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A note of the fractional integral operators in generalized Morrey spaces on the Heisenberg group

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Abstract. *We shall give a characterization for the strong and weak type boundedness of the fractional integral operator I_α on Heisenberg group $\mathbb{H}_n$ in the generalized Morrey spaces $M_{p,\varphi}(\mathbb{H}_n)$.*

Keywords. Heisenberg group, fractional integral operator, generalized Morrey space.

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1 Introduction

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. We state some basic results about Heisenberg group. More detailed information can be found in [1–3] and the references therein. Let $\mathbb{H}_n$ be the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication

$$(z,t)\cdot(w,s) = (z+w, t+s+2Im(z\cdot\bar{w})),$$

where $z\cdot\bar{w} = \sum_{j=1}^{n} z_j\bar{w}_j$. The inverse element of $u=(z,t)$ is $u^{-1} = (-z,-t)$ and we write the identity of $\mathbb{H}_n$ as $0=(0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r>0$, by $\delta_r(z,t) = (rz, r^2t)$. These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q=2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|(z,t)| = (|z|^2+|t|)^{1/2}.$$

With this norm, we define the Heisenberg ball centered at $u=(z,t)$ with radius r by $B(u,r) = \{v\in\mathbb{H}_n : |u^{-1}v|<r\}$, and we denote by $B_r = B(0,r) = \{v\in\mathbb{H}_n : |v|<r\}$

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the open ball centered at 0, the identity element of $\mathbb{H}_n$, with radius r. The volume of the ball $B(u,r)$ is $C_Q r^Q$, where C_Q is the volume of the unit ball B_1.

Using coordinates $u = (z,t) = (x+iy,t)$ for points in $\mathbb{H}_n$, the left-invariant vector fields X_j, Y_j and T on $\mathbb{H}_n$ equal to $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ \ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ \ T = \frac{\partial}{\partial t},$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}_n$ with commutation relations

$$[Y_j, X_j] = 4T$$

for $j = 1, ..., n$, and all other commutators equal to 0.

Let $f \in L_1^{\rm loc}(\mathbb{H}_n)$. The maximal operator M and the fractional integral operator I_α are defined by

$$Mf(u) = \sup_{r>0} |B(u,r)|^{-1} \int_{B(u,r)} |f(v)| dV(v),$$

$$I_\alpha f(u) = \int_{\mathbb{H}_n} \frac{f(v) dV(v)}{|u^{-1}v|^{Q-\alpha}}, \qquad 0 < \alpha < Q,$$

where Q is the homogeneous dimension of the homogeneous Heisenberg group $\mathbb{H}_n$ and $|B(u,r)|$ is the Haar measure of the $\mathbb{H}_n$- ball $B(u,r)$.

The operators M and I_α play an important role in real and harmonic analysis and applications (see, for example [1] and [2]).

In the present work, we shall give a characterization for the Spanne type boundedness of the operator I_α on the generalized Morrey spaces, including weak versions.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [6]. The Morrey space in a Heisenberg group is defined as follows: for $1 \leq p \leq \infty$, $0 \leq \lambda \leq Q$, a function $f \in L_{p,\lambda}(\mathbb{H}_n)$ if $f \in L_p^{\rm loc}(\mathbb{H}_n)$ and

$$\|f\|_{L_{p,\lambda}} := \sup_{u \in \mathbb{H}_n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(u,r))} < \infty.$$

If $\lambda = 0$, then $L_{p,0}(\mathbb{H}_n) = L_p(\mathbb{H}_n)$; if $\lambda = Q$, then $L_{p,Q}(\mathbb{H}_n) = L_\infty(\mathbb{H}_n)$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{H}_n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{H}_n$.

We also denote by $WL_{p,\lambda}(\mathbb{H}_n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{H}_n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{H}_n)} = \sup_{u \in \mathbb{H}_n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(u,r))} < \infty,$$

where $WL_p(B(u,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(B(u,r))} = \sup_{t>0} t \left|\{y \in B(u,r) :\ |f(y)| > t\}\right|^{1/p}. \tag{2.1}$$

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1 *Let $1 \leq p < \infty$ and $\varphi(u,r)$ be a positive measurable function on $\mathbb{G} \times (0,\infty)$. The generalized Morrey space $M_{p,\varphi}(\mathbb{H}_n)$ is defined of all functions $f \in L_p^{loc}(\mathbb{H}_n)$ by the finite norm*

$$\|f\|_{M_{p,\varphi}} = \sup_{u\in\mathbb{H}_n, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} \|f\|_{L_p(B(u,r))}.$$

Also the weak generalized Morrey space $WM_{p,\varphi}(\mathbb{H}_n)$ is defined of all functions $f \in L_p^{loc}(\mathbb{H}_n)$ by the finite norm

$$\|f\|_{WM_{p,\varphi}} = \sup_{u\in\mathbb{H}_n, r>0} \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} \|f\|_{WL_p(B(u,r))}.$$

Lemma 2.1 *Let $\varphi(u,r)$ be a positive measurable function on $\mathbb{H}_n \times (0,\infty)$.*

(i) If

$$\sup_{t<r<\infty} \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} = \infty \quad \text{for some } t>0 \text{ and for all } u \in \mathbb{H}_n, \tag{2.2}$$

then $M_{p,\varphi}(\mathbb{H}_n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(u,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } u \in \mathbb{H}_n, \tag{2.3}$$

then $M_{p,\varphi}(\mathbb{H}_n) = \Theta$.

Proof. (i) Let (2.2) be satisfied and f be not equivalent to zero. Then $\sup_{u\in\mathbb{H}_n} \|f\|_{L_p(B(u,t))} > 0$, hence

$$\begin{aligned}\|f\|_{M_{p,\varphi}} &\geq \sup_{u\in\mathbb{H}_n} \sup_{t<r<\infty} \varphi(u,r)^{-1} r^{-\frac{Q}{p}} \|f\|_{L_p(B(u,r))} \\ &\geq \sup_{u\in\mathbb{H}_n} \|f\|_{L_p(B(u,t))} \sup_{t<r<\infty} \varphi(u,r)^{-1} r^{-\frac{Q}{p}}.\end{aligned}$$

Therefore $\|f\|_{M_{p,\varphi}} = \infty$.

(ii) Let $f \in M_{p,\varphi}(\mathbb{H}_n)$ and (2.3) be satisfied. Then there are two possibilities:

Case 1. $\sup_{0<r<t} \varphi(u,r)^{-1} = \infty$ for all $t > 0$.

Case 2. $\sup_{0<r<t} \varphi(u,r)^{-1} < \infty$ for some $t \in (0,\tau)$.

For Case 1, by Lebesgue differentiation theorem, for almost all $u \in \mathbb{H}_n$,

$$\lim_{r\to 0+} \frac{\|f\chi_{B(u,r)}\|_{L_p}}{\|\chi_{B(u,r)}\|_{L_p}} = |f(u)|. \tag{2.4}$$

We claim that $f(u) = 0$ for all those x. Indeed, fix u and assume $|f(u)| > 0$. Then by Lemma 2.2 and (2.4) there exists $t_0 > 0$ such that

$$r^{-\frac{Q}{p}} \|f\|_{L_p(B(u,r))} \geq 2^{-1} c_2^{\frac{1}{p}} |f(u)|$$

for all $0 < r \leq t_0$. Consequently,

$$\|f\|_{M_{p,\varphi}} \geq \sup_{0<r<t_0} \varphi(u,r)^{-1} r^{-\frac{Q}{p}} \|f\|_{L_p(B(u,r))} \geq 2^{-1} c_2^{\frac{1}{p}} |f(u)| \sup_{0<r<t_0} \varphi(u,r)^{-1}.$$

Hence $\|f\|_{M_{p,\varphi}} = \infty$, so $f \notin M_{p,\varphi}(\mathbb{H}_n)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{s<r<\tau} \varphi(u,r)^{-1} = \infty$, hence

$$\sup_{s<r<\infty} \varphi(u,r)^{-1} r^{-\frac{Q}{p}} \geq \sup_{s<r<\tau} \varphi(u,r)^{-1} r^{-\frac{Q}{p}} \geq \tau^{-\frac{Q}{p}} \sup_{s<r<\tau} \varphi(u,r)^{-1} = \infty,$$

which is the case in (i).

Remark 2.1 We denote by Ω_p the sets of all positive measurable functions φ on $\mathbb{H}_n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{u\in\mathbb{H}_n} \Big\| \frac{r^{-\frac{Q}{p}}}{\varphi(u,r)} \Big\|_{L_\infty(t,\infty)} < \infty, \quad \text{and} \quad \sup_{u\in\mathbb{H}_n} \Big\| \varphi(u,r)^{-1} \Big\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_p$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C>0$ such that

$$\varphi(r) \leq C\varphi(s) \qquad (\text{resp. } \varphi(r) \geq C\varphi(s)) \quad \text{for } r \leq s.$$

Let $1 \leq p < \infty$. Denote by $\mathcal{G}_p$ the the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{\frac{Q}{p}} \varphi(t) \in (0,\infty)$ is almost increasing.

Seemingly the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{n/p} \leq \rho(T)T^{n/p}$ for all $0 < t \leq T < \infty$ and that $M_{p,\phi}(\mathbb{H}_n) = M_{p,\rho}(\mathbb{H}_n)$.

By elementary calculations we have the following, which shows particularly that the spaces $M_{p,\varphi}(\mathbb{H}_n)$ and $WM_{p,\varphi}(\mathbb{H}_n)$ are not trivial.

Lemma 2.2 *Let $\varphi \in \mathcal{G}_p$, $1 \leq p < \infty$, $B_0 = B(u_0, r_0)$ and χ_{B_0} is the characteristic function of the ball B_0, then $\chi_{B_0} \in M_{p,\varphi}(\mathbb{H}_n)$. Moreover, there exists $C>0$ such that*

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{B_0}\|_{WM_{p,\varphi}} \leq \|\chi_{B_0}\|_{M_{p,\varphi}} \leq \frac{C}{\varphi(r_0)}.$$

Proof. Let $\varphi \in \mathcal{G}_p$, $1 \leq p < \infty$, $B_0 = B(u_0, r_0)$ denote an arbitrary ball in $\mathbb{H}_n$. It is easy to see that

$$\|\chi_{B_0}\|_{WM_{p,\varphi}} = \sup_{u\in\mathbb{H}_n, r>0} \frac{1}{\varphi(r)} \Big(\frac{|B(u,r)\cap B_0|}{|B(u,r)|} \Big)^{1/p} \geq \frac{1}{\varphi(r_0)} \Big(\frac{|B_0 \cap B_0|}{|B_0|} \Big)^{1/p} = \frac{1}{\varphi(r_0)}.$$

Now, if $r \leq r_0$, then $\varphi(r_0) \leq C\varphi(r)$ and

$$\frac{1}{\varphi(r)} \Big(\frac{|B(u,r)\cap B_0|}{|B(u,r)|} \Big)^{1/p} \leq \frac{1}{\varphi(r)} \leq \frac{C}{\varphi(r_0)}$$

for all $x \in \mathbb{H}_n$.

On the other hand, if $r_0 \leq r$, we have $\varphi(r_0) r_0^{Q/p} \leq C\varphi(r) r^{Q/p}$ for all $x \in \mathbb{H}_n$ and

$$\frac{1}{\varphi(r)} \Big(\frac{|B(u,r)\cap B_0|}{|B(u,r)|} \Big)^{1/p} = \frac{|B(u,r)\cap B_0|^{1/p}}{c_2^{1/p} \varphi(r) r^{Q/p}} \leq \frac{|B_0|^{1/p}}{c_2^{1/p} \varphi(r) r^{Q/p}} = \frac{r_0^{Q/p}}{\varphi(r) r^{Q/p}} \leq \frac{C}{\varphi(r_0)}$$

for all $x \in \mathbb{H}_n$. This completes the proof.

The following theorem was proved in [5].

Theorem 2.1 *Let* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfies the condition*

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(u,s)\, s^{\frac{Q}{p}} \le C\, \varphi_2(u,r), \tag{2.5}$$

where C *does not depend on* u *and* r. *Then for* $p > 1$, *the operator* M *is bounded from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{p,\varphi_2}(\mathbb{H}_n)$ *and for* $p = 1$, *the operator* M *is bounded from* $M_{1,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{1,\varphi_2}(\mathbb{H}_n)$.

3 Fractional integral operator in the spaces $M_{p,\varphi}(\mathbb{H}_n)$

Following theorem were proved in [4, Theorem 5.2].

Theorem 3.1 *Let* $1 \le p < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ *and the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_t^{\infty} \frac{\operatorname*{ess\,sup}_{r<s<\infty} \varphi_1(u,s) s^{\frac{Q}{p}}}{r^{\frac{Q}{q}}} \frac{dr}{r} \le C\, \varphi_2(u,t), \tag{3.1}$$

where C *does not depend on* u *and* r. *Then for* $p > 1$ *the operator* I_α *is bounded from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{q,\varphi_2}(\mathbb{H}_n)$ *and for* $p = 1$ *the operator* I_α *is bounded from* $M_{1,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{q,\varphi_2}(\mathbb{H}_n)$.

For proving our main result, we need the following estimate.

Lemma 3.1 *If* $B_0 := B(u_0, r_0)$, *then* $r_0^{\alpha} \le c_2 \big(2c_0\big)^{Q-\alpha} I_\alpha \chi_{B_0}(u)$ *for every* $x \in B_0$.

Proof. If $u, v \in B_0$, then $\left|u^{-1}v\right| \le c_0\big(\left|u^{-1}u_0\right| + \left|u_0^{-1}v\right|\big) < 2c_0 r_0$. Since $0 < \alpha < Q$, we get $r_0^{\alpha-Q} \le \big(2c_0\big)^{Q-\alpha}\left|u^{-1}v\right|^{\alpha-Q}$. Therefore

$$I_\alpha \chi_{B_0}(u) = \int_{\mathbb{H}_n} \chi_{B_0}(y)\left|u^{-1}v\right|^{\alpha-Q} dy = \int_{B_0} \left|u^{-1}v\right|^{\alpha-Q} dy \ge c_2\big(2c_0\big)^{Q-\alpha} r_0^{\alpha}.$$

Our main result is the following theorem.

Theorem 3.2 *Let* $0 < \alpha < Q$, $p, q \in [1, \infty)$, $\varphi_1 \in \Omega_p$ *and* $\varphi_2 \in \Omega_q$.

1. *If* $1 \le p < \frac{Q}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, *then the condition* (3.1) *is sufficient for the boundedness of* I_α *from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{q,\varphi_2}(\mathbb{H}_n)$. *Moreover, if* $1 < p < \frac{Q}{\alpha}$, *the condition* (3.1) *is sufficient for the boundedness of* I_α *from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{q,\varphi_2}(\mathbb{H}_n)$.
2. *If the function* $\varphi_1 \in \mathcal{G}_p$, *then the condition*

$$t^{\alpha}\varphi_1(t) \le C\varphi_2(t), \tag{3.2}$$

for all $t > 0$, *where* $C > 0$ *does not depend* t, *is necessary for the boundedness of* I_α *from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{q,\varphi_2}(\mathbb{H}_n)$ *and* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{q,\varphi_2}(\mathbb{H}_n)$.

3. *Let* $1 \le p < \frac{Q}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. *If* $\varphi_1 \in \mathcal{G}_p$ *satisfies the regularity condition*

$$\int_t^{\infty} r^{\alpha-1}\, \varphi_1(r) dr \le C t^{\alpha} \varphi_1(t), \tag{3.3}$$

for all $t > 0$, *where* $C > 0$ *does not depend* t, *then the condition* (3.2) *is necessary and sufficient for the boundedness of* I_α *from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{q,\varphi_2}(\mathbb{H}_n)$. *Moreover, if* $1 < p < \frac{Q}{\alpha}$, *then the condition* (3.2) *is necessary and sufficient for the boundedness of* I_α *from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{q,\varphi_2}(\mathbb{H}_n)$.

Proof. The first part of the theorem proved in Theorem 3.1.

We shall now prove the second part. Let $B_0 = B(u_0, t_0)$ and $u \in B_0$. By Lemma 3.1 we have $t_0^\alpha \leq C I_\alpha \chi_{B_0}(u)$. Therefore, by Lemma 2.2 and Lemma 3.1

$$t_0^\alpha \lesssim |B_0|^{-\frac{1}{p}} \|I_\alpha \chi_{B_0}\|_{L_q(B_0)} \lesssim \varphi_2(t_0) \|I_\alpha \chi_{B_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(t_0) \|\chi_{B_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)}$$

or

$$t_0^\alpha \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)} \text{ for all } t_0 > 0 \Longleftrightarrow t_0^\alpha \varphi_1(t_0) \lesssim \varphi_2(t_0) \text{ for all } t_0 > 0.$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem.

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