and for operators in Lebesgue space L_p, $1 \le p < \infty$.

Let u be a complex valued function. We will say that u is a positive function, i.e. $u \ge 0$, if $Reu \ge 0 \wedge Imu \ge 0$.

Let T be an operator, acting from L_p to L_p. We will say that T is a positive operator if

$$T(f) \ge 0, \forall f \in L_p : f \ge 0.$$

Let us consider the ordinary Lebesgue space L_p on $[-\pi,\pi]$ and $\|.\|_{L_p}$ is an ordinary norm in this space.

Now, we define harmonic functions and denote by h_p the Lebesgue space of harmonic functions.

Definition 2.1 *Let $H(\omega)$ be the set of all harmonic functions on ω and denote*

$$h_p = \left\{ u \in H(\omega) : \sup_{r \in (0,1)} \left\| u_r\left(e^{it}\right) \right\|_{L_p} < \infty \right\}, 1 < p < +\infty,$$

where $u_r\left(e^{it}\right) = u\left(re^{it}\right)$.

Let us define the positivity of operators in h_p.

Definition 2.2 *The operator $t : h_p \to h_p$ is positive, if*

$$t(u) \ge 0, \forall u \in h_p : u \ge 0.$$

Let Γ denotes the restriction operator on $\partial\omega$, i.e.

$$\Gamma(u) = u^+, \forall u \in h_p,$$

where $u^+\left(e^{it}\right) = \lim_{r \to 1-0} u\left(re^{it}\right), t \in [-\pi,\pi]$.

Let $P_r(\theta)$ be a Poisson kernel, i.e.

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}, 0 < r < 1, -\pi < \theta < \pi.$$

For $f \in L_p$ let us define the operator $P_r : L_p \to h_p$ by the following formula

$$P_r(f)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta) f(\theta) d\theta.$$

Let X, Y be Banach spaces and $B(X;Y)$ denote the Banach space of bounded operators acting from X to Y.

We will use the following Kantorovich polynomials

$$K_n(f)(x) =: \sum_{k=0}^{n} \left[(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt\right] \binom{n}{k} x^k (1-x)^{n-k},$$

for every $n \geq 1, f \in L_p, 0 \leq x \leq 1$. The following theorem is proved in [2].

Theorem 2.1 *[2] If $f \in L_p([0,1]), 1 \leq p < +\infty$, then $\lim_{n\to\infty} K_n(f) = f$ in $L_p([0,1])$.*

Periodical version of this theorem has the following form.

Theorem 1$'$ *[2] Let $T_n : C_{2\pi}[-\pi,\pi] \to C[-\pi,\pi], \forall n \in N$ be positive operators and $T_n(g) \to g, n \to \infty$ in $C[-\pi,\pi]$, $\forall g \in \{1; cos; sin\}$. Then $\|T_n(f) - f\|_p \to 0, n \to \infty, \forall f \in L_p(-\pi,\pi)$.*

Firstly we define the concept of an asymptotic density.

Definition 2.3 *Let $A \subset N$ be some set and denote $A(n) = \sum_{a \leq n, a \in A} 1$. If there exists $\lim_{n\to\infty} \frac{A(n)}{n}$, it will be called the asymptotic density of the set A and will be denoted by $\delta(A)$.*

Now, we define statistical convergence.

Definition 2.4 *The sequence $\{a_n\}_{n\in N} \subset R$ is said to converge statistically to the $a \in R$ if $\delta(A_\varepsilon) = 0$, for $\forall\varepsilon > 0$, where $A_\varepsilon = \{n \in N : |a_n - a| \geq \varepsilon\}$.*

Let us define the statistical positivity of the sequences of numbers and operators.

Definition 2.5 *We will say that the sequence $\{a_n\}_{n\in N} \subset R$ is a statistical positive sequence if*

$$\delta(\{n \in N : a_n \geq 0\}) = 1,$$

or

$$\delta(K_+^c) = 0,$$

where $K_+ = \{n \in N : a_n \geq 0\} \wedge K_+^c = N \backslash K_+$.

Definition 2.6 *$\{T_n\}_{n\in N} \subset B(L_p; L_p)$ is said to be a sequence of the statistical positive operators if for any $f \geq 0$:*

$$\delta(\{n : T_n f \geq 0\}) = 1.$$

By $C_M[a,b]$ we denote the space of all functions f which are continuous at every point of the interval $[a,b]$ and are bounded on the entire real axis, that is,

$$|f(x)| \le M_f, -\infty < x < +\infty,$$

where M_f is a constant depending on f.

Let $\{A_n\}_{n\in N}$ be a sequence of positive linear operators (that is, $A_n(f,x) \ge 0$ if $f(x) \ge 0$), acting from $C_M[a,b]$ to the space $B[a,b]$ of all bounded functions on $[a,b]$.

Recall that $B[a,b]$ is a Banach space with a norm

$$\|f\|_B =: \sup_{a\le x\le b} |f(x)|, f \in B[a,b].$$

As usual, we write $A_n(f,x)$ instead of $A_n(f(t),x)$.

We also need the following known theorem from the work [9].

Theorem 2.2 *[9] If the sequence of positive linear operators $A_n : C_M[a,b] \to B[a,b]$ satisfies the conditions*

(a) $\|A_n(1,x) - 1\|_B \to 0, n \to \infty$;

(b) $\|A_n(t,x) - t\|_B \to 0, n \to \infty$;

(c) $\|A_n(t^2,x) - t^2\|_B \to 0, n \to \infty$,

then for any function $f \in C_M[a,b]$, we have

$$\|A_n(f,x) - f(x)\|_B \to 0, n \to \infty.$$

It should be noted that this theorem is also true for complex-valued functions.

Let us give the following definition of k-positivity for operators (see e.g. [7,8]).

Set

$$A(\omega) = \{f : \omega \to C : f \quad \text{is an analytic function on} \quad \omega\}.$$

A linear operator T, acting from $A(\omega)$ to $A(\omega)$, is called k-positive if it preserves the class of function with non-negative Taylor coefficient, i.e. $T_{k,m} \ge 0$, for $\forall k, m$, where $Tf(z) = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} f_m T_{k,m}$.

We will use the following theorem from the work [8].

Theorem 2.3 *[8] Let $g_k \ge 1, k = 0,1,2,...$, $\limsup_{k\to\infty} g_k^{1/k} = 1$ and $g(z) = \sum_{k=0}^{\infty} g_k z^k$ be an analytic function on ω. Let $\{T_n\}_{n\in N}$ be a sequence of linear k-positive operators from $A(\omega)$ into itself. If for any r < 1:*

$$\|T_n g(z) - g(z)\|_{A(\omega),r} \to 0, n \to \infty,$$

$$\|T_n(zg'(z)) - zg'(z)\|_{A(\omega),r} \to 0, n \to \infty,$$

$$\|T_n(z^2 g''(z)) - z^2 g''(z)\|_{A(\omega),r} \to 0, n \to \infty,$$

then for any function $f \in A(\omega)$ with Taylor coefficients satisfying $|f_k| \le M g_k, k = 0,1,2,...$

$$\left\| \frac{d^p T_n f(z)}{dz^p} - f^{(p)}(z) \right\|_{A(\omega),r} \to 0, n \to \infty, p = 0,1,2\ldots,$$

where f_k is a Taylor coefficient of $f \in A(\omega)$, $M > 0$ and $\|.\|_{A(\omega),r}$ is an ordinary norm $A(\omega)$, i.e.

$$\|f\|_{A(\omega),r} = \max_{|z|\le r} |f(z)|, \forall f \in A(\omega), 0 < r < 1.$$

Let us define the statistical version of k-positivity for operator sequence.

Definition 2.7 *Let $\{T_n\}_{n\in N}$ be a sequence of k-positive operators acting from $A(\omega)$ to $A(\omega)$. $A(\omega)$ is an ordinary Frechet space of analytic functions with the family of above mentioned norms. If $\delta(\{n : T_n \text{ is } k-positive\}) = 1$, then we say that $\{T_n\}_{n\in N}$ is a statistical k-positive sequence.*

3 Korovkin-type Theorem in Lebesgue Space of Harmonic Functions

In this section we will consider Korovkin-type theorem in h_p space. So, the following theorem is true.

Theorem 3.1 *Let $\{T_n\}_{n\in \mathrm{N}} \subset B(L_p; L_p), 1 < p < +\infty$, be a sequence of the positive operators such that*

$$T_n(g) \to g, n \to \infty, \text{ in } L_p(-\pi, \pi), \tag{3.1}$$

for any $g \in \{1; \cos; \sin\}$. Then for the operators $t_n : h_p \to h_p$, defined by expression

$$t_n = P_r \circ T_n \circ \Gamma, \forall n \in N,$$

it is valid the following properties:

1. *$\{t_n\}_{n\in N} \subset B(h_p, h_p)$;*
2. *$\{t_n\}_{n\in N}$ are positive operators for $\forall n \in N$;*
3. *$\|t_n(u) - u\|_{h_p} \to 0$, $n \to \infty$, $\forall u \in h_p$.*

Proof. Firstly, we show that $\{t_n\}_{n\in N}$ are bounded operators. Let us take any u in h_p. Then

$$\|t_n(u)\|_{h_p} = \left\|P_r\left(T_n u^+\right)\right\|_{h_p} = \left\|\Gamma\left(P_r\left(T_n u^+\right)\right)\right\|_{L_p}$$

$$= \left\|T_n u^+\right\|_{L_p} \le \|T_n\| \left\|u^+\right\|_{L_p} = \|T_n\| \|u\|_{h_p},$$

from this follows that

$$\|t_n\| \le \|T_n\| \Rightarrow \{t_n\}_{n\in N} \subset B(h_p, h_p).$$

Let us show that $\{t_n\}_{n\in N}$ are positive operators, i.e.
$Re\, t_n(u) \ge 0 \wedge Im\, t_n(u) \ge 0, \forall n \in N, \forall u \in h_p$: u is a positive function.
We have

$$u^+\left(e^{it}\right) = \lim_{r\to 1-0} u\left(re^{it}\right)$$

$$= \lim_{r\to 1-0} Re\, u\left(re^{it}\right) + i \lim_{r\to 1-0} Im\, u\left(re^{it}\right) \ge 0.$$

Since, $\{T_n\}_{n\in N}$are positive operators, it follows that

$$T_n\left(u^+\right) = T_n\left(Re\, u^+\right) + iT_n\left(Im\, u^+\right)$$

$$\Rightarrow T_n\left(Re\, u^+\right) \ge 0 \wedge T_n\left(Im\, u^+\right) \ge 0.$$

Consequently

$$t_n(u) = (P_r \circ T_n \circ \Gamma)(u) = P_r\left(T_n u^+\right) = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{T_n\left(u^+\right)}{1 - 2r\cos(t-\theta) + r^2} d\theta =$$

$$= \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{T_n\left(Reu^+\right)}{1-2r\cos\left(t-\theta\right)+r^2}d\theta + i\frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{T_n\left(Imu^+\right)}{1-2r\cos\left(t-\theta\right)+r^2}d\theta \geq 0$$

Hence, $t_n\left(u\right) \geq 0, \forall n \in N \Rightarrow \{t_n\}$ are positive operators for any $n \in N$.

Let us prove

$$\left\|t_n\left(u\right)-u\right\|_{h_p} \to 0, n \to +\infty.$$

Take any $u \in h_p$ and consider

$$\left\|t_n\left(u\right)-u\right\|_{h_p} = \left\|\Gamma\left(t_n\left(u\right)-u\right)\right\|_{L_p} = \left\|\Gamma\left(t_n\left(u\right)-\Gamma\left(u\right)\right)\right\|_{L_p}$$

$$= \left\|\Gamma\left(P_r\left(T_n u^+\right)\right)-u^+\right\|_{L_p} = \left\|T_n\left(u^+\right)-u^+\right\|_{L_p}. \tag{3.2}$$

From the results of monograph [10] it is follows that $u^+ \in L_p$. From the relation (3.1) and by Theorem $1^{'}$, we obtain

$$\left\|T_n\left(u^+\right)-u^+\right\|_{L_p} \to 0, n \to \infty.$$

Then from the relation (3.2) we obtain

$$\left\|t_n\left(u\right)-u\right\|_{h_p} = \left\|T_n\left(u^+\right)-u^+\right\|_{L_p} \to 0, n \to \infty.$$

◀

Example 1 Let us consider the following Kantorovich operators

$$\left(K_n f\right)\left(x\right) = \sum_{k=0}^{n}\binom{n}{k}\left(n+1\right)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f\left(t\right)dt x^k(1-x)^{n-k}, 0 < x < 1.$$

These operators are positive in $L_p\left(0,1\right)$, and it is fact that

$$\left\|K_n\left(f\right)-f\right\|_{L_p(0,1)} \to 0, n \to \infty, \forall f \in L_p\left(0,1\right).$$

It follows from this that

$$\left\|K_n^{\pi}\left(f\right)-f\right\|_{L_p(0,2\pi)} \to 0, n \to \infty, \forall f \in L_p\left(0,2\pi\right),$$

where

$$\left(K_n^{\pi} f\right)\left(x\right) = \frac{1}{(2\pi)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\left(n+1\right)\int_{\frac{2k\pi}{n+1}}^{\frac{(k+1)2\pi}{n+1}} f\left(t\right)dt x^k(2\pi-x)^{n-k}, 0 < x < 2\pi.$$

Then from Theorem 3.1 immediately it follows that

$$\left\|t_n^{\pi}\left(u\right)-u\right\|_{h_p} \to 0, n \to \infty,$$

where $t_n^{\pi} = P_r \circ K_n^{\pi} \circ \Gamma, \forall n \in N$.

4 Statistical Version of Korovkin-type Theorem in h_p

Let us consider the statistical version of Theorem 2.1. So, firstly we need some auxiliary facts. It is true the following fact from the work T.Salat [21].

Lemma 4.1 *The sequence $\{x_n\}_{n\in N}$ of real numbers converges statistically to the real number x if and only if there exists such a set*

$$K = \{k_1, k_2, ..., k_n, ...\} \subset N,$$

that $\delta(K) = 1 \wedge \lim_{n\to\infty} x_{k_n} = x.$

Also we need the following lemma from the work of B.T.Bilalov, T.Y. Nazarova [3].

Lemma 4.2 *Let* $\delta(K_j) = 1, j = 1, 2 \Rightarrow \delta(K_1 \bigcap K_2) = 1.$

From this lemma immediately follows the following

Corollary 4.1 *Let* $\delta(K_j) = 1, j = 1, 2, ..., m \Rightarrow \delta\left(\bigcap_{j=1}^{m} K_j\right) = 1.$

Denote

$$L_p^+ =: \{f \in L_p : f \geq 0\}.$$

Let us prove the following

Theorem 4.1 *Let $\{T_n\}_{n\in N} \subset B(L_p, L_p)$ is a statistical positive and $\delta(K^+) = 1$, where*

$$K^+ = \bigcap_{f\in L_p^+} K_f^+, K_f^+ = \{n : T_n f \geq 0\}.$$

If the following relation holds

$$\|T_n(g) - g\|_{L_p} \xrightarrow{st} 0, n \to \infty,$$

for any $g \in \{1, \cos, \sin\}$, then it is valid the following :

1. *$\{t_n\}_{n\in N} \subset B(h_p, h_p)$ is a statistical positive ;*
2. *$\|t_n(u) - u\|_{h_p} \xrightarrow{st} 0, n \to \infty, \forall u \in h_p,$*

where $t_n = P_r \circ T_n \circ \Gamma, \forall n \in N$.

Proof. Let $\{T_n\}_{n\in N}$ is a statistical positive operators which satisfied the conditions of theorem. Then it is obvious that

$$T_n f \geq 0, \forall n \in K^+, \forall f \in L_p^+. \tag{4.1}$$

From the condition of the theorem it follows

$$\|T_n(1) - 1\|_{L_p} \xrightarrow{st} 0, n \to \infty.$$

Then from the Lemma 4.1 it follows that

$$\exists K_1 \subset N : \lim_{\substack{n\to\infty \\ n\in K_1}} \|T_n(1) - 1\|_{L_p} = 0 \wedge \delta(K_1) = 1. \tag{4.2}$$

Similarly

$$\|T_n(\cos)-\cos\|_{L_p} \overset{st}{\to} 0, n\to\infty.$$

Then

$$\exists K_{\cos}\subset N: \lim_{\substack{n\to\infty \\ n\in K_{\cos}}} \|T_n(\cos)-\cos\|_{L_p}=0\wedge\delta(K_{\cos})=1, \tag{4.3}$$

and from the relation

$$\|T_n(\sin)-\sin\|_{L_p} \overset{st}{\to} 0, n\to\infty,$$

it follows that

$$\exists K_{\sin}\subset N: \lim_{\substack{n\to\infty \\ n\in K_{\sin}}} \|T_n(\sin)-\sin\|_{L_p}=0\wedge\delta(K_{\sin})=1. \tag{4.4}$$

Let K be a set like the following

$$K=K^+\cap K_1\cap K_{\cos}\cap K_{\sin}.$$

Then from the Corollary 4.1 it is follows that

$$\delta(K)=1.$$

Set

$$A_n=:T_{k_n}, \forall n\in N, \forall k_n\in K.$$

It is evidently that operators$\{A_n\}_{n\in N}$ satisfy the following conditions :

1 $\{A_n\}_{n\in N}$ is positive for $\forall n\in N$;
2 $A_n(1)\to 1, n\to\infty$;
3 $A_n(\cos)\to\cos, n\to\infty$;
4 $A_n(\sin)\to\sin, n\to\infty$.

Set

$$\tilde{t}_n=P_r\circ A_n\circ\Gamma, \forall n\in N.$$

Then it follows from Theorem 3.1 that

$$\left\|\tilde{t}_n(u)-u\right\|_{h_p}\to 0, n\to\infty.$$

Therefore, with respect to the operators $\{t_n\}$ we obtain

$$\|t_n(u)-u\|_{h_p} \overset{st}{\to} 0, n\to\infty.$$

Hence, we have proved the statistical version of Korovkin-type theorem in h_p.

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Now, we compare the positive and statistical positive operators. We know that if $\{T_n\}_{n\in N}$ is a positive sequence then $\{T_n\}_{n\in N}$ is a statistical positive. But the converse is not always true.

Example 2 Let $\{T_n\}_{n \in N}$ be a sequence of the positive operators and

$$K_e = \left\{ n : n = 2^k, k \in N \right\}.$$

It is not difficult to show that

$$\delta (K_e) = 0.$$

Define

$$\tilde{T}_n = \begin{cases} T_n, n \in N \backslash K_e \,, \\ -T_n, n \in K_e \,, \end{cases}$$

It is evident that these operators $\left\{ \tilde{T}_n \right\}_{n \in N}$ are statistical positive, but are not positive.

Let us consider statistical version of Korovkin-type theorem for $C_M [a, b]$. Denote

$$C_M^+ [a, b] = \{ f \in C_M [a, b] : f \geq 0 \}.$$

Theorem 4.2 *Let T_n be statistical positive linear operators and $\delta (K_c^+) = 1$, where*

$$K_c^+ = \bigcap_{f \in C_M^+ [a,b]} K_f^+, K_f^+ = \{ n : T_n f \geq 0 \}.$$

If $\{T_n\}_{n \in N}$ satisfies the following conditions

1. $\| T_n (1, x) - 1 \|_B \xrightarrow{st} 0, n \to \infty;$
2. $\| T_n (t, x) - x \|_B \xrightarrow{st} 0, n \to \infty;$
3. $\left\| T_n \left(t^2, x \right) - x^2 \right\|_B \xrightarrow{st} 0, n \to \infty,$

then for any function $f \in C_M [a, b]$ we have

$$\| T_n (f, x) - f (x) \|_B \xrightarrow{st} 0, n \to \infty.$$

Proof. We take a sequence of the statistical positive linear operators $\{T_n\}_{n \in N}$ and let $T_n : C_M [a, b] \to B [a, b]$ provides all conditions of theorem. Then it is obvious that
T_n are positive linear operators for $n \in K_c^+ \wedge \delta (K_c^+) = 1$.
Define

$$K_c =: K_c^+ \cap K_c^0 \cap K_c^1 \cap K_c^2,$$

where
$\delta \left(K_c^0 \right) = 1 : \exists K_c^0 \subset N$ from condition a);
$\delta \left(K_c^1 \right) = 1 : \exists K_c^1 \subset N$ from condition b) ;
$\delta \left(K_c^2 \right) = 1 : \exists K_c^2 \subset N$ from condition c) .
From Corollary 4.1 it follows that

$$\delta (K_c) = 1.$$

Let

$$K_c =: \{ 1 \leq k_1 < k_2 < ... \}.$$

Set

$$A_n =: T_{k_n}, \forall n \in N.$$

It is clear that $\{A_n\}_{n \in N}$ satisfies the following conditions

1. $\{A_n\}_{n \in N}$ is a positive linear$\forall n \in N$
2. $\| A_n (1, x) - 1 \|_B \to 0, n \to \infty;$

3 $\|A_n(t,x) - x\|_B \to 0, n \to \infty;$
4 $\left\|A_n\left(t^2,x\right) - x^2\right\|_B \to 0, n \to \infty.$

So, $\{A_n\}_{n\in N}$ provides the conditions of Theorem 2.2. Then for any function $f \in C_M[a,b]$, we have

$$\|A_n(f,x) - f(x)\|_B \to 0, n \to \infty.$$

It means that

$$\|T_n(f,x) - f(x)\|_B \overset{st}{\to} 0, n \to \infty.$$

◄

5 Statistical k-positivity

In this section we consider statistical k-positive operators $\{T_n\}_{n\in N}$, acting from $A(\omega)$ to $A(\omega)$. The following theorem is true.

Theorem 5.1 *Let $g_k \geq 1, k = 0,1,2,...$ $\limsup\limits_{k\to\infty} g_k^{1/k} = 1$ and $g(z) = \sum_{k=0}^{\infty} g_k z^k$ be an analytic function in ω. Let $\{T_n\}_{n\in N}$ be a sequence of linear statistical k-positive operators from $A(\omega)$ into itself. If for any $r < 1$:*

$$\|T_n g(z) - g(z)\|_{A(\omega),r} \overset{st}{\to} 0, n \to \infty, \tag{5.1}$$

$$\left\|T_n\left(zg'(z)\right) - zg'(z)\right\|_{A(\omega),r} \overset{st}{\to} 0, n \to \infty, \tag{5.2}$$

$$\left\|T_n\left(z^2g''(z)\right) - z^2g''(z)\right\|_{A(\omega),r} \overset{st}{\to} 0, n \to \infty, \tag{5.3}$$

then for any function with Taylor coefficients $|f_k| \leq Mg_k, k = 0,1,2,...$ holds

$$\left\|\frac{d^pT_nf(z)}{dz^p} - f^{(p)}(z)\right\|_{A(\omega),r} \overset{st}{\to} 0, n \to \infty, p = 0,1,2\dots.$$

Proof. We take a sequence of the statistical k-positive operators $\{T_n\}_{n\in N}$, under which all conditions of theorem are fulfilled. Then from Definition 2.5 it follows $\delta(\{n : T_n$ is $k - positive\}) = 1$.
If we denote $\tilde{K}_+ =: \{n : T_n$ is $k - positive\}$, then it is obvious that

$$\delta\left(\tilde{K}_+\right) = 1 \wedge \{T_n\}_{n\in N} \text{ is } k-\text{positive for any } n \in \tilde{K}_+. \tag{5.4}$$

Define

$$\tilde{K} =: \tilde{K}_+ \cap K_0 \cap K_1 \cap K_2,$$

where
$\delta(K_0) = 1 : \exists K_0 \subset N$ from (5.1) ;
$\delta(K_1) = 1 : \exists K_1 \subset N$ from (5.2) ;
$\delta(K_2) = 1 : \exists K_2 \subset N$ from (5.3) .
From Corollary 4.1 it follows that

$$\delta\left(\tilde{K}\right) = 1.$$

Let

$$\tilde{K} =: \{1 \leq k_1 < k_2 < ...\} .$$

Set

$$\tilde{A}_n =: T_{k_n}, \forall n \in N.$$

It is obviously that $\left\{\tilde{A}_n\right\}_{n\in N}$ satisfy the following conditions

1 $\left\{\tilde{A}_n\right\}_{n\in N}$ are k-positive $\forall n \in N$
2 $\lim\limits_{n\to\infty} \left\|\tilde{A}_n g(z) - g(z)\right\|_{A(\omega),r} = 0;$
3 $\lim\limits_{n\to\infty} \left\|\tilde{A}_n (zg'(z)) - zg'(z)\right\|_{A(\omega),r} = 0;$
4 $\lim\limits_{n\to\infty} \left\|\tilde{A}_n \left(z^2 g''(z)\right) - z^2 g''(z)\right\|_{A(\omega),r} = 0.$

So $\left\{\tilde{A}_n\right\}_{n\in N}$ provided the conditions of Theorem 2.3. Then

$$\left\|\frac{d^p \tilde{A}_n f(z)}{dz^p} - f^{(p)}(z)\right\|_{A(\omega),r} \to 0, n \to \infty, p = 0, 1, 2, ...$$

It means that

$$\left\|\frac{\partial^p T_n f(z)}{\partial z^p} - f^p(z)\right\|_{A(\omega),r} \overset{st}{\to} 0, n \to \infty, \forall p \geq 0.$$

◀

The converse is not always true.

Example 3 Let $\{T_n\}_{n\in N}$be a linear k-positive sequence of operators and

$$K_s = \left\{n : n = k^2, k \in N\right\} .$$

It follows immediately from the definition that

$$\delta(K_s) = 0.$$

Define

$$\tilde{T}_n = \begin{cases} T_n, n \in N \backslash K_s , \\ -T_n, n \in K_s . \end{cases}$$

Then it is obvious that $\left\{\tilde{T}_n\right\}_{n\in N}$ is a sequence of the linear statistical k-positive operators, but is not k-positive.

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