Transactions of NAS of Azerbaijan, Issue Mathematics, **37** (1), 30–36 (2017).
Series of Physical-Technical and Mathematical Sciences.

On stability of the solutions of matrix game and the capital preservation problem to the perturbations of initial data

Soltan A. Aliyev · Yaroslav I. Yeleyko · Katerina V. Kosarevych · Oleq I. Karpa

Received: 19.09.2016 / Revised: 15.04.2017/ Accepted: 25.05.2017

Abstract. *The stability of the solution of the capital preservation problem to the perturbations of initial data given by the matrix of rates of profits of assets is proved in terms of the semi-continuous multi-valued maps. It is also shown that the solution of the corresponding matrix game is also stable to the perturbations. Finally, it is proved that solutions of both perturbed problems are the same.*

Keywords. matrix game, completely mixed optimal strategy, upper semicontinuous multi-valued mapping.

Mathematics Subject Classification (2010): 91A13 · 91A23 · 91A43

1 Introduction

This article focuses on the issue of equality of solutions of the capital preservation problem [3] and corresponding matrix game [1], and also with the problem of stability of solutions to the perturbations of initial data given by the matrix of rates of return of assets. Research will be carried out in two stages. The first phase takes place if the second player's (the economic environment) optimal strategies are completely mixed [1]. In this case, we will define the main constraints to the initial data necessary for the existence of such strategies, and show that the solution of capital preservation problem coincides with the matrix game solution.

Obviously, the question of whether the solutions (of the capital preservation problem and matrix game) are equal requires research even if the second player's optimal strategies are not completely mixed. That is why the purpose of this study is also to determine whether the solutions of perturbed capital preservation problem and perturbed matrix game are equal (second stage). We will show that in the perturbed situation, there is continuous dependence of the solution of the capital preservation problem on the initial data given by

S.A. Aliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: soltanaliyev@yahoo.com

YA.I. Yeleyko
Ivan Franko National University of Lviv, Ukraine
E-mail: mstat@franko.lviv.ua

K.V. Kosarevych
Ivan Franko National University of Lviv, Ukraine
E-mail: katiakosarevych@mail.ru

O.I. Karpa
Ivan Franko National University of Lviv, Ukraine
E-mail: yits@yahoo.com

the matrix of rates of return. We will check the solution of capital preservation problem for stability to perturbations of the initial data. Required proof will be carried out in terms of semicontinuous multivalued mappings. We will also show that solution of corresponding matrix game is stable to the perturbation of initial data. The proof will be carried out on the basis of the method for solving matrix games by using differential equations [2].

2 Unperturbed case

Let m be amount of risky assets, E $= (e_1, ..., e_n)$ be the set of states of economic environment, g_j be a priori probability distribution of occurrence of state j ($j = \overline{1,n}$), r_{ij} be the rate of return of i-type asset provided economic environment is in state j ($i = \overline{1,m}$), $R = \{r_{ij}\}$ be matrix of rates of return. Consider one-criterion capital preservation problem (problem of risk minimization for the portfolio on the set of allowable portfolios):

$$D = D(R) = \sum_{i=1}^{m} \sum_{l=1}^{m} x_i x_l \sigma_{il} \to \min_X, \tag{2.1}$$

$$\sum_{i=1}^{m} x_i = 1, \tag{2.2}$$

$$x_i \geq 0, \quad i = 1, ..., m, \tag{2.3}$$

where random variable R is asset portfolio with the structure $X = (x_1, ..., x_m) \in S_m$, $S_m = \left\{ x \in \tilde{y}^m : \sum\limits_{i=1}^{m} x_i = 1;\ x_i \geq 0,\ i = \overline{1,m} \right\}$ is m-dimensional simplex in which optimization is carried out, $R = \sum\limits_{i=1}^{m} x_i R_i$, R_i – i-throw of matrix R of rates of return, D is the appropriate risk, x_i is the share of capital invested in i-th asset; $\sigma_{il} = cov(R_i, R_l) = \sum\limits_{j=1}^{n} g_j (r_{ij} - E_i)(r_{lj} - E_l), i, l = \overline{1,m}.$

Consider now the capital preservation problem as pair game (economic environment plays the role of second player) with zero sum and payoff matrix R under the assumption that this game has no saddle point. Solving matrix game in mixed strategies, we obtain optimal strategies for both players and the price of the game, that is (P^*, Q^*), $P^* \in S_m$, $Q^* \in S_n$, and V^*, respectively.

Statement 2.1 *Let (P^*, Q^*) be solution of pair zero-sum game with matrix $R = \{r_{ij}\}$, $i = \overline{1,m}$, $j = \overline{1,n}$. Let optimal strategy Q^* of second player be completely mixed, that is $q_j^* > 0$, $\forall j = \overline{1,n}$. Then*

(i) $P^* = X^*$ (in the sense of $p_i^* = x_i^*$, $i = \overline{1,m}$), where X^* is a solution of the capital preservation problem (2.1)-(2.3)*;*

(ii) for portfolio $R^* = \sum\limits_{i=1}^{m} p_i^* R_i$ equality $D^* = D(R^*) = 0$ is valid.

Proof. If $q_{j_0}^* > 0$ for some j_0 in optimal strategy Q^* of second player, then $\sum\limits_{i=1}^{m} r_{ij_0} p_i^* = V^*$. According to conditions of statement 2.1 optimal strategy Q^* of second player is completely mixed, that is $q_j^* > 0$, $\forall\ j = \overline{1,n}$. Therefore $\forall\ j = \overline{1,n}$ equality $\sum\limits_{i=1}^{m} r_{ij} p_i^* = V^*$ is valid. Remind that each row R_i of matrix R is discrete (by assumption) random variable with values from the set $\{r_{i1}, ..., r_{in}\}$. Hence portfolio $R^* = \sum\limits_{i=1}^{m} p_i^* R_i$ is random variable with possible values $\left\{\sum\limits_{i=1}^{m} r_{i1}p_i^*, ..., \sum\limits_{i=1}^{m} r_{in}p_i^*\right\}$. It is easy to see that random variable R^* takes values $\{V^*, ..., V^*\}$, that is it is constant. On the other hand for any portfolio with the structure $X = (x_1, ..., x_m)$ its variance is $D(R) \geq 0$, so $\min\limits_{X \in S_m} D(R) = 0 = D(R^*)$. Thus, portfolio with the structure $P^* = X^*$ has the smallest (zero) risk. The statement is proved.

So we have established the following. If optimal strategy Q^* of second player is completely mixed in pair zero-sum matrix game then solution of games theory problem for first player and the structure X^* of portfolio (where X^* is the solution of the capital preservation problem (2.1)-(2.3)) coincide.

Remark 2.1 In order for optimal strategy Q^* of the second player to be completely mixed the following conditions are necessary:

(i) column of matrix R which dominates any other column or some convex combination of other columns of matrix R is missing, that is $\bar{\exists}\ j = \overline{1,n}$ that $r_{ij} \geq r_{ik}$ $(k = \overline{1,n},\ k \neq j)$ for all $i = \overline{1,m}$, and $r_{ij} > r_{ik}$ at least for one $i = \overline{1,m}$;

(ii) $n - 1 \leq rang(R) \leq m - 1$.

Remark 2.2 The assumption about zero sum in matrix game with matrix R is not obligatory. It is easy to see that by selection $k > 0$ and $l \in$ R you can always get zero price of game with matrix R', where $r'_{ij} = kr_{ij} + l$. For games with matrices R and R' the sets of optimal strategies of players coincide.

3 Solution of capital preservation problem and its stability to the perturbations of initial data

Consider a more general case. Assume that in solution (P^*, Q^*) of matrix game with matrix R the optimal strategy Q^* of second player is not completely mixed. Assume also that there exists a family $\{R^\varepsilon\}$ of matrices $m \times n$ for which conditions of remark 2.1 are satisfied. Let for all ε optimal strategy $Q^{\varepsilon *}$ of second player be completely mixed in game with matrix R^ε. If matrices $\{R^\varepsilon\}$ are such that $R^\varepsilon \to R$, $\varepsilon \to 0$ (in terms of element-by-element convergence) then for R solutions of capital preservation problem and matrix game coincide.

Really, on the basis of matrix R^ε capital preservation problem can be solved for all ε on the set of allowable portfolios. Denote solution of problem (2.1)-(2.3) by $X^{\varepsilon *} = (x_1^{\varepsilon *}, ..., x_m^{\varepsilon *})$ for every ε. On the other hand define optimal strategies $(P^{\varepsilon *}, Q^{\varepsilon *})$ of both players and the price $V^{\varepsilon *}$ of the game with payoff matrix R^ε. Within the assumptions above and according to statement 1 we get $P^{\varepsilon *} = X^{\varepsilon *}, \forall \varepsilon$.

Let us show that if $\varepsilon \to 0$, then the solution of capital preservation problem for R^ε converges to the solution of a similar problem for R, that is actually there is continuous dependence of this solution on initial data given by the matrix of rates of return of assets. Thus, we obtain the following problem for every ε:

$$D^\varepsilon = D(R^\varepsilon) = \sum_{i=1}^{m} \sum_{l=1}^{m} x_i^\varepsilon \sigma_{il}^\varepsilon x_l^\varepsilon = X^\varepsilon C^\varepsilon (X^\varepsilon)^T \to \min_{X^\varepsilon}, \tag{3.1}$$

$$\sum_{i=1}^{m} x_i^\varepsilon = 1, \tag{3.2}$$

$$x_i^\varepsilon \geq 0, \ \ i = 1, ..., m, \tag{3.3}$$

where C^ε is a covariance matrix with elements which are determined from matrix R^ε by using formula $\sigma_{il}^\varepsilon = cov(R_i^\varepsilon, R_l^\varepsilon)$.

For $\varepsilon = 0$ we get corresponding problem based on matrix R:

$$D = D(R) = \sum_{i=1}^{m}\sum_{l=1}^{m} x_i \sigma_{il} x_l = XC(X)^T \to \min_X, \tag{3.4}$$

$$\sum_{i=1}^{m} x_i = 1, \tag{3.5}$$

$$x_i \geq 0, \quad i = 1, ..., m, \tag{3.6}$$

where C is a covariance matrix with elements which are determined from matrix R.

Consider the general optimization problem. Let $f : X \to \tilde{y}$ be continuous function, $X \subset \tilde{y}^n, T \subset \tilde{y}^s$. Let also $C : T \to X$ be multi-valued mapping. We obtain the following problem:$f(x,t) \to \max_x$; $x \in C(t)$; $C(t) \neq \emptyset$, $t = (t_1, ..., t_s) \in T$; $T \subset \tilde{y}^s$ is the set of parameters.

Lemma 3.1 *(Berge's maximum theorem [3], [1]). Let function $f : X \times T \to \tilde{y}$ be continuous in all arguments and let multi-valued mapping $C : T \to X$ (which determines the set of constraints) be continuous. Then:*

(i) function $M : T \to \tilde{y}$ determined by $M(t) = \max\{D(x,t) : x \in C(t)\}$ is continuous;

(ii) mapping $m : T \to X$ determined by $m(t) = \{x \in C(t) : D(x,t) = M(t)\}$ is upper semicontinuous.

Consider optimization problem (3.4)-(3.6) it terms of mappings:

$$D(x,t) \to \min_x, x \in C(t), C(t) \subset S_m, C(t) \neq \emptyset, t \in T.$$

The set T of parameters is the set of matrices of norms of returns and the element $t \in T$ is a given matrix R. It is easy to see that objective function $D_\Pi : X \times T \to \tilde{y}D : X \times T \to \mathrm{R}$ of that problem is continuous in all arguments. A non-empty set of constraints does not visibly depend on parameter, corresponding mapping $C : T \to X$ is continuous. So problem (3.4)-(3.6) satisfies the conditions of lemma 1. Consequently mapping $m : T \to X$ that puts the set of vectors (minimizing the objective function) in accordance to matrix R is upper semicontinuous.

Next consider the sequence $\{X^{\varepsilon *}\}$ of solutions of problem (3.1)-(3.3) for each ε. Since $X^{\varepsilon *} \in S_m, \forall \varepsilon$, and the set S_m is compact then convergent subsequence can be separated from the sequence $\{X^{\varepsilon *}\}$. Denote the limit of that subsequence by X^*. Then from upper semicontinuity of mapping $m : T \to X$ we get the following statement.

Statement 3.1 *(i) $R^\varepsilon \to R$ when $\varepsilon \to 0$;*

(ii) $X^{\varepsilon *} \to X^*$, where $X^{\varepsilon *} \in m(R^\varepsilon)$, $\forall \varepsilon$, $\varepsilon \to 0$;

(iii) X^* is the solution of problem (3.4)-(3.6)*, $X^* \in m(R)$.*

4 Stability of solution of matrix game to the perturbations of initial data

It remains to show that the set of optimal strategies in matrix game is also upper semi continuous mapping of payoff matrices. Of course, this can be done as described above by using Berge's theorem since matrix game is reduced to linear programming problem. But we will use the method for solving of matrix games by using differential equations [2].

Note that searching for optimal strategies one can be restricted by symmetric games (with skew-symmetric payoff matrix). Really, there are methods of representation of arbitrary game as a symmetric game with $V^* = 0$ [1].

Let A be a skew-symmetric $n \times n$ matrix and let $P = (p_1, ..., p_n)$ be arbitrary strategy of one player (obviously sets of optimal strategies of both players are the same). Denote $u_k = \sum\limits_{j=1}^{n} a_{kj} p_j$, $k = \overline{1, n}$, $\varphi(u_k) = \max\{0,\ u_k\}$, $\Phi(P) = \sum\limits_{k=1}^{n} \varphi(u_k)$, $\Psi = \Psi(P) = \sum\limits_{k=1}^{n} \varphi^2(u_k)$. Consider the system of differential equations

$$\frac{dp_k}{dt} = \varphi(u_k) - \Phi(P) p_k, k = \overline{1, n}, \tag{4.1}$$

under certain initial conditions

$$p_k(t_0) = p_k^0, k = 1, ..., n, P^0 = (p_1^0, ..., p_n^0) \in S_n. \tag{4.2}$$

Cauchy problem for the system (4.1) with initial conditions (4.2) has a unique continuous solution. Existence of such a solution follows from the fact that $\forall k$ functions $f_k(t, p_1, ..., p_n) \equiv \varphi(u_k) - \Phi(P) p_k$ are continuous in all arguments and satisfy the Lipschitz condition. It is easy to make sure that number $L = 4n \max\limits_{i,j}\{a_{ij}\}$ can be a Lipschitz constant.

Establish two important properties of that solution. If $p_k^0 \geq 0$, then $p_k(t) \geq 0$ for all $t > 0$. Really, if $p_k(t') = 0$, then $\frac{dp_k}{dt} = \varphi(u_k) \geq 0$, function p_k grows. Let now $\sum\limits_{k=1}^{n} p_k^0 = 1$. Let show that $\sum\limits_{k=1}^{n} p_k(t) = 1$ for all $t > 0$. Assume that $\sum\limits_{k=1}^{n} p_k(t') > 1\, \forall t'$. Then there exists $t'' \in [0,\ t')$ such that $\sum\limits_{k=1}^{n} p_k(t'') = 1$ and $\sum\limits_{k=1}^{n} p_k(t) > 1$ for all $t \in (t'',\ t']$. Since for these values t $\sum\limits_{k=1}^{n} p_k(t) > \sum\limits_{k=1}^{n} p_k(t'')$, then $\frac{d}{dt} \sum\limits_{k=1}^{n} p_k(t''') > 0$ at least for one $t''' \in (t'',\ t)$. By adding all the equations of system (4.1) we obtain $\frac{d}{dt} \sum\limits_{k=1}^{n} p_k(t) = \Phi(P)\left(1 - \sum\limits_{k=1}^{n} p_k(t''')\right) \leq 0$, that leads to contradiction. The case of $\sum\limits_{k=1}^{n} p_k(t) < 1$ is refuted by similar explanation. Hence, by taking value 1 for some $t = t_0$, the sum $\sum\limits_{k=1}^{n} p_k(t)$ will still remain equal to 1. So if vector P^0 is a strategy, then given property is kept for $P(t)$ for all t.

Next, if $\varphi(u_k) = u_k > 0$ we obtain $\frac{d\varphi(u_k)}{dt} = \sum\limits_{l=1}^{n} a_{kl} \frac{dp_l}{dt} = \sum\limits_{l=1}^{n} a_{kl} \varphi(u_l) - \Phi(P)\varphi(u_k)$. Therefore

$$\frac{d\varphi^2(u_k)}{dt} = 2\varphi(u_k)\frac{d\varphi(u_k)}{dt} = 2\varphi(u_k)\left(\sum_{l=1}^{n} a_{kl}\varphi(u_l) - \Phi(P)\varphi(u_k)\right). \tag{4.3}$$

If $u_k \leq 0$ then $\varphi(u_k)$ is constant and equals zero, so the derivatives $\dfrac{d\varphi(u_k)}{dt} = 0$ and $\dfrac{d\varphi^2(u_k)}{dt} = 0$. Hence equality (4.3) remains true. It follows that for all k

$$\sum_{k=1}^{n} \frac{d\varphi^2(u_k)}{dt} = \frac{d\Psi(P)}{dt} = 2\sum_{k=1}^{n} \varphi(u_k)\left(\sum_{l=1}^{n} a_{kl}\varphi(u_l) - \Phi(P)\varphi(u_k)\right)$$

$$= 2\sum_{k=1}^{n}\ \sum_{l=1}^{n} a_{kl}\varphi(u_k)\varphi(u_l) - 2\Phi(P)\sum_{k=1}^{n}\varphi^2(u_k),$$

and using the fact that matrix A is skew-symmetric

$$\frac{d\Psi(P)}{dt} = -2\Phi(P)\Psi(P). \qquad (4.4)$$

Since summands of $\Phi(P)$ are not negative, then it is easy to see that $\sqrt{\Psi(P)} \le \Phi(P)$. By substituting obtained assessment for $\Phi(\cdot)$ in (4.4) we get $\frac{d\Psi(P)}{dt} \le -2\left(\Psi(P)\right)^{3/2}$ or $\frac{d}{dt}\left(\Psi^{-1/2}(P)\right) \ge 1$.

By integrating obtained inequality in an arbitrary interval $(0,\ t)$, in which $\Psi(\cdot) > 0$, we get $\Psi^{-1/2}(P) \ge t + \Psi^{-1/2}(P^0)$ or $\Psi(P) \le \frac{\Psi(P^0)}{\left(1+t\sqrt{\Psi(P^0)}\right)^2}$. Since $\varphi^2(u_k) \le \Psi(P)$, then we get $u_k \le \varphi(u_k) \le \frac{\sqrt{\Psi(P^0)}}{1+t\sqrt{\Psi(P^0)}}$. Hence $\lim\limits_{t\to\infty} u_k(t) \le \lim\limits_{t\to\infty} \frac{\sqrt{\Psi(P^0)}}{1+t\sqrt{\Psi(P^0)}} = 0$ for $k = \overline{1,n}$, that is

$$\lim_{t\to\infty}\sum_{j=1}^{n} a_{kj}p_j(t) \le 0 = V_A^*. \qquad (4.5)$$

Consider unlimited growing sequence of values $t_1,\ t_2,\ ...$ and corresponding sequence of strategies $P(t_1), P(t_2), ...$ Since the set S_n of mixed strategies is compact, then convergent subsequence can be always separated from the given sequence. Let P^* be its limit. From (4.5) it follows that $\sum\limits_{j=1}^{n} a_{kj}p_j^* \le 0 = V^*$, $\forall\ k = 1, ..., n$, that is P^* is optimal strategy of second player (and also of first player due to symmetry of the game).

Let us turn directly to the issue of stability of solution of matrix game. First of all since the solution of Cauchy problem (4.1)-(4.2) has the property of uniqueness, then this solution is stable to the perturbations of initial data and right part $f_k(t, p_1, ..., p_n) \equiv \varphi(u_k) - \Phi(P)p_k$.

In the given problem the perturbations in the right part are represented by payoff matrices or rather by matrices obtained from R^ε and R by symmetrizating the games. Since right part is continuous in all parameters, then it follows from $R^\varepsilon \to R$, $\varepsilon \to 0$, that $f_k^\varepsilon(t, p_1, ..., p_n) \to f_k(t, p_1, ..., p_n)$, $\varepsilon \to 0$. So the solution of (4.1)-(4.2) with parameter R^ε (vector function $P^\varepsilon(t) = (p_1^\varepsilon(t), ..., p_n^\varepsilon(t))$) converges to the solution of corresponding unperturbed system, that is to vector function $P(t) = (p_1(t), ..., p_n(t))$ with $\varepsilon \to 0$ if $R^\varepsilon \to R$.

Next on the basis of continuous functions $P(t)$ and $P^\varepsilon(t)$, $\forall\varepsilon$, we can construct convergent sequences $\{P(t_1),\ P(t_2), ...\}$ and $\{P^\varepsilon(t_1),\ P^\varepsilon(t_2), ...\}$, $\forall\varepsilon$, which converge to the optimal strategies P^* and $P^{\varepsilon *}$, $\forall\varepsilon$, respectively.

Remark 4.1 In terms of set of the functions depending on parameter ε and due to the stability of the solution of problem (4.1)-(4.2) to the perturbations of initial data, $P(t)$ uniformly converges to $P^\varepsilon(t)$ with $\varepsilon \to 0$, $t \in \tilde{y}$ ($P(t) \rightrightarrows P^\varepsilon(t)$). On the other hand the mapping $P^\varepsilon(t) \equiv P(t, \varepsilon)$ can be considered as a set of functions depending on parameter t. So, $\forall\varepsilon \in \tilde{y}$ $P(t, \varepsilon) \to P(\varepsilon) \equiv P^{\varepsilon *}$, $t \to \infty$, where $P^{\varepsilon *}$ is an optimal mixed strategy. Hence $P^\varepsilon(t) \to P^{\varepsilon *}$ when $\varepsilon \in \tilde{y}$ and $t \to \infty$. An important property of the set of functions depending on parameter is represented by the following lemma.

Lemma 4.1 *(about permutation of limits). Consider* $f \underset{X}{\rightrightarrows} \varphi$, $t \to \tau$, *and* $f \underset{T}{\to} \psi$, $x \to a$, *where* $f : T \times X \to \tilde{y}^n$, $\varphi : X \to \tilde{y}^n$, $\psi : T \to \tilde{y}^n$, $a \in X$, $\tau \in T$. *Then:*

(i) $\exists \lim\limits_{x\to a}\varphi(x)$;

(ii) $\exists \lim_{t\to\tau} \psi(t)$*;*
(iii) $\lim_{x\to a} \varphi(x) = \lim_{t\to\tau} \psi(t)$.

Due to lemma 4.1 and remark 4.1 we obtain the following statement.

Statement 4.1 (i) $\exists \lim_{t\to\infty} P(t)$, furthermore $\lim_{t\to\infty} P(t) = P^*$, where P^* is an optimal strategy of first player;
(ii) $\exists \lim_{\varepsilon\to 0} P^{\varepsilon *}$;
(iii) $\lim_{\varepsilon\to 0} P^{\varepsilon *} = \lim_{t\to\infty} P(t)$.

Remark 4.2 From the statement 4.1 we get that $P^{\varepsilon *} \to P^*$ with $\varepsilon \to 0$, that is the solution of matrix game is stable to the perturbations of initial data.

References

1. Ausubel, L.M., Deneckere, R.J.: *A generalized theorem of the maximum*. Econom. Theory, **3** (1), 99–107 (1993).
2. Brown, G.W., Neumann, J.V.: *Solutions of games by differential equations*. Contributions to the Theory of Games, 7379. Annals of Mathematics Studies, no. 24. Princeton University Press, Princeton, N. J. (1950).
3. Berzh, K.: Obshchaja teorija igr neskolkih liz, *M.: Fizmatgiz*, 126 p. (1961).
4. Gitman, L. Dzh., Dzhonk M.D.: Osnovy investirovanija, *M.: Delo*, 810 p. (1997).
5. Vorobjov, N.N.: Osnovy teorii igr. *Beskoalicionnyje igry, M.: Nauka*, 496 p. (1984).