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On approximation theorems for two-dimensional Szasz operator

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Abstract. *In the paper, we study some properties of two-dimensional Szasz operator and we find certain moments of this operator. In particular, we proved point-wise convergence of the derivative of Szasz operator to the corresponding derivatives of two-variable function.*

Keywords. linear positive operator, two-variable function, Szasz operator, approximation theorem.

Mathematics Subject Classification (2010): 41A36, 41A65

1 Introduction

Approximation theory plays an important role in mathematical analysis and other branches of mathematics. The results of theory of approximation is generally related to positive linear operators, and deals with rate of convergence and order of approximation. Weierstrass was the first who gave an important theorem, namely, Weierstrass approximation theorem in this regard. The aim of this theorem is to minimize the maximum value of $|f(x) - P_n(x)|$ for the continuous functions $f(x)$ on $[a, b]$, where $P_n(x)$ is the polynomial of degree n. The proof of this theorem was considered very difficult until Bernstein gave an elegant and simple proof of it. Bernstein [5] defined the positive linear operators using binomial distribution in the following way

$$B_n(f;x) = \sum_{k=0}^{n} P_{n,k}(x) f\left(\frac{k}{n}\right), n = 1, 2, 3, ..., k = 0, 1, 2, 3, ... \tag{1.1}$$

where $P_{n,k}(x) = \binom{n}{k} (x)^k (1-x)^{n-k}$ and proved pointwise and uniform approximation in the space of continuous functions on $[0, 1]$. These operators provide the powerful tool for numerical analysis, computer added geometric design (CAGD) and solutions of differential equations. But these operators are not suitable for discontinuous functions. Later on,

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Kantorovich [11] generalized the Bernstein operators for integrable functions as

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt, k = 0,1,2,3,...,n = 1,2,3,...,$$

where $P_{n,k}(x) = \binom{n}{k}(x)^k(1-x)^{n-k}$, $0 \le x \le 1$. Szasz [18] introduced linear positive operators in the sense of exponential growth on non-negative semiaxis

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}, n = 1,2,... \tag{1.2}$$

where $f \in C[0,\infty)$. Several generalizations of this operator have been studied by different researchers ([1]-[4], [6], [15], [16]). A generalization of operator (1.2) was given by Stancu [17] depending on the parameters α and β such that $0 \le \alpha \le \beta$ on $[0,1]$. Many operators preserve the constant and linear functions, but these operators do not preserve x^2. King [12] introduced a method in order to preserve x^2 for the Bernstein operators. In [7]-[9], [13], [14] various variants of the sequences of linear positive operator and its particular case was considered and some approximation theorems in non-weighted and weighted spaces of continuous functions were proved.

2 Preliminaries and auxilary results

Consider the two-dimensional Szasz operator defined in [19] as

$$S_{n,m}(f;x,y) = e^{-nx}e^{-my}\sum_{k,l=0}^{\infty} f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!},\ n,m = 1,2,..., \tag{2.1}$$

where $f(t,\tau) \in C(R_+^2)$.

Lemma 2.1 *If the r-th derivative $f^{(r)}(x,y)$ exists and $f^{(r)}(x,y) = O\left((x+y)^k\right)$ as $x \to \infty, y \to \infty$, for some $k > 0$, and if $f^{(r)}(x,y)$ is continuous at the point $x = \xi,\ y = \eta$, then $S_{n,m}^{(r)}(x,y)$ approaches $f^{(r)}(x,y)$ at $x = \xi,\ y = \eta$.*

The proof of Lemma 2.1 is analogous to the proof of result in [20].

Lemma 2.2 *The following identity holds*

$$\sum_{k,l=0}^{\infty}(k-n)^2(l-m)^2\frac{n^k}{k!}\frac{m^l}{l!} = nme^{n+m}.$$

Proof.

$$\sum_{k,l=0}^{\infty}\left(k^2-2kn+n^2\right)\left(l^2-2lm+m^2\right)\frac{n^k}{k!}\frac{m^l}{l!}$$

$$= \sum_{k=0}^{\infty}\left(k^2\frac{n^k}{k!} - 2kn\frac{n^k}{k!} + n^2\frac{n^k}{k!}\right)\sum_{l=0}^{\infty}\left(l^2\frac{m^l}{l!} - 2lm\frac{m^l}{l!} + m^2\frac{m^l}{l!}\right)$$

$$= \sum_{k=0}^{\infty}\left(k\frac{n^k}{(k-1)!} - 2n\frac{n^k}{(k-1)!} + n^2\frac{n^k}{k!}\right)\sum_{l=0}^{\infty}\left(l\frac{m^l}{(l-1)!} - 2m\frac{m^l}{(l-1)!} + m^2\frac{m^l}{l!}\right)$$

$$= \left(\sum_{k=0}^{\infty}\left((k+1)\frac{n^{k+1}}{k!} - 2n\frac{n^{k+1}}{k!}\right) + n^2e^n\right)$$

$$\times\left(\sum_{l=0}^{\infty}\left((l+1)\frac{m^{l+1}}{l!} - 2m\frac{m^{l+1}}{l!}\right) + m^2e^m\right)$$

$$= \left(\sum_{k=0}^{\infty}\left(n(k+1)\frac{n^k}{k!} - 2n^2\frac{n^k}{k!}\right) + n^2e^n\right)$$

$$\times\left(\sum_{l=0}^{\infty}\left(m(l+1)\frac{m^l}{l!} - 2m^2\frac{m^l}{l!}\right) + m^2e^m\right)$$

$$= \left(\sum_{k=0}^{\infty}\left(nk\frac{n^k}{k!} + n\frac{n^k}{k!}\right) - 2n^2e^n + n^2e^n\right)$$

$$\times\left(\sum_{l=0}^{\infty}\left(ml\frac{m^l}{l!} + m\frac{m^l}{l!}\right) - 2m^2e^m + m^2e^m\right)$$

$$= \left(\sum_{k=0}^{\infty} n\frac{n^k}{(k-1)!} + ne^n - 2n^2e^n + n^2e^n\right)$$

$$\times\left(\sum_{l=0}^{\infty} m\frac{m^l}{(l-1)!} + me^m - 2m^2e^m + m^2e^m\right)$$

$$= \left(\sum_{k=0}^{\infty} n^2\frac{n^k}{k!} + ne^n - 2n^2e^n + n^2e^n\right)$$

$$\times\left(\sum_{l=0}^{\infty} m^2\frac{m^l}{l!} + me^m - 2m^2e^m + m^2e^m\right)$$

$$= \left(n^2e^n + ne^n - 2n^2e^n + n^2e^n\right)\left(m^2e^m + me^m - 2m^2e^m + m^2e^m\right) = nme^ne^m.$$

The proof of Lemma 2.2 is complete.

Lemma 2.3 *For* $0 < \delta < 1$

$$\sum_{|k-n|>n\delta} e^{-n}\frac{n^k}{k!} = O\left(e^{-\frac{\delta^2 n}{3}}\right), \quad n\to\infty,$$

$$\sum_{|l-m\eta|>m\delta} e^{-m}\frac{m^l}{l!} = O\left(e^{-\frac{\delta^2 m}{3}}\right), \quad m\to\infty.$$

Let $u_m = u_m(x) = e^{-x}\frac{x^m}{m!}$, $(m = 0,1,2,...)$, *so that* $\sum\limits_{m=0}^{\infty} u_m(x) = 1$.

If $m = M + h$ *and* $0 < \delta < 1$ *, then*

$$\sum_{|h|>\delta x} u_m = O\left(e^{-\frac{1}{3}\delta^2 x}\right), h = m - M.$$

Proof. It is obvious that $\frac{u_{m+1}}{u_m} = \frac{x}{m+1}$.

Next, we divide $\sum u_m$ or $\sum u_{M+h}$ into the five pieces:

$$\sum_{m=0}^{\infty} u_m = \sum_{h=-M} u_{M+h} = \sum_{-M\le h<-\delta x} u_{m+h} + \sum_{-\delta x\le h<-x^{\xi}} u_{m+h}$$

$$+ \sum_{|h|\le x^{\xi}} u_{m+h} + \sum_{x^{\xi}<h\le\delta x} u_{m+h} + \sum_{h>\delta x} u_{m+h}$$

$$= \sum_{m=0}^{M_1-1} u_m + \sum_{m=M_1}^{M_2-1} u_m + \sum_{m=M_2}^{M_3} u_m + \sum_{m=M_3+1}^{M_4} u_m$$

$$+ \sum_{m=M_4+1}^{\infty} u_m = S_1 + S_2 + S_3 + S_4 + S_5.$$

(x being large enough to make $x^{\xi} < \delta x$), where

$$M_1 = [x] - [\delta x]\,, M_2 = [x] - \left[x^{\xi}\right], M_3 = [x] + \left[x^{\xi}\right], M_4 = [x] + [\delta x]\,.$$

It is easy to see that

$$S_1 = O\left(xu_{M_1}\right), S_2 = O\left(xu_{M_2}\right), S_4 = O\left(xu_{M_3}\right).$$

Also $M_4 + 2 > x + \delta x$, and so

$$S_5 = u_{M_4+1}\left\{1 + \frac{x}{M_4+2} + \frac{x^2}{(M_4+2)(M_4+3)} + ...\right\}$$

$$< u_{M_4}\left(1 + \frac{1}{1+\delta} + \frac{1}{(1+\delta)^2} + ...\right) = O\left(u_{M_4}\right).$$

Hence, in order to prove Lemma 2.3, it is enough to prove that

$$u_{M_1} = O\left(e^{-\frac{\delta^2 x}{3}}\right), u_{M_4} = O\left(e^{-\frac{\delta^2 x}{3}}\right).$$

Now

$$u_{M_1} = e^{-x}\frac{x^{M_1}}{M_1!} < e^{-x+M_1}\left(\frac{x}{M_1}\right)^{M_1}$$

and

$$x - \delta x - 1 < M_1 = [x] - [\delta x] < x - \delta x + 1.$$

Hence

$$u_{M_1} = O\left(e^{-\delta x}\left(\frac{x}{x-\delta x-1}\right)^{x-\delta x+1}\right)$$

$$= O\left(e^{-\delta x}\left(\frac{1}{1-\delta}\right)^{x-\delta x}\right) = O\left(e^{-\Delta x}\right),$$

where

$$\Delta = \delta - (1-\delta)\ln\frac{1}{1-\delta} = \frac{\delta^2}{1\cdot 2} + \frac{\delta^3}{2\cdot 3} + \frac{\delta^4}{3\cdot 4} + ... > \frac{\delta^2}{2}.$$

Similarly

$$u_{M_4} = O\left(e^{\delta x}\left(\frac{1}{1+\delta}\right)^{x+\delta x}\right) = O\left(e^{-\Delta' x}\right),$$

where

$$\Delta' = -\delta + (1+\delta)\ln(1+\delta) = \frac{\delta^2}{1\cdot 2} - \frac{\delta^3}{2\cdot 3} + \frac{\delta^4}{3\cdot 4} - ... > \frac{\delta^2}{3}.$$

This completes the proof of Lemma 2.3.

Remark 2.1 We note that for Lemma 2.3 was proved in [10]. We need some application of Lemma 2.3.

3 Main results

Now we formulate the following theorem.

Theorem 3.1 *Let* $f(x,y)$ *is bounded in the interval* $0 \le x \le R_1$, $0 \le y \le R_2$ *for every* $R > 0$, $f(x,y) = O\left((x+y)^u\right), x \to \infty,\ y \to \infty,$ *for some* $u > 0$, *and let at every point* ξ, η *the derivatives of second order is exists. Then*
a) $\lim\limits_{n\to\infty} \frac{\partial}{\partial x} S_{n,m}(f;x,y)\left|_{(x,y)=(\xi,\eta)}\right. = \frac{\partial}{\partial x} f(x,y)\left|_{(x,y)=(\xi,\eta)}\right.$,
b) $\lim\limits_{m\to\infty} \frac{\partial}{\partial y} S_{n,m}(f;x,y)\left|_{(x,y)=(\xi,\eta)}\right. = \frac{\partial}{\partial y} f(x,y)\left|_{(x,y)=(\xi,\eta)}\right.$,
c) $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} \frac{\partial^2}{\partial x \partial y} S_{n,m}(f;x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y)_{(x,y)=(\xi,\eta)}$.

Proof. We must prove that

$a)$ $\frac{\partial}{\partial x} S_{n,m}(f;x,y) \to f'_x(\xi, y),\ n \to \infty$, for fixed $y \in [0,\infty)$;

$b)$ $\frac{\partial}{\partial y} S_{n,m}(f;x,y) \to f'_y(x, \eta),\ m \to \infty$, for fixed $x \in [0,\infty)$;

$c)$ $\frac{\partial^2}{\partial x \partial y} S_{n,m}(f;x,y) \to \frac{\partial^2 f}{\partial x \partial y}\left|_{(x,y)=(\xi,\eta)}\right.,\ n,m \to \infty.$

By (2.1), we find the partial derivative of operator $S_{n,m}(f;x,y)$ with respect to the variable x :

$$\frac{\partial}{\partial x} S_{n,m}(f;x,y) = -ne^{-nx}e^{-my}\sum_{k,l=0}^{\infty} f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+ne^{-nx}e^{-my}\sum_{k,l=0}^{\infty} f\left(\frac{k}{n},\frac{l}{m}\right)k\frac{(nx)^{k-1}}{k!}\frac{(my)^l}{l!}$$

$$= -ne^{-nx}e^{-my}\sum_{k,l=0}^{\infty} f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+ne^{-nx}e^{-my}\sum_{k,l=0}^{\infty} f\left(\frac{k+1}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$= ne^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l}{m}\right) - f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}.$$

Now we find the mixed derivative of operator $S_{n,m}(f;x,y)$ with respect to the variable y :

$$\frac{\partial^2}{\partial x\partial y}S_{n,m}(f;x,y)$$

$$=-nme^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+ne^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}ml\frac{(my)^{l-1}}{l!}$$

$$=-nme^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+nme^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l+1}{m}\right)-f\left(\frac{k}{n},\frac{l+1}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$=mne^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l+1}{m}\right)-f\left(\frac{k}{n},\frac{l+1}{m}\right)\right.$$

$$\left.-f\left(\frac{k+1}{n},\frac{l}{m}\right)+f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}.$$

Similarly, we get that

$$\frac{\partial}{\partial y}S_{n,m}(f;x,y)=-me^{-nx}e^{-my}\sum_{k,l=0}^{\infty}f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+me^{-nx}e^{-my}\sum_{k,l=0}^{\infty}f\left(\frac{k}{n},\frac{l}{m}\right)l\frac{(nx)^k}{k!}\frac{(my)^{l-1}}{l!}$$

$$=-me^{-nx}e^{-my}\sum_{k,l=0}^{\infty}f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+me^{-nx}e^{-my}\sum_{k,l=0}^{\infty}f\left(\frac{k}{n},\frac{l+1}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$=me^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k}{n},\frac{l+1}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!},$$

and

$$\frac{\partial^2}{\partial y\partial x}S_{n,m}(f;x,y)$$

$$=-nme^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k}{n},\frac{l+1}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$+me^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k}{n},\frac{l+1}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]kn\frac{(nx)^{k-1}}{k!}\frac{(my)^l}{l!}$$

$$= -nme^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k}{n},\frac{l+1}{m}\right)-f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}+$$

$$+mne^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l+1}{m}\right)-f\left(\frac{k+1}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}$$

$$= mne^{-nx}e^{-my}\sum_{k,l=0}^{\infty}\left[f\left(\frac{k+1}{n},\frac{l+1}{m}\right)-f\left(\frac{k+1}{n},\frac{l}{m}\right)\right.$$

$$\left.-f\left(\frac{k}{n},\frac{l+1}{m}\right)+f\left(\frac{k}{n},\frac{l}{m}\right)\right]\frac{(nx)^k}{k!}\frac{(my)^l}{l!}. \tag{3.1}$$

Thus

$$\frac{\partial^2}{\partial x\partial y}S_{n,m}(f;x,y)=\frac{\partial^2}{\partial y\partial x}S_{n,m}(f;x,y).$$

Since at the point $x=\xi$, $y=\eta$ the function f have the partial derivative, then

$$f(x,y)=f(\xi,y)+(x-\xi)\,f_x^{/}(\xi,y)+\alpha(x,y)\,(x-\xi)\,,\ \alpha(x,y)\to 0,\ x\to\xi,$$

$$f(x,y)=f(x,\eta)+(y-\eta)\,f_y^{/}(x,\eta)+\beta(x,y)\,(y-\eta)\,,\ \beta(x,y)\to 0,\ y\to\eta.$$

In particular, for $k=0$, $l=0$, $x=0$, $y=0$, from (3.1) we get

$$mn\left[f\left(\frac{1}{n},\frac{1}{m}\right)-f\left(\frac{1}{n},0\right)-\left(f\left(0,\frac{1}{m}\right)-f(0,0)\right)\right]$$

$$=m\left[\frac{f\left(\frac{1}{n},\frac{1}{m}\right)-f\left(0,\frac{1}{m}\right)}{\frac{1}{n}}-\frac{f\left(\frac{1}{n},0\right)-f(0,0)}{\frac{1}{n}}\right]$$

$$\overset{n\to\infty}{\to} m\left[f_x^{/}\left(0,\frac{1}{m}\right)-f_x^{/}(0,0)\right]\overset{m\to\infty}{\to}\frac{f_x^{/}\left(0,\frac{1}{m}\right)-f_x^{/}(0,0)}{\frac{1}{m}}\to f_{xy}^{//}(0,0)\,,$$

$$\lim_{n\to\infty}\frac{\partial}{\partial x}S_{n,m}(f;x,y)\to\frac{\partial}{\partial x}f(x,y)\,|_{x=\xi}\ ,$$

$$\lim_{n\to\infty}\frac{\partial}{\partial y}S_{n,m}(f;x,y)\to\frac{\partial}{\partial y}f(x,y)\,|_{y=\eta}\ ,$$

$$(\xi,0)\,,\ \ (0,\eta)\,,\ \ (\xi,\eta)\,,\ \xi,\eta\neq 0.$$

After some standard calculations, we have

$$\frac{\partial^2}{\partial x\partial y}S_{n,m}(f;x,y)=mne^{-nx}e^{-my}$$

$$\times\sum_{k,l=0}^{\infty}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left[\frac{1}{n}f_x^{/}\left(\xi_{k,n},\frac{l+1}{m}\right)-\frac{1}{n}f_x^{/}\left(\eta_{k,n},\frac{l}{m}\right)\right],$$

$$f\left(\frac{k+1}{n},\frac{l+1}{m}\right)=f\left(\frac{k}{n},\frac{l+1}{m}\right)+f_x^{/}\left(\xi,\frac{l+1}{m}\right)\cdot\frac{1}{n}+\varepsilon_{k,n}\cdot\frac{1}{n},$$

$$f\left(x,\frac{l+1}{m}\right)=f\left(\xi,\frac{l+1}{m}\right)+(x-\xi)\,f_x^{/}\left(\xi,\frac{l+1}{m}\right)+\varepsilon\left(x\right)(x-\xi)\,,$$

$$f\left(x,\frac{l}{m}\right)=f\left(\xi,\frac{l}{m}\right)+(x-\xi)\,f_x^{/}\left(\xi,\frac{l}{m}\right)+\theta\left(x\right)(x-\xi)\,,$$

$$P\left(x,y\right)=f\left(\xi,y\right)+(x-\xi)\,f_x^{/}\left(\xi,y\right),\ g\left(x,y\right)=\alpha\left(x,y\right)(x-\xi)\,,$$

$$\frac{\partial}{\partial x}S_{n,m}(f;x,y)=\frac{\partial}{\partial x}S_{n,m}(P;x,y)+\frac{\partial}{\partial x}S_{n,m}(g;x,y),$$

$$\frac{\partial}{\partial\xi}S_{n,m}(g;\xi,y)=\frac{e^{-n\xi}}{n\xi}\frac{e^{-m\eta}}{m\eta}\sum_{k,l=0}^{\infty}\alpha_{k,l}\left(n,m\right)(k-n\xi)^2\left(l-m\eta\right)^2\frac{(n\xi)^k}{k!}\frac{(m\eta)^l}{l!},$$

where $\alpha_{k,l}\left(n,m\right)<\gamma(\delta)$, for $\left(\frac{k}{n}-\xi\right)^2+\left(\frac{l}{m}-\eta\right)^2\leq\delta^2$.

We write

$$\sum_{|k-n\xi|\leq n\delta}(k-n\xi)^2\frac{(n\xi)^k}{k!}\sum_{|l-m\eta|\leq m\delta}\alpha_{k,l}\left(n,m\right)(l-m\eta)^2\frac{(m\eta)^l}{l!}$$
$$+\sum_{|k-n\xi|>n\delta}(k-n\xi)^2\frac{(n\xi)^k}{k!}\sum_{|l-m\eta|>m\delta}\alpha_{k,l}\left(n,m\right)(l-m\eta)^2\frac{(m\eta)^l}{l!}$$
$$+\sum_{|k-n\xi|>n\delta}(k-n\xi)^2\frac{(n\xi)^k}{k!}\sum_{|l-m\eta|\leq m\delta}\alpha_{k,l}\left(n,m\right)(l-m\eta)^2\frac{(m\eta)^l}{l!}$$
$$+\sum_{|k-n\xi|\leq n\delta}(k-n\xi)^2\frac{(n\xi)^k}{k!}\sum_{|l-m\eta|>m\delta}\alpha_{k,l}\left(n,m\right)(l-m\eta)^2\frac{(m\eta)^l}{l!}$$
$$=A_1+A_2+A_3+A_4.$$

Obviously,

$$\frac{e^{-n\xi}}{n\xi}\frac{e^{-m\eta}}{m\eta}\left|A_1\right|<\gamma(\delta)\frac{e^{-n\xi}}{n\xi}e^{n\xi}n\xi\frac{e^{-m\eta}}{m\eta}e^{m\eta}m\eta=\gamma(\delta).$$

Denoting

$$\sup_{\substack{x\leq\xi\\ y\leq\eta}}\left|f(x,y)\right|=M(\xi,\eta)$$

and taking into account that

$$\alpha_{k,l}\left(n,m\right)(k-n\xi)^2\left(l-m\eta\right)^2=nm\left(k-n\xi\right)\left(l-m\eta\right)\left[f\left(\frac{k}{n},\frac{l}{m}\right)-f(\xi,\eta)\right]$$
$$-\left(k-n\xi\right)^2\left(l-m\eta\right)^2f^{//}(\xi,\eta),$$

we obtain

$$\left|A_2\right|\leq\left[2\xi\eta M(\xi,\eta)+\xi^2\eta^2\left|f^{//}(\xi,\eta)\right|\right]\sum_{|k-n\xi|>n\delta}\sum_{|l-m\eta|>m\delta}n^2m^2\frac{(n\xi)^k}{k!}\frac{(m\eta)^l}{l!}.$$

From Lemma 2.3, we conclude that

$$\sum_{|k-n\xi|>n\delta} e^{-n}\frac{n^k}{k!} = O\left(e^{-\frac{\delta^2 n}{3}}\right),$$

$$\sum_{|l-m\eta|>m\delta} e^{-m}\frac{m^l}{l!} = O\left(e^{-\frac{\delta^2 m}{3}}\right).$$

Since

$$f(x,y) = O\left((x+y)^u\right),\ x,y\to\infty,$$

$$\alpha_{k,l}(n,m)(k-n\xi)^2(l-m\eta)^2 = O\left(\left(\frac{k^{u+1}}{n^{u-1}}\frac{l^{u+1}}{m^{u-1}}\right)\right),$$

$$|A_3| \le \gamma(\delta_1) O\left(\frac{k^{u+1}}{n^{u-1}}\right).$$

In the same way we can prove that

$$|A_4| \le \gamma(\delta_2) O\left(\frac{l^{u+1}}{m^{u-1}}\right).$$

This completes the proof of Theorem 3.1.

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