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Approximate solution of hypersingular integral equations with Cauchy kernel

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Abstract. *In the present paper, the hypersingular integral operator with Cauchy kernel H is approximated by a sequence of operators of the special form, it is proved that, the approximating operators H_n strongly converges to the operator H, for an algebraic polynomial of degree not higher than n, the operators H_n and H coincide, and is given a new method for the approximate solution of hypersingular integral equations with Cauchy kernel.*

Keywords. hypersingular integral operator, Cauchy kernel, hypersingular integral equation, best mean-square approximation.

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1 Introduction

An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. (see [4, 5, 10, 13, 20, 22, 23, 26, 27]). Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works [4-9, 11, 12, 14, 16-19, 21-25, 27-31] are devoted to their development. In the present paper, the hypersingular integral operator with Cauchy kernel H is approximated by a sequence of operators of the special form, it is proved that, the approximating operators H_n strongly converges to the operator H, for an algebraic polynomial of degree not higher than n, the operators H_n and H coincide, and is given a new method for the approximate solution of hypersingular integral equations with Cauchy kernel. The advantages of this method are as follows: using this method, we obtain an immediate estimate of the convergence rate for functions of the appropriate class, while other methods used earlier to approximate solutions of hypersingular integral equations require the study of the specific properties of the function classes under consideration; moreover, the estimate established in this paper yields more exact results in terms of the convergence rate than the methods used

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earlier (see [4, 5, 7, 9, 11, 12, 16-19, 22, 23, 27-31]); approximate solutions of the system of linear algebraic equations can be found explicitly (not at isolated points) and, therefore, given a fixed convergence rate, this method requires smaller computational resources.

It is known (see [4, 7, 9, 22]) that, the solution of the inner Neumann problem for Laplace equation reduced to the hypersingular integral equation

$$(R\varphi)(t) = \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau$$

$$+\int_{\gamma_0} K(t, \tau)\varphi(\tau)\,d\tau = f(t)\,, t \in \gamma_0 = \{t \in C:\ |t| = 1\}\,, \tag{1.1}$$

where $K(t, \tau) = \frac{\partial}{\partial t}F(t, \tau)$ is continuous function.

In the present paper, the operator R is approximated by a sequence of operators of the form

$$(R_n\varphi)(t) = \sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right)\,, t \in \gamma_0, n \in N,$$

where the functions $\alpha_k^{(n)}(t)$ is continuous functions, $k = 0, 1, 2, ..., n \in N$, are expressed in terms of the given functions, and is given a new method for the approximate solution of hypersingular integral equations of the first kind (1.1). It should be noted that, for the operators of the form $(R_n\varphi)(t) = \sum\limits_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right)$, the determination of the inverse operator is equivalent to the study of the equation

$$\sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right) = f(t)$$

at the points $t = \tau_0^{(t)}, \tau_1^{(t)}, ..., \tau_{2n-1}^{(t)}$, because solving the resulting system of linear algebraic equations

$$\begin{cases} \alpha_0^{(n)}(t)\,\varphi\left(\tau_0^{(t)}\right) + \alpha_1^{(n)}(t)\,\varphi\left(\tau_1^{(t)}\right) + ... + \alpha_{2n-1}^{(n)}(t)\,\varphi\left(\tau_{2n-1}^{(t)}\right) = f\left(\tau_0^{(t)}\right)\,, \\ \alpha_{2n-1}^{(n)}\left(\tau_1^{(t)}\right)\varphi\left(\tau_0^{(t)}\right) + \alpha_0^{(n)}\left(\tau_1^{(t)}\right)\varphi\left(\tau_1^{(t)}\right) + ... \\ \qquad\qquad + \alpha_{2n-2}^{(n)}\left(\tau_1^{(t)}\right)\varphi\left(\tau_{2n-1}^{(t)}\right) = f\left(\tau_1^{(t)}\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_1^{(n)}\left(\tau_{2n-1}^{(t)}\right)\varphi\left(\tau_0^{(t)}\right) + \alpha_2^{(n)}\left(\tau_{2n-1}^{(t)}\right)\varphi\left(\tau_1^{(t)}\right) + ... \\ \qquad\qquad + \alpha_0^{(n)}\left(\tau_{2n-1}^{(t)}\right)\varphi\left(\tau_{2n-1}^{(t)}\right) = f\left(\tau_{2n-1}^{(t)}\right), \end{cases}$$

with respect to $\left(\varphi\left(\tau_0^{(t)}\right)\,,\ \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$, we obtain the function $\varphi(t)$.

Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and its application to singular integral equations are given in the papers [1] and [2].

2 Approximation of the hypersingular integral operators with Cauchy kernel

Consider the integral

$$\int_{\gamma_0} \frac{\varphi(\tau)\, d\tau}{(\tau - t)^2}, t \in \gamma_0, \tag{2.1}$$

where φ is Lebesgue integrable on γ_0. If we define this integral similar to the Cauchy integral, even if $\varphi \equiv 1$, we get the divergent integral.

Therefore, using the idea of Hadamard [15], we will define the integral (2.1) as follows:

Definition 1.1 *If a finite limit* $\lim\limits_{\varepsilon \to 0+} \left(\int_{\gamma_\varepsilon} \frac{\varphi(\tau) d\tau}{(\tau - t)^2} - \frac{2\varphi(t)}{\varepsilon\, i\, t} \right)$ *exist, then the value of this limit is referred to as the hypersingular Cauchy integral of the function* φ *on* γ_0*, and is denoted by* $\int_{\gamma_0} \frac{\varphi(\tau) d\tau}{(\tau - t)^2}$*, where* $\gamma_\varepsilon = \{\tau \in \gamma_0 : |\tau - t| > \varepsilon\}$.

From the definitions 1.1 it follows that,

$$\int_{\gamma_0} \frac{d\tau}{(\tau - t)^2} = \lim_{\varepsilon \to 0+} \left(\frac{1}{t - t \cdot e^{-i\delta(\varepsilon)}} - \frac{1}{t - t \cdot e^{i\delta(\varepsilon)}} + \frac{2i}{\varepsilon t} \right)$$

$$= \frac{1}{t} \lim_{\varepsilon \to 0+} \left(\frac{e^{i\delta(\varepsilon)} + 1}{e^{i\delta(\varepsilon)} - 1} + \frac{2i}{\varepsilon} \right) = 0, \tag{2.2}$$

where $\delta(\varepsilon) = 2 \arcsin \frac{\varepsilon}{2} \sim \varepsilon$ as $\varepsilon \to 0+$, and, therefore,

$$\int_{\gamma_0} \frac{\varphi(\tau)\, d\tau}{(\tau - t)^2} = \int_{\gamma_0} \frac{\varphi(\tau) - \varphi(t)}{(\tau - t)^2} d\tau,$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

In the paper [3], it is proved that, if the function φ absolutely continuous on γ_0, then the hypersingular integral (2.1) exist for almost all $t \in \gamma_0$, and the equation

$$\int_{\gamma_0} \frac{\varphi(\tau)\, d\tau}{(\tau - t)^2} = \int_{\gamma_0} \frac{\varphi'(\tau)\, d\tau}{\tau - t} \tag{2.3}$$

holds, where the integral standing in the right side is understood in the sense of the Cauchy principal value.

Let $L_2 = L_2(\gamma_0)$ be the space of the functions square-integrable on γ_0 with the norm

$$\|\varphi\|_{L_2} = \left(\frac{1}{2\pi} \int_{\gamma_0} |\varphi(\tau)|^2 |d\tau| \right)^{1/2},$$

and let $W_2^1 = W_2^1(\gamma_0)$ be the space of absolutely continuous on γ_0 functions, which derivative belongs on the space L_2, with the norm

$$\|\varphi\|_{W_2^1} = \|\varphi\|_{L_2} + \|\varphi'\|_{L_2}.$$

From the equation (2.3) it follows that, the hypersingular integral operator

$$(H\varphi)(t) \equiv \frac{1}{\pi i} \int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau, t \in \gamma_0$$

is bounded from the space W_2^1 into the space L_2, and $\|H\|_{W_2^1 \to L_2} = 1$.

Consider the sequence of operators

$$(H_n\varphi)(t) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left(\tau_{2k+1}^{(t)}-t\right)^2}\Delta\tau_{2k+1}^{(t)}, \ \ t\in\gamma_0, \quad n=1,2,..,$$

where $\tau_k^{(t)} = e^{k\theta i}\cdot t$, $\Delta\tau_k^{(t)} = \left(\tau_{k+1}^{(t)}-\tau_{k-1}^{(t)}\right)\frac{\theta}{\sin\theta} = 2ie^{k\theta i}\cdot t\cdot\theta$, $k=\overline{0,2n}$, $\theta=\frac{\pi}{n}$.

In the paper [3], it is proved that if $\varphi(t) = \sum\limits_{k=-\infty}^{+\infty} c_k t^k \in W_2^1$, then $(H_n\varphi)(t) = \sum\limits_{k=-\infty}^{+\infty} c_k\mu_k^{(n)}t^{k-1}$, where $\mu_m^{(n)} = m$ for $m=\overline{0,n}$, $\mu_m^{(n)} = 2n-m$ for $m=\overline{n+1,2n}$, $\mu_{m\pm 2n}^{(n)} = \mu_m^{(n)}$ for all $m\in Z$.

Properties A [3] *The operators H_n, $n=1,2,..$ is bounded from the space W_2^1 into the space L_2, and $\|H_n\|_{W_2^1\to L_2}\le 1$, and for any polynomial $q(t) = \sum\limits_{k=-n}^{n} q_k t^k$ the following relation holds*

$$(H_n q)(t) = (Hq)(t)\,.$$

Suppose that $E_n\left(\varphi;\,W_2^1\right) = \inf\limits_{q\in T_n}\|\varphi(\cdot)-q_n(\cdot)\|_{W_2^1}$ is the best approximation of the function $\varphi\in W_2^1$ by polynomials T_n, where T_n is the set of polynomials of the form $\sum\limits_{k=-n}^{n}\alpha_k t^k$, $\alpha_k\in C$.

Theorem B [3] *The sequence of operators $\{H_n\}$ strongly converges to the operator H and, for any $\varphi\in W_2^1$, the following estimate holds:*

$$\|H\varphi - H_n\varphi\|_{L_2}\le 2E_n\left(\varphi;\,W_2^1\right).$$

3 The approximate solution of the hypersingular integral equations of the first kinds

In [3] it is considered the simple hypersingular integral equation of the first kind

$$(H\varphi)(t) = f(t)\,, t\in\gamma_0, \tag{3.1}$$

where $f\in L_2$. The equation (3.1) is solvable only if

$$\int_{\gamma_0} f(\tau)\,d\tau = 0, \tag{3.2}$$

If the condition (3.2) is satisfies, then the equation (3.1) has infinitely many solutions in the general form

$$\varphi^*(t) = d_0 + \sum_{k\in Z\backslash\{-1\}}\frac{c_k(f)}{|k+1|}t^{k+1}\in W_2^1, \tag{3.3}$$

where $c_k(f) = \frac{1}{2\pi i}\int_{\gamma_0}\tau^{-k-1}f(\tau)\,d\tau$ is Fourier's coefficients of function $f\in L_2$, and d_0 is a constant. Therefore if we consider the equation

$$(H\varphi)(t) = f(t) - \frac{1}{2\pi\, i\, t}\int_{\gamma_0} f(\tau)\,d\tau, t\in\gamma_0, \tag{3.4}$$

$$\frac{1}{2\pi\, i}\int_{\gamma_0}\frac{\varphi\left(\tau\right)}{\tau}d\tau=d_0, \tag{3.5}$$

then the equation (3.4)-(3.5) is unique solvable for any $(f;\, d_0)\in L_2\times C$, and the solution of (3.4)-(3.5) is the function, defined by (3.3).

Now we consider the equation

$$\left(H_n\varphi\right)(t)=f\left(t\right), t\in\gamma_0, \tag{3.6}$$

where $f\in L_2$. Considering equation (3.6) at the points $t=\tau_0^{(t)}, \tau_1^{(t)}, ..., \tau_{2n-1}^{(t)}$ we obtain the following system of linear algebraic equations:

$$\left(H_n\varphi\right)\left(\tau_p^{(t)}\right)=f\left(\tau_p^{(t)}\right), p=\overline{0,2n-1}, t\in\gamma_0 \tag{3.7}$$

with respect to $\left(\varphi\left(\tau_0^{(t)}\right), \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$. Since for any $t\in\gamma_0$ the equation

$$\sum_{p=0}^{2n-1}\left(H_n\varphi\right)\left(\tau_p^{(t)}\right)\cdot\tau_p^{(t)}=0,$$

holds, then we obtain that, the equation (3.6) is solvable only if

$$\sum_{p=0}^{2n-1}f\left(\tau_p^{(t)}\right)\cdot\tau_p^{(t)}=0. \tag{3.8}$$

If the condition (3.8) is satisfies, then the equation (3.6) has infinitely many solutions. And if we consider the equations

$$\left(H_n\varphi\right)(t)=f\left(t\right)-\frac{1}{2n}\sum_{p=0}^{2n-1}e^{p\theta i}f\left(\tau_p^{(t)}\right), t\in\gamma_0, \tag{3.9}$$

$$\frac{1}{2n}\sum_{p=0}^{2n-1}\varphi\left(\tau_p^{(t)}\right)=d_0, \tag{3.10}$$

we will receive that, the equations (3.9)-(3.10) is unique solvable for any $(f;\, d_0)\in L_2\times C$, and the solutions of (3.9)-(3.10) is the function

$$\varphi_n^*\left(t\right)=d_0+\sum_{\substack{k=-\infty\\ k\neq-1\ (mod\, 2n)}}^{+\infty}\frac{c_k\left(f\right)}{\mu_{k+1}^{(n)}}t^{k+1}\in L_2. \tag{3.11}$$

Theorem C [3] *For any* $(f;\, d_0)\in L_2\times C$ *the system of linear algebraic equations* (3.9)-(3.10) *unique solvable with respect to* $\left(\varphi\left(\tau_0^{(t)}\right), \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$*; the solutions* $\varphi_n^*\left(t\right)=\varphi_n^*\left(\tau_0^{(t)}\right)$ *of the equations* (3.9)-(3.10) *converge in the norm of the space* L_2 *to the solution* $\varphi^*\left(t\right)$ *of the equation* (3.4)-(3.5)*, and the following estimate is holds:*

$$\left\|\varphi_n^*-\varphi^*\right\|_{L_2}\leq\left(1+\frac{1}{n}\right)E_n\left(f;\, L_2\right). \tag{3.12}$$

We consider the hypersingular integral equation of the first kind

$$\left(R\varphi\right)(t)=\left(H\varphi\right)(t)+\left(\mathrm{K}\varphi\right)(t)=f\left(t\right), t\in\gamma_0, \tag{3.13}$$

where $(\mathrm{K}\varphi)(t)=\int_{\gamma_0} K(t,\tau)\,\varphi(\tau)\,d\tau$, and $K(t,\tau)=\frac{\partial}{\partial t}F(t,\tau)$ is continuous function. The equation (3.13) is also solvable only if the condition (3.2) is satisfied. Therefore, we consider the equation

$$(R\varphi)(t)=f(t)-\frac{1}{2\pi\, i\, t}\int_{\gamma_0} f(\tau)\,d\tau, t\in\gamma_0, \tag{3.14}$$

$$\frac{1}{2\pi\, i}\int_{\gamma_0}\frac{\varphi(\tau)}{\tau}d\tau=d_0. \tag{3.15}$$

Since the solution of the equation (3.4)-(3.5) is a function

$$(\varphi)(t)=d_0+(Bf)(t),$$

where $(Bf)(t)=\frac{1}{\pi i}\int_{\gamma_0}\ln\frac{1}{|\tau-t|}f(\tau)\,d\tau$ is bounded linear operator acting in L_2, then the equation (3.14)-(3.15) is equivalent to the equation

$$(\varphi)(t)+(B\mathrm{K})\,\varphi(t)=d_0+(Bf)(t), t\in\gamma_0. \tag{3.16}$$

Consider the equation

$$(H_n\varphi)(t)+(\mathrm{K}_n\varphi)(t)-\frac{1}{2n}\sum_{p=0}^{2n-1}e^{p\theta i}\sum_{k=0}^{2n-1}K\left(\tau_p^{(t)},\tau_k^{(t)}\right)\varphi\left(\tau_k^{(t)}\right)\left(\frac{1}{2}\Delta\tau_k^{(t)}\right)$$

$$=f(t)-\frac{1}{2n}\sum_{p=0}^{2n-1}e^{p\theta i}f\left(\tau_p^{(t)}\right), t\in\gamma_0, \tag{3.17}$$

$$\frac{1}{2n}\sum_{p=0}^{2n-1}\varphi\left(\tau_p^{(t)}\right)=d_0, \tag{3.18}$$

where $(\mathrm{K}_n\varphi)(t)=\sum_{k=0}^{2n-1}K\left(t,\tau_k^{(t)}\right)\varphi\left(\tau_k^{(t)}\right)\left(\frac{1}{2}\Delta\tau_k^{(t)}\right)$. By theorem C for any $(f;\, d_0)\in L_2\times C$ the equation (3.9)-(3.10) has the unique solution

$$(\varphi)(t)=d_0+(B_nf)(t),$$

the operators B_n are the form $(B_nf)(t)=\sum\limits_{k=0}^{2n-1}\beta_k^{(n)}(t)\,f\left(\tau_k^{(t)}\right)$, and the sequence of operators $\{B_n\}$ strongly converges to the operator B in L_2. Therefore equation (3.17)-(3.18) is equivalent to the equation

$$(\varphi)(t)+(B_n\mathrm{K}_n)\,\varphi(t)=d_0+(B_nf)(t). \tag{3.19}$$

We need the following theorem and lemma proved in [1].

Theorem D [1] *The sequence of operators* $\{\mathrm{K}_n\}$ *strongly converges to the operator* K *in* L_2 *and, for any* $\varphi\in L_2$*, the following estimate holds:*

$$\|\mathrm{K}\varphi-\mathrm{K}_n\varphi\|_{L_2}\le 4\pi\,\|K\|_\infty\,E_{n-1}(\varphi;\,L_2)+4\pi E_{n-1}(K)\left\{E_{n-1}(\varphi;\,L_2)+\|\varphi\|_{L_2}\right\},$$

where $\|K\|_\infty=\max\limits_{t,\tau\in\gamma_0}|K(t,\tau)|$, $E_{n-1}(K)=\inf\left\|K(t,\tau)-\sum\limits_{k=-n+1}^{n-1}\alpha_k(t)\,\tau^k\right\|_\infty$ *and infimum is taken on all algebraic polynomials* $\alpha_k(t)$, $k=\overline{-n+1,\,n-1}$ *is degree not higher than* $n-1$.

Lemma E [1] *Suppose that B, B_n, $n = 1, 2, ...,$ are bounded linear operators acting in L_2; moreover, the operators B_n are of the form $(B_n f)(t) = \sum_{k=0}^{2n-1} \beta_k^{(n)}(t) f\left(\tau_k^{(t)}\right)$ and the sequence of operators $\{B_n\}$ strongly converges to the operator B in L_2. If the inverse operator $(I + B\mathrm{K})^{-1}$ exist, then, for large values of n, the operators $(I + B_n\mathrm{K}_n)$ are also invertible and the sequence of operators $\left\{(I + B_n\mathrm{K}_n)^{-1}\right\}$ strongly converges to the operator $(I + B\mathrm{K})^{-1}$ in L_2.*

Theorem 2.1 *If for any $(f;\, d_0) \in L_2 \times C$ the equation* (3.14)-(3.15) *unique solvable, then for large values of n, the systems of linear algebraic equations* (3.17)-(3.18) *are also unique solvable for any $(f;\, d_0) \in L_2 \times C$ with respect to $\left(\varphi\left(\tau_0^{(t)}\right),\ \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$; the solutions $\varphi_n^*(t)$ of the equations* (3.17)-(3.18) *converge in the norm of the space L_2 to the solution $\varphi^*(t)$ of the equation* (3.14)-(3.15), *and the following estimate is holds:*

$$\|\varphi_n^* - \varphi^*\|_{L_2} \le const \cdot \{E_n(f;\, L_2) + E_n(\mathrm{K}\varphi^*;\, L_2)$$

$$+4\pi \|K\|_\infty E_{n-1}(\varphi^*;\, L_2) + 4\pi E_{n-1}(K)\left[E_{n-1}(\varphi^*;\, L_2) + \|\varphi^*\|_{L_2}\right]\}. \qquad (3.20)$$

Proof. Since the equation (3.14)-(3.15) is equivalent to the equation (3.16), we will get that, the operator $I + B\mathrm{K}$ invertible in L_2. Then from lemma B it follows that, the operators $(I + B_n\mathrm{K}_n)$ are also invertible and the sequence of operators $\left\{(I + B_n\mathrm{K}_n)^{-1}\right\}$ strongly converges to the operator $(I + B\mathrm{K})^{-1}$ in L_2. Since the equation (3.17)-(3.18) is equivalent to the equation (3.19), we will get that, the equations (3.17)-(3.18) are also unique solvable for any $(f;\, d_0) \in L_2 \times C$ with respect to $(\varphi(\tau_0^{(t)}),\ \varphi(\tau_1^{(t)}), ..., \varphi(\tau_{2n-1}^{(t)}))$ and the solutions $\varphi_n^*(t)$ of the equations (3.17)-(3.18) converge in the norm of the space L_2 to the solution $\varphi^*(t)$ of the equation (3.14)-(3.15).

Estimate (3.20) follows from the inequality

$$\|\varphi_n^* - \varphi^*\|_{L_2} = \left\|(I + B_n\mathrm{K}_n)^{-1}(d_o + B_n f) - (I + B\mathrm{K})^{-1}(d_o + Bf)\right\|_{L_2} \le$$

$$\le \left\|(I + B_n\mathrm{K}_n)^{-1}\right\|_{L_2 \to L_2} \cdot \|B_n f - Bf\|_{L_2}$$

$$+ \left\|(I + B_n\mathrm{K}_n)^{-1}\right\|_{L_2 \to L_2} \cdot \|(B\mathrm{K} - B_n\mathrm{K}_n)\varphi^*\|_{L_2},$$

and from the theorems C and D. This completes the proof of the theorem.

4 The approximate solution of the hypersingular integral equations of the second kinds

Now we consider the hypersingular integral equation of the second kind

$$(T\varphi)(t) = \lambda \cdot \frac{\varphi(t)}{t} + (H\varphi)(t) = f(t),\, t \in \gamma_0. \qquad (4.1)$$

If $\varphi(t) = \sum_{k=-\infty}^{+\infty} c_k t^k \in W_2^1$, then $(T\varphi)(t) = \sum_{k=-\infty}^{+\infty} (\lambda + |k|)\, c_k t^{k-1}$. Therefore, the equation (4.1) is solvable only if $\lambda \notin Z_- = Z \backslash N$ and the solution of the equation (4.1) is

$$(\varphi^*)(t) = \sum_{k=-\infty}^{+\infty} \frac{c_k(f)}{(\lambda + |k|)} t^{k-1}.$$

Consider the equations

$$(T_n\varphi)(t) = \lambda \cdot \frac{\varphi(t)}{t} + (H_n\varphi)(t) = f(t), t \in \gamma_0, n \in N. \tag{4.2}$$

Theorem 3.1 *If $\lambda \notin Z_-$, then for any values of n, equations* (4.2) *also unique solvable for any $f \in L_2$ with respect to $\left(\varphi\left(\tau_0^{(t)}\right), \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$, the solutions $\varphi_n^*(t)$ of the equations* (4.2) *converge in the norm of the space L_2 to the solution $\varphi^*(t)$ of the equation* (4.1)*, and the following estimate is holds:*

$$\|\varphi_n^* - \varphi^*\|_{L_2} \le \frac{2}{\rho(\lambda; Z_-)} E_n(f; L_2), \tag{4.3}$$

where $\rho(\lambda; Z_-)$ – distance from λ to the set Z_-.
Proof. Since

$$T_n\left(\sum_{k=-\infty}^{+\infty} c_k t^k\right) = \sum_{k=-\infty}^{+\infty}\left(\lambda + \mu_k^{(n)}\right) c_k t^{k-1},$$

then equations (4.2) also is solvable for any values of n, and the solution of the equation (4.2) is

$$(\varphi_n^*)(t) = \sum_{k=-\infty}^{+\infty} \frac{c_k(f)}{\left(\lambda + \mu_k^{(n)}\right)} t^{k-1},$$

where $\mu_m^{(n)} = m$ for $m = \overline{0,n}$, $\mu_m^{(n)} = 2n - m$ for $m = \overline{n+1, 2n}$, $\mu_{m\pm 2n}^{(n)} = \mu_m^{(n)}$ for all $m \in Z$. Therefore

$$\varphi_n^*(t) - \varphi^*(t) = \sum_{|k|>n}\left(\frac{1}{\left(\lambda + \mu_k^{(n)}\right)} - \frac{1}{(\lambda + |k|)}\right) c_k(f)\, t^{k-1}.$$

From this it follows that the solutions $\varphi_n^*(t)$ of the equations (4.2) converge in the norm of the space L_2 to the solution $\varphi^*(t)$ of the equation (4.1) and estimate (4.3) holds. This completes the proof of the theorem.

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