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Oscillation theorems for half-linear Sturm-Liouville problems with spectral parameter in the boundary condition

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Abstract. *We consider half-linear Sturm-Liouville problem with spectral parameter in the boundary condition. We prove that there exists two sequences of simple real half-eigenvalues for this problem,* $\lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots$, *and* $\lambda_1^- < \lambda_2^- < \ldots < \lambda_k^- < \ldots$, *and no other half-eigenvalues. Moreover, we also find the number of zeros of the corresponding half-eigenfunctions.*

Keywords. Half-linear Sturm-Liouville equation, half-eigenvalue, half-eigenfunction, Sturm comparison theorem, global continua

Mathematics Subject Classification (2010): 35J15, 35J25, 35P05, 35P30, 47J10, 47J15.

1 Introduction

We consider following "half-linear" Sturm-Liouville equation

$$\ell(y) \equiv -\left(p(x)y'\right)' + q(x)y = \lambda r(x)y + \alpha(x)y^+ + \beta(x)y^-,\ x \in (0,\pi), \tag{1.1}$$

subject the boundary conditions

$$b_0 y(0) = d_0 y'(0), \tag{1.2}$$

$$(a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi), \tag{1.3}$$

where $y^+ = \max\{y,0\}$ $y^- = \max\{-y,0\}$, p is a positive, continuously differentiable function on $[0,1]$, r is a positive continuous function on $[0,1]$, q, α, β are the continuous functions on $[0,1]$, b_0, d_0, a_1, b_1, c_1, d_1 are real numbers with $|b_0| + |d_0| > 0$ and

$$a_1 d_1 - b_1 c_1 > 0. \tag{1.4}$$

We note that the problem (1.1)-(1.3) is non-linear, but is positively homogeneous (in the sense that if y is a solution of this problem, then αy is also a solution for all $\alpha > 0$) and

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linear in the cones $y > 0$ and $y < 0$. Problems of this type have been termed *half-linear* by Berestycki [3].

The following definitions are given in [3] (see also [8]). If there exists a nontrivial solution (λ, y_λ) of problem (1.1)-(1.3), then a number λ is said a *half-eigenvalue* of this problem; the function y_λ be called a *half-eigenfunction*. In this situation the set $\{(\lambda, ty_\lambda) : t > 0\}$ is a half-line of nontrivial solutions of (1.1)-(1.3). There may exist other half-lines of solutions $\{(\lambda, v_\lambda) : t > 0\}$. A half-eigenvalue λ is said to be simple if there is only one such half-line or there are exactly two such half-lines $\{(\lambda, ty_\lambda) : t > 0\}$ and $\{(\lambda, tv_\lambda) : t > 0\}$, with y_λ and v_λ having opposite signs on a deleted neighborhood of 0.

Problem (1.1)-(1.3) in the case $a_1 = c_1 = 0$ (i.e., when the spectral parameter is not involved in the boundary conditions) was considered in the works [3, 6]. Berestycki [3] by using global bifurcation techniques and Sturm oscillation theorems, for these problems showed that for each $\nu \in \{+, -\}$ there exists sequence of half-eigenvalues, $\lambda_1^\nu < \lambda_2^\nu < \ldots < \lambda_k^\nu < \ldots$, where the corresponding half-eigenfunction y_k^ν has $k-1$ simple nodal zeros in $(0, \pi)$ and satisfies $\nu y_k^\nu > 0$ in a neighborhood of 0. Later Browne [6] established this result by using Prüfer angle techniques.

In [6] it was considered also problem (1.1)-(1.3) under condition (1.4). But it should be noted that in [6] the location of eigenvalues on real axis and the oscillatory properties of the eigenfunctions of this problem have not been studied completely.

In [2] author showed that a sequence of half-eigenvalues exists, with certain properties and proved various results regarding the existence and multiplicity of solutions of positively homogeneous, half-linear problem for $2m$th order ordinary differential operator.

In this paper we consider the location of eigenvalues on real axis and the oscillatory properties of eigenfunctions of problem (1.1)-(1.3) under condition (1.4).

2 Prelinimaries

Alongside problem (1.1)-(1.3) we consider the spectral problems

$$\begin{cases} \ell(y)(x) = \lambda r(x) y(x),\ x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ y(1) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \ell(y)(x) = \lambda r(x) y(x),\ x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1 \lambda + b_1) y(\pi) = (c_1 \lambda + d_1) y'(\pi). \end{cases} \tag{2.2}$$

The eigenvalues of the Sturm-Liouville problem (2.1) are denoted by $\mu_k,\ k \in \mathbb{N}$, where $\mu_1 < \mu_2 < \ldots < \mu_k \mapsto +\infty$.

For $c_1 \neq 0$ let N_0 be an integer such that $\mu_{N_0 - 1} < -\frac{d_1}{c_1} \leq \mu_{N_0}$, where $\mu_0 = -\infty$.

It is known [4] that the eigenvalues of the boundary value problem (2.2) are real, simple, and form an infinitely increasing sequence $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$. The corresponding eigenfunctions $y_1(x), y_2(x), \ldots, y_k(x), \ldots$ have the following oscillation properties: (a) if $c_1 = 0$, then $y_k(x),\ k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in $(0, \pi)$; (b) if $c_1 \neq 0$, then $y_k(x)$ has exactly $k-1$ simple nodal zeros for $k \leq N_0$, and exactly $k-2$ simple nodal zeros for $k > N_0$ in the interval $(0, \pi)$.

Let $E = \{y \in C^1[0, \pi] : b_0 y(0) = d_0 y'(0)$ is a Banach space with the norm $||y||_1 = \max\limits_{x \in [0,\pi]} |y(x)| + \max\limits_{x \in [0,\pi]} |y'(x)|$. Let S_k^+ be the set of functions $y \in E$ which have exactly $k-1$ simple nodal zeros in $(0, \pi)$ and which are positive near $x = 0$, and set $S_k^- = -S_k^+$, and

$S_k = S_k^+ \cup S_k^-$. The sets S_k^+ and S_k^- are disjoint and open in E. From now on ν will denote an element of $\{+\,,\,-\}$ that is, either $\nu = +$ or $\nu = -$.

For $c_1 = 0$ let $\mathcal{S}_k^\nu = S_k^\nu$, and for $c_1 \neq 0$ let

$$\mathcal{S}_k^\nu = \begin{cases} S_k^\nu & \text{if } \ k \leq N_0, \\ S_{k-1}^\nu & \text{if } \ k > N_0, \end{cases} \ k \in \mathbb{N}, \text{ and } \nu \in \{+\,,\,-\}.$$

Let $J_k = \left[\lambda_k - \frac{M}{r_0}, \lambda_k + \frac{M}{r_0}\right]$, $k \in \mathbb{N}$, where $r_0 = \min\limits_{x \in [0,\pi]} |\, r(x)|$. For $c_1 = 0$ let $I_k = J_k$, and for $c_1 \neq 0$ let

$$I_k = \begin{cases} J_k & if \ \ k < N_0 \\ J_k \cup J_{k+1} & if \ \ k = N_0, \\ J_{k+1} & if \ \ k > N_0 \end{cases}$$

Now we consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda r(x) y(x) + f(x, y(x), y'(x), \lambda), \ x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1 \lambda + b_1) y(\pi) = (c_1 \lambda + d_1) y'(\pi), \end{cases} \tag{2.3}$$

where f is continuous function on $[0, \pi] \times \mathbb{R}^3$, satisfying the following condition: there exists a constant $M > 0$ such that

$$|f(x, u, v, \lambda)| \leq M|u|, \ \ (x, u, v, \lambda) \in [0, \pi] \times \mathbb{R}^3.$$

We denote by $\mathfrak{L}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of problem (2.2), and by $\mathfrak{L}_k^\nu$ the closure in $\mathbb{R} \times E$ of the set of all solutions (λ, y) of (2.2) with $y \in \mathcal{S}_k^\nu$ (the norm in $\mathbb{R} \times E$ is determined as follows: $||(\lambda, y)|| = \left\{|\lambda|^2 + ||y||_1^2\right\}^{\frac{1}{2}}$). We also let $\Phi_k^\nu = \mathbb{R} \times S_k^\nu$ under the product topology.

The following result which we will need in the further proved in [2].

Theorem 2.1 (see [2, Theorem 3.5]) *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there exists a connected component $\mathcal{D}_k^\nu$ of the set of solutions of problem* (2.3) *in $\mathfrak{L}_k^\nu \cup (I_k \times \{0\})$ which contain $I_k \times \{0\}$, lies in $I_k \times (\mathcal{S}_k^\nu \cup \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

Let $c_1 \neq 0$ and

$$M < \frac{1}{2}\, r_0 \gamma_0, \ \text{ where } \ \gamma_0 = \min\left\{\lambda_{k+1} - \lambda_k : \ k \in \mathbb{N}\right\}. \tag{2.4}$$

If condition (2.4) holds, then it is obvious that $J_k \cap J_m = \emptyset$ for any different $k, m \in \mathbb{N}$. Hence from Theorem 2.1 it follows the following result.

Corollary 2.1 *Suppose that $c_1 \neq 0$ and* (2.4) *holds. Then for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the connected component $\mathcal{D}_k^\nu$ of $\mathfrak{L}_k^\nu \cup (J_k \times \{0\})$ contain $J_k \times \{0\}$, lies in $J_k \times (\mathcal{S}_k^\nu \cup \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

3 Oscillatory properties of half-eigenfunctions of problem (1.1)-(1.3) in the case $c_1 = 0$

In this section we consider oscillation properties of problem (1.1)-(1.3) for $c_1 = 0$. Without loss generality in this case we assume that $d_1 = 1$, so that we can consider the following boundary value problem

$$\begin{cases} \ell(y)(x) = \lambda r(x)y(x) + \alpha(x)y^+(x) + \beta(x)y^-(x),\ x \in (0,\pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)y(\pi) = y'(\pi). \end{cases} \tag{3.1}$$

Theorem 3.1 *There exist two sequences of simple half-eigenvalues of problem* (3.1)*, $\lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots$, and $\lambda_1^- < \lambda_2^- < \ldots < \lambda_k^- < \ldots$. The corresponding half-lines of solutions are in $\{\lambda_k^+\} \times S_k^+$ and $\{\lambda_k^-\} \times S_k^-$. Furthermore, aside from these solutions and the trivial ones, there are no other solutions of problem* (3.1).

In the proof of this theorem, we require the following simple statement from [2].

Lemma 3.1 *Let j and k be integers, $j \geq k \geq 2$. Suppose there exist two families of real numbers*

$$\xi_0 = 0 < \xi_1 < \xi_2 < \ \ldots \ < \xi_{k-1} < \xi_k = \pi,$$

$$\eta_0 = 0 < \eta_1 < \eta_2 < \ \ldots \ < \eta_{j-1} < \eta_j = \pi,$$

Then, if $\xi_1 \leq \eta_1$, there exist integers p and l having the same parity, $1 \leq p \leq k-1$, $1 \leq l \leq j-1$, such that $\xi_p \leq \eta_l < \eta_{l+1} \leq \xi_{p+1}$.

Proof of Theorem 3.1 Let $M^* = \max\limits_{x\in[0,\pi]} \{|\alpha(x)| + |\beta(x)|\}$. Then it follows by Theorem 2.1 that for every $k \in \mathbb{N}$ and $\nu \in \{+\,,\,-\}$ there exists at least one solution $(\lambda_k^\nu, y_k^\nu) \in J_k^* \times S_k^\nu$ of problem (3.1), where $J_k^* = \left[\lambda_k - \frac{M^*}{r_0},\ \lambda_k + \frac{M^*}{r_0}\right]$. From the positive homogeneity of equation in (3.1) we have that $\{(\lambda_k^\nu, ty_k^\nu) : t > 0\}$ are half-lines of solutions in $\{\lambda_k^\nu\} \times S_k^\nu$.

It follows by the uniqueness of the solution of the initial value problem (the right-hand side of equation in (3.1) being Lipschitz continuous in y) that for any nontrivial solution (λ, y) of problem (3.1) the function y has only simple nodal zeros in $[0,\pi]$. Hence y lies in some S_k^ν. Now let we have two solutions (λ, y) and (μ, v) of problem (3.1) such that $y \in S_k^\nu$ and $v \in S_k^\nu$. Without loss of generality we may assume that the first zero of function $y(x)v(x)$ to occur in $(0,\pi]$ is a zero of y. Let $\xi(0,\pi]$ is a first zero of function $y(x)$. Then $y(\xi) = 0$, the functions $y(x)$ and $v(x)$ do not vanish and are positive in $(0,\xi)$. Since $y(\xi) = 0$, the function $y(x)$ is decreasing in a neighborhood of the point ξ. Hence $y'(\xi) < 0$. By (3.1) we have

$$v(x)\ell(y)(x) - v(x)\ell(y)(x) = (\lambda - \mu)r(x)y(x)\,v(x) \tag{3.2}$$

Integrating by parts of (3.2) from 0 to ξ, we obtain

$$\left[p(x)\left\{v'(x)y(x) - y'(x)v(x)\right\}\right]_0^\xi = (\lambda - \mu)\int\limits_0^\xi r(x)y(x)\,v(x)\,dx.$$

Hence it follows by condition (1.2) from this identity that

$$-p(\xi)y'(\xi)v(\xi) = (\lambda - \mu)\int\limits_0^\xi r(x)y(x)\,v(x)\,dx,$$

which implies that $\lambda \le \mu$. On the other hand, by Lemma 3.1, there must exist an interval $[\zeta, \eta] \subset [0, \pi]$ such that $y(x)$ and $v(x)$ do not vanish and have the same sign in (ζ, η), and either $v(\zeta) = v(\eta) = 0$, or $v(\zeta) = 0$ and $\eta = \pi$, or $\zeta = 0$ and $\xi = \eta = \pi$ (the latter occurring when $k = 1$). Consider, for example, the case of $v(\zeta) = 0$ and $\eta = \pi$. Without loss of generality we may suppose that $y(x) > 0$ and $v(x) > 0$ in the interval (ζ, π).Since $v(\zeta) = 0$, the function $y(x)$ is increasing in a neighborhood of the point ζ. Hence $v'(\zeta) > 0$. By (3.1) we have

$$\int\limits_{\zeta}^{\pi} \{v(x)\ell(y)(x) - y(x)\ell(v)(x)\}\, dx = (\lambda - \mu) \int\limits_{\zeta}^{\eta} r(x)y(x)\, v(x)\, dx.$$

Integrating by parts of equation (3.2) from ζ to π, we have

$$\big[p(x)\left\{v'(x)y(x) - y'(x)v(x)\right\}\big]_{\zeta}^{\pi} = (\lambda - \mu) \int\limits_{\zeta}^{\pi} r(x)y(x)\, v(x)\, dx. \tag{3.3}$$

Taking into account boundary condition (1.3) from (3.3) we obtain

$$-\,p(\zeta)y(\zeta)v'(\zeta) = (\lambda - \mu) \left\{ \int\limits_{0}^{\xi} r(x)y(x)\, v(x)\, dx + a_1 p(\pi)y(\pi)\, v(\pi) \right\}$$

which implies that $\lambda \ge \mu$ (since $c_1 = 0$ and $d_1 = 1$ it follows by condition (1.4) that $a_1 > 0$). Therefore $\lambda = \mu$. From the uniqueness in the initial value problem implies that there exists of a positive constant c such that $v = cy$. Thus the λ_k^ν, $k \in \mathbb{N}$, are simple half-eigenvalues and aside from the trivial solutions and the half-lines $\{(\lambda_k^\nu, ty_k^\nu) : t > 0\}$, there are no other solutions of problem (3.1).

Now we show that for each $\nu \in \{+\,,\, -\}$ the sequence $\{\lambda_k^\nu\}_{k=1}^\infty$ is increasing. If $k < j$, then for the half-eigenpair (λ_k^ν, y_k^ν) and (λ_j^ν, y_j^ν) of problem (3.1) the first zero of function $y_k^\nu(x)y_j^\nu(x)$ to occur in the interval $(0, \pi)$ is a zero of function $y_j^\nu(x)$. Indeed, otherwise using the above arguments and Lemma 3.1 we obtain $\lambda_k^\nu = \lambda_j^\nu$, which is impossible, since the half-eigenvalues λ_k^ν, $k \in \mathbb{N}$ of problem (3.1) are simple. Again using the above arguments we can show that $\lambda_k^\nu < \lambda_j^\nu$. The proof of theorem is complete.

4 Oscillatory properties of half-eigenfunctions of problem (1.1)-(1.3) in the case $c_1 \neq 0$

In this section we consider the problem (1.1)-(1.3) for $c_1 \neq 0$.

Theorem 4.1 *Suppose that the condition $M^* < \frac{1}{2}\gamma_0 r_0$ is satisfied. Then there exist two sequences of simple half-eigenvalues of problem* (1.1)-(1.3), $\lambda_1^+, \lambda_2^+, \ldots, \lambda_k^+, \ldots$ *, and* $\lambda_1^-, \lambda_2^-, \ldots, \lambda_k^-, \ldots$ *, such that if $k > k' \ge 1$, then $\lambda_k^\nu > \lambda_{k'}^{\nu'}$ for each $\nu, \nu' \in \{+\,,\, -\}$. The corresponding half-lines of solutions are in $\{\lambda_k^+\} \times \mathcal{S}_k^+$ and $\{\lambda_k^-\} \times \mathcal{S}_k^-$. Furthermore, aside from these solutions and the trivial ones, there are no other solutions of problem* (1.1)-(1.3).

Proof. Since $M^* < \frac{1}{2}\gamma_0 r_0$ it follows by Corollary 2.1 that for every $k \in \mathbb{N}$ and $\nu \in \{+\,,\, -\}$ there exists at least one solution $(\lambda_k^\nu, y_k^\nu) \in J_k^* \times \mathcal{S}_k^\nu$ of problem (1.1)-(1.3), where $J_k^* = \left[\lambda_k - \frac{M^*}{r_0},\ \lambda_k + \frac{M^*}{r_0}\right]$. From the positive homogeneity of equation (1.1) we have that $\{(\lambda_k^\nu, ty_k^\nu) : t > 0\}$ are half-lines of solutions in $\{\lambda_k^\nu\} \times S_k^\nu$.

We show that for each $\nu \in \{+\,,\,-\}$ in each interval J_k^*, $k \in \mathbb{N}$, is contained exactly one half-eigenvalue of problem (1.1)-(1.3). Let $k \neq N_0, N_0+1$ and we have two solutions (λ, y) and (μ, v) of problem (3.1) such that $y \in S_k^\nu$ and $v \in S_k^\nu$. Then it is obvious that $\lambda, \mu \in J_k^*$. From the proof of [1, Theorem 1] it is clear that $\lambda_{N_0} < -\frac{d_1}{c_1} \leq \lambda_{N_0+1}$. Hence if $k < N_0$, then $\lambda, \mu < -\frac{d_1}{c_1}$, and if $k > N_0 + 1$, then $\lambda, \mu > -\frac{d_1}{c_1}$. Without loss of generality we may assume that the first zero of function $y(x)v(x)$ to occur in $(0, \pi]$ is a zero of y. Then as in the proof of Theorem 3.1 we can show that $\lambda \leq \mu$. On the other hand by Lemma 3.1 there must exist an interval $[\zeta, \eta] \subset [0, \pi]$ such that $y(x)$ and $v(x)$ do not vanish and have the same sign in (ζ, η), and either $v(\zeta) = v(\eta) = 0$, or $v(\zeta) = 0$ and $\eta = \pi$, or $\zeta = 0$ and $\xi = \eta = \pi$ (the latter occurring when $k = 1$). Consider again, for example, the case of $v(\zeta) = 0$ and $\eta = \pi$ (the remaining cases are treated in the same way). Then taking into account boundary condition (1.3) from (3.3) we obtain

$$-p(\zeta)y(\zeta)v'(\zeta) = (\lambda - \mu)\left\{\int_0^\xi r(x)y(x)\,v(x)\,dx + \frac{\sigma_1}{(c_1\lambda + d_1)(c_1\mu + d_1)}\,y(\pi)\,v(\pi)\right\}.$$

Since by condition (1.4) $\sigma_1 > 0$ and by above remark $c_1\lambda + d_1$ and $c_1\mu + d_1$ have same sign it follows from the last identity that $\lambda \geq \mu$; hence $\lambda = \mu$. Therefore, for each $\nu \in \{+\,,\,-\}$ in each interval J_k^* for $k \neq N_0, N_0+1$ the problem (1.1)-(1.3) possesses one simple half-eigenvalue of λ_k^ν.

Now in each interval $J_{N_0}^*$ and $J_{N_0+1}^*$ is contained at least one half-eigenvalue of problem (1.1)-(1.3). It follows from the above arguments that if (λ, y) and (μ, v) are solutions of boundary value problem (1.1)-(1.3) such that $\lambda < \mu$, $y \in S_{N_0}^\nu$ and $v \in S_{N_0}^\nu$, then $\lambda < -\frac{d_1}{c_1} < \mu$. Hence $\lambda \in J_{N_0}^*$ and $\mu \in J_{N_0+1}^*$. Thus $\lambda = \lambda_{N_0}^\nu$ and $\mu = \lambda_{N_0+1}^\nu$.

It follows by the condition $M^* < \frac{1}{2}\,\gamma_0 r_0$ that $J_k^* \cap J_{k'}^* = \emptyset$ for any different $k, k' \in \mathbb{N}$. Hence for each $\nu \in \{+\,,\,-\}$ the half-eigenvalues λ_k^ν, $k \in \mathbb{N}$ of problem (1.1)-(1.3) are simple and aside from the trivial solutions and the half-lines $\{(\lambda_k^\nu, ty_k^\nu) : t > 0\}$, there are no other solutions of this problem.

Since $\lambda_k^+, \lambda_k^- \in J_k^*$ and $J_k^* \cap J_{k'}^* = \emptyset$ for any different $k, k' \in \mathbb{N}$ it follows that $\lambda_k^\nu > \lambda_{k'}^{\nu'}$ for every $k > k'$ and for each $\nu, \nu' \in \{+\,,\,-\}$. The proof of this theorem is complete.

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