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Some extension and generalization of a polynomial inequality

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Abstract. *The objective of this article is to obtained some extension and generalization of inequalities analogous to maximum modulus principle [9] by considering the class of polynomial*

$$\left|p(Rz) - \alpha p(z) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} p(z)\right| \text{ for } R \geq 1,$$

and for all real or complex number α, β *with* $|\alpha| \leq 1, |\beta| \leq 1$.

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1 Introduction

Let $P_{n,s}$ is a class of polynomial of degree n with an $s-$fold zeros at origin, which is define as

$$P_{n,s} :\equiv \left\{ p(z);\ p(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu, a_{n-s} \neq 0 \right\}.$$

For $s = 0$, $P_{n,0} = P_n$, where P_n is a class of n^{th} degree polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$.

If $p \in P_n$, then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1.1}$$

and for $R \geq 1$

$$\max_{|z|=R>1} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

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Inequality (1.1) known as Bernstein's Inequality [5] and inequality (1.2) is a simple deduction from Maximum modulus principle [9]. The inequalities (1.1) and (1.2) are best possible and equality holds for the polynomial having all its zeros at the origin.

It was shown by Ankeny and Rivlin [1] that if $p \in P_n$ and $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=R>1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

Aziz and Dawood [3] obtained following result concerning the minimum modulus of polynomial $p(z)$ analogous to (1.2) on $|z| = 1$ by applying a restriction on $p \in P_n$. Basically, they proved that if $p \in P_n$ and having all its zeros in $|z| \leq 1$, then

$$\min_{|z|=R\geq 1} |p(z)| \geq R^n \min_{|z|=1} |p(z)|. \tag{1.4}$$

In the same article, Aziz and Dawood [3] obtained a refinement of inequality (1.3) by considering $m = \min_{|z|=1} |p(z)|$ and proved that if $p \in P_n$ and $p(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{2}\right) \min_{|z|=1} |p(z)| \tag{1.5}$$

The inequality (1.3) and (1.5) are best possible and equality holds for $p(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta| = 1$.

Aziz and Rather [4] consider the polynomial

$$\left| p(Rz) - \alpha p(z) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} p(z) \right|, \quad R \geq 1$$

and investigating the dependence of it on $\max_{|z|=1} |p(z)|$ for all real or complex number α, β with $|\alpha| \leq 1, |\beta| \leq 1$. As a generalization of inequality (1.2), they proved that for $R \geq 1$

$$\begin{aligned} &\left| p(Rz) - \alpha p(z) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} p(z) \right| \\ &\quad \leq \left| (R^n - \alpha) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} \right| \max_{|z|=1} |p(z)| \quad \text{for } |z| \geq 1 \end{aligned} \tag{1.6}$$

As a corresponding generalization of inequality (1.3), they [4] also proved that for $p \in P_n, p(z) \neq 0$ in $|z| < 1$

$$\begin{aligned} &\left| p(Rz) - \alpha p(z) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} p(z) \right| \\ &\quad \leq \frac{1}{2} \left[\left| (R^n - \alpha) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} \right| \right. \\ &\qquad \left. + \left| (1 - \alpha) + \beta \left\{ \left(\frac{R+1}{2}\right)^n - |\alpha| \right\} \right| \right] \max_{|z|=1} |p(z)| \text{ for } |z| \geq 1 \end{aligned} \tag{1.7}$$

where $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$.

Recently Dewan and Hans [7] proved the following generalization of inequality (1.4) due to Aziz and Dawood [3] by showing that if $p \in P_n$ and having all its zeros in $|z| \leq 1$, then for all real or complex α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$

$$\min_{|z|=1}\left|p(Rz) - \alpha p(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}p(z)\right| \geq \left|(R^n - \alpha) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| \min_{|z|=1}|p(z)|, \tag{1.8}$$

and on using it, Dewan and Hans [7] replaced inequality (1.7) by following improved inequality

$$\begin{aligned}\left|p(Rz) - \alpha p(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}p(z)\right| &\leq \frac{1}{2}\left[\left\{\left|(R^n - \alpha) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|\right.\right.\\ &\quad \left. + \left|(1-\alpha) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|\right\}M\\ &\quad - \left\{\left|(R^n - \alpha) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|\right.\\ &\quad \left.\left. - \left|(1-\alpha) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|\right\}m\right]\end{aligned} \tag{1.9}$$

for $|z| \geq 1$. Where $M = \max_{|z|=1}|p(z)|$ and $m = \min_{|z|=1}|p(z)|$.

In this paper, we obtained an extension of inequality (1.8) and (1.9) for the class of polynomial $p \in P_{n,s}$ of degree n and present some other extension and generalizations of above mention results and to obtained other related results.

2 Lemma

For the proof of our results, we require following lemmas.

Lemma 2.1 [2]*If $p \in P_n$ and $p(z) \neq 0, |z| < k, k \geq 1$, then for $0 < r \leq 1$*

$$\max_{|z|=r}|p(z)| \geq \left(\frac{r+k}{1+k}\right)^n \max_{|z|=1}|p(z)|. \tag{2.1}$$

Lemma 2.2 *If $p \in P_{n,s}$ and having $n-s$ zeros in $|z| \leq k, k \leq 1$, then for $R \geq 1$*

$$\max_{|z|=R}|p(z)| \geq R^s\left(\frac{R+k}{1+k}\right)^{n-s} \max_{|z|=1}|p(z)| \tag{2.2}$$

Proof. Let $p \in P_n$ and having all its zeros in $|z| \leq k, k \leq 1$ with $s-$fold zeros at origin $0 \leq s < n$. Define

$$q(z) = z^n\overline{p(1/\overline{z})}$$

Clearly $q(z)$ is of degree $n-s$ with $q(0) \neq 0$ and having no zeros in $|z| < (1/k)$, then from Lemma 2.1 for $0 < r \leq 1$

$$\max_{|z|=r}|q(z)| \geq \left(\frac{r+1/k}{1+1/k}\right)^{n-s} \max_{|z|=1}|q(z)|. \tag{2.3}$$

Equivalently

$$r^n \max_{|z|=1/r} |p(z)| \geq \left(\frac{r+1/k}{1+1/k} \right)^{n-s} \max_{|z|=1} |q(z)|. \tag{2.4}$$

By taking $1/r = R$, we have

$$\max_{|z|=R} |p(z)| \geq R^s \left(\frac{R+k}{1+k} \right)^{n-s} \max_{|z|=1} |p(z)|, \tag{2.5}$$

where $R \geq 1$. This prove Lemma 2.2.

Lemma 2.3 *If $f \in P_{n,s}$ is a polynomial of degree n and having $n-s$ zeros in $|z| \leq k, k \leq 1$ and $p(z)$ is a polynomial of degree at most n with an $s-$fold zeros at origin, $0 \leq s < n$, such as $|p(z)| \leq |f(z)|$ for $|z| = 1$, then for all α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k} p(z)| \leq |f(Rz) + \Lambda_{\alpha,\beta,s,k} f(z)|, \tag{2.6}$$

for $|z| \geq k$. Where

$$\Lambda_{\alpha,\beta,s,k} = \beta \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} - \alpha \tag{2.7}$$

Proof. For $|z| = 1$, $|p(z)| \leq |f(z)|$, then for any δ with $|\delta| > 1$ it follows from Rouche's Theorem that the polynomial $G(z) = p(z) - \delta f(z)$ has all its zeros in $|z| \leq k, k \leq 1$ with an $s-$fold zeros at origin. Using Lemma 2.2 for $G(z)$, we get for $|z| = 1$

$$|G(Rz)| \geq R^s \left(\frac{R+k}{1+k} \right)^{n-s} |G(z)|.$$

For any α with $|\alpha| \leq 1$, $|G(Rz) - \alpha G(z)| \geq |G(Rz)| - |\alpha||G(z)|$, therefore we have from above

$$|G(Rz) - \alpha G(z)| \geq \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} |G(z)|, \tag{2.8}$$

Since all the zeros of $G(Rz)$ lies in $|z| \leq (k/R) < k$ with s number of zeros lies on $z = 0$, then form Rouche's Theorem, all the zeros of $G(Rz) - \alpha G(z)$ lies in $|z| < k$ with an $s-$fold zeros at origin, where $|\alpha| \leq 1$. For any real or complex β with $|\beta| \leq 1$, applying Rouche's Theorem again, it follows from inequality (2.8) that all the zeros of the polynomial $S(z)$ lies in $|z| < k$ with an $s-$fold zeros at origin, where

$$S(z) = G(Rz) - \alpha G(z) + \beta \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} G(z), \tag{2.9}$$

i.e.

$$S(z) = p(Rz) + \Lambda_{\alpha,\beta,s,k} p(z) - \delta \left[f(Rz) + \Lambda_{\alpha,\beta,s,k} f(z) \right] \neq 0 \tag{2.10}$$

for $|z| \geq k$ and $R \geq 1$. Where $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7). For $|\delta| > 1$, this implies that

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k} p(z)| \leq |f(Rz) + \Lambda_{\alpha,\beta,s,k} f(z)| \tag{2.11}$$

for $|z| \geq k$. Where $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$. If inequality (2.11) is not true, then for any $z = z_0$ with $|z_0| \geq k$ such that for $R \geq 1$

$$|p(Rz_0) + \Lambda_{\alpha,\beta,s,k}\ p(z_0)| > |f(Rz_0) + \Lambda_{\alpha,\beta,s,k}\ f(z_0)| \tag{2.12}$$

Define

$$\delta = \frac{p(Rz_0) + \Lambda_{\alpha,\beta,s,k}\ p(z_0)}{f(Rz_0) + \Lambda_{\alpha,\beta,s,k}\ f(z_0)}.$$

Therefore form (2.12), $|\delta| > 1$ and with this choice of δ, we have from (2.10) $S(z_0) = 0$. Which contradict the fact that $S(z) \neq 0$ for all $|z| \geq k$. Hence inequality (2.6) is true for all $|z| \geq k$.

Lemma 2.4 *If $p \in P_{n,s}$ is a n^{th} degree polynomial and having $n - s$ zeros in $|z| \geq k \geq 1$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k}p(z)| \leq |q(Rz) + \Lambda_{\alpha,\beta,s,k}\ q(z)| \tag{2.13}$$

for $|z| \geq k$, where $q(z) = z^{n+s}\overline{p(1/\overline{z})}$ and $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7).

Proof. By the hypothesis that $p(z)$ has all its zeros in $|z| \geq k, k \geq 1$ except an $s-$fold zeros at origin, $0 \leq s < n$, then all the zeros of $q(z) = z^{n+s}\overline{p(1/\overline{z})}$ lies in $|z| < k$ with s number of zeros at origin and for $|z| = 1$, $|p(z)| = |q(z)|$. Therefore form Lemma 2.3, we get for $R \geq 1$

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k}p(z)| \leq |q(Rz) + \Lambda_{\alpha,\beta,s,k}q(z)|\,,$$

where $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq k$. This proves the Lemma 2.4.

3 Main Result

Theorem 3.1 *If $p \in P_{n,s}$ and having $n - s$ zeros in $|z| \leq k, k \leq 1$, then for every α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k}p(z)| \geq k^{-n}\,|R^n + \Lambda_{\alpha,\beta,s,k}|\,|z|^n m, \tag{3.1}$$

for $|z| \geq k$, where $m = \min_{|z|=k} |p(z)|$ and $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7). Equality in above holds for $p(z) = \lambda(z/k)^n, \lambda \geq 0$.

Proof. If $p(z)$ has a zeros on $|z| = k$, then inequality (3.1) is trivial. So, we suppose all the zeros of $p(z)$ lies in $|z| < k, k \leq 1$ with an $s-$fold zeros at origin, $0 \leq s < n$. If $m = \min_{|z|=k} |p(z)|$, then $0 < m \leq |p(z)|$ for $|z| = k$. For any λ with $|\lambda| < 1$, it follows from Rouche's Theorem that the polynomial $G(z) = p(z) - \lambda m(z/k)^n$ of degree n with s number of zeros at origin, has $n - s$ zeros in $|z| < k$. Therefore, applying Lemma 2.2 to the polynomial $G(z)$, we get for $R > 1$

$$|G(Rz)| \geq R^s \left(\frac{R+k}{1+k}\right)^{n-s} |G(z)|. \tag{3.2}$$

Hence for every α with $|\alpha| \leq 1$, we have for $R > 1$ and $|z| = 1$

$$\begin{aligned}|G(Rz) - \alpha G(z)| &\geq |G(Rz)| - |\alpha||G(z)| \\ &> \left\{R^s \left(\frac{R+k}{1+k}\right)^{n-s} - |\alpha|\right\} |G(z)| \end{aligned} \tag{3.3}$$

Now, from (3.2)

$$|G(e^{i\theta})| \le R^{-s}\left(\frac{1+k}{R+k}\right)^{n-s}|G(Re^{i\theta})| \tag{3.4}$$

for $R > 1, k \le 1$ and $0 \le \theta < 2\pi$. Since $R^{-s}\left(\frac{1+k}{R+k}\right)^{n-s} \le 1$ and $G(Re^{i\theta}) \neq 0$, therefore form (3.4)

$$|G(e^{i\theta})| \le |G(Re^{i\theta})|,$$

i.e. for $R > 1$ and $|z| = 1$

$$|G(z)| \le |G(Rz)|.$$

$G(Rz)$ has all its zeros in $|z| \le (k/R) < k$ except s number of zeros at origin, then by direct application of Rouche's Theorem its shows that for any α with $|\alpha| \le 1$, $G(Rz) - \alpha G(z)$ has all its zeros in $|z| < k$ except an $s-$fold zeros at origin. Applying Rouche's Theorem again, it follows from (3.3) that for every β with $|\beta| \le 1$ and $R > 1$ all the zeros of the polynomial $T(z)$ lies in $|z| < k$ with an $s-$fold zeros at origin, where

$$T(z) = G(Rz) - \alpha G(z) + \beta\left\{R^s\left(\frac{R+k}{1+k}\right)^{n-s} - |\alpha|\right\}G(z), \tag{3.5}$$

i.e.

$$T(z) = [p(Rz) + \Lambda_{\alpha,\beta,s,k}\, p(z)] - \lambda k^{-n}\left[R^n + \Lambda_{\alpha,\beta,s,k}\right] mz^n \neq 0 \tag{3.6}$$

for $|z| \ge k$ and $R > 1$. Since $|\lambda| < 1$, therefore (3.6) implies for every α, β with $|\alpha| \le 1, |\beta| \le 1$,

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k}\, p(z)| \ge k^{-n}\left|R^n + \Lambda_{\alpha,\beta,s,k}\right||z|^n m, \tag{3.7}$$

for $|z| \ge k$ and $R > 1$. This complete the proof of Theorem 3.1.

Remark 3.1 If $s = 0$ and $k = 1$, then Theorem 3.1 reduces to inequality (1.8) due to Dewan and Hans [7] and by taking $\beta = 0$ and $k = 1$, we get a result due to Aziz and Rather [4].

We get following extension of inequality (1.8) due to Dewan and Hans [7] by taking $k = 1$ in Theorem 3.1.

Corollary 3.1 *If $p \in P_{n,s}$ and having $n - s$ zeros in $|z| \le 1$, then for every α, β with $|\alpha| \le 1, |\beta| \le 1$ and $R \ge 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,1}\, p(z)| \ge \left|R^n + \Lambda_{\alpha,\beta,s,1}\right||z|^n \min_{|z|=1}|p(z)|, \text{ for } |z| \ge 1, \tag{3.8}$$

where $\Lambda_{\alpha,\beta,s,1} = \beta\left\{R^s\left(\frac{R+1}{2}\right)^{n-s} - |\alpha|\right\} - \alpha$. Equality in above holds for $p(z) = \lambda z^n, \lambda \ge 0$.

Corollary 3.2 *If $p \in P_{n,s}$ and having $n - s$ zeros in $|z| \le k, k \le 1$, then for every β with $|\beta| \le 1$ and $R \ge 1$*

$$|p(Rz) + \Lambda_{0,\beta,s,1} p(z)| \ge k^{-n}\left|R^n + \Lambda_{0,\beta,s,1}\right||z|^n m, \text{ for } |z| \ge k, \tag{3.9}$$

where $m = \min_{|z|=k}|p(z)|$ and $\Lambda_{0,\beta,s,1} = \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}$. Equality in above holds for $p(z) = \lambda(z/k)^n, \lambda > 0$.

The above Corollary 3.1.2 was also an application of the result due to Dewan and Hans [6], which has been obtained by taking $\alpha = 0$ and by making $\beta = 0$ in inequality (3.1), we have the following result.

Corollary 3.3 *If $p \in P_{n,s}$ and having $n-s$ zeros in $|z| \le k, k \le 1$, then for every α with $|\alpha| \le 1$*

$$k^n |p(Rz) - \alpha p(z)| \ge |R^n - \alpha| \min_{|z|=k} |p(z)|. \tag{3.10}$$

The result is best possible and equality holds for $p(z) = \lambda (z/k)^n, \lambda > 0$.

From the inequality (2.8), we have for $R \ge 1$

$$|p(Rz) - \alpha p(z)| \ge \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} |p(z)|.$$

Therefore, with suitable choice of argument of β, we get

$$\begin{aligned} &\left| p(Rz) - \alpha p(z) + \beta \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} p(z) \right| \\ &= |p(Rz) - \alpha p(z)| - |\beta| \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} |p(z)|. \end{aligned}$$

Now from Theorem 3.1, we obtained for $R \ge 1$

$$\begin{aligned} &|p(Rz) - \alpha p(z)| - |\beta| \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} |p(z)| \\ &= \left| p(Rz) - \alpha p(z) + \beta \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} p(z) \right| \\ &\ge k^{-n} \left| (R^n - \alpha) + \beta \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} \right| m \\ &\ge k^{-n} |R^n - \alpha| - |\beta| k^{-n} \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} m \end{aligned}$$

and by taking $\beta \to 1$, we have the following result.

Corollary 3.4 *If $p \in P_{n,s}$ and having $n-s$ zeros in $|z| \le k, k \le 1$, then for $R \ge 1$*

$$\begin{aligned} \max_{|z|=1} |p(Rz) - \alpha p(z)| \ge &\left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} M \\ &+ k^{-n} \left[|R^n - \alpha| - \left\{ R^s \left(\frac{R+k}{1+k} \right)^{n-s} - |\alpha| \right\} \right] m, \end{aligned} \tag{3.11}$$

where $M = \max_{|z|=1} |p(z)|$ and $m = \min_{|z|=k} |p(z)|$. The equality in above holds for $p(z) = \lambda z^n, \lambda \ge 0$.

Next, we prove the following extended generalization of inequality (1.6) on the polynomial $p \in P_{n,s}$.

Theorem 3.2 *If $p \in P_{n,s}$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k}\, p(z)| \leq |R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n \max_{|z|=1} |p(z)|, \tag{3.12}$$

for $|z| \geq k$. Where $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7). The result is sharp and equality in above holds for $p(z) = \lambda z^n, \lambda \neq 0$.

Proof. By taking $f(z) = Mz^n$, where $M = \max_{|z|=1} |p(z)|$ in Lemma 2.3, we have Theorem 3.2.

If we take $k = 1$ in inequality (3.12), an application of inequality (1.6) has been followed.

Corollary 3.5 *If $p \in P_{n,s}$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,1}\, p(z)| \leq |R^n + \Lambda_{\alpha,\beta,s,1}|\, |z|^n \max_{|z|=1} |p(z)|, \tag{3.13}$$

for $|z| \geq 1$. Where $\Lambda_{\alpha,\beta,s,1}$ is define in Corollary 3.1.1. The result is sharp and equality in above holds for $p(z) = \lambda z^n, \lambda \neq 0$.

By taking $\alpha = 1$, divide inequality (3.12) by $R - 1$ and letting $R \to 1$, we have

$$\left| p'(z) + \beta \frac{n+s}{2} p(z) \right| \leq \left| n + \beta \frac{n+s}{2} \right| |z|^n \max_{|z|=1} |p(z)| \text{ for } |z| \geq 1,$$

which is an extension of an inequality due to Jain [8, Lemma 2] on the polynomial $p \in P_{n,s}$.

Now we prove following result as an application of Lemma 2.4.

Theorem 3.3 *If $p \in P_{n,s}$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$\begin{aligned} &|p(Rz) + \Lambda_{\alpha,\beta,s,k} p(z)| + |q(Rz) + \Lambda_{\alpha,\beta,s,k} q(z)| \\ &\quad \leq [|R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n + |R^s + \Lambda_{\alpha,\beta,s,k}|\, |z|^s] \max_{|z|=1} |p(z)|, \end{aligned} \tag{3.14}$$

for $|z| \geq k$. Where $q(z) = z^{n+s}\overline{p(1/\overline{z})}$ and $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7).

Proof. Since $M = \max_{|z|=1} |p(z)|$, therefore $|p(z)| \leq M$ for $|z| = 1$. It follows from the Rouche's Theorem that for every real or complex δ with $|\delta| > 1$, the polynomial $G(z) = p(z) + \delta z^s M$ has all its zeros in $|z| \geq k, k \geq 1$ except an $s-$fold zeros at origin. Applying Lemma 2.3 for $G(z)$, we have for $|z| \geq k$ and $R > 1$

$$|G(Rz) + \Lambda_{\alpha,\beta,s,k} G(z)| \leq |H(Rz) + \Lambda_{\alpha,\beta,s,k} G(z)|, \tag{3.15}$$

where $|\alpha| \leq 1, |\beta| \leq 1$ and

$$\begin{aligned} H(z) &= z^{n+s}\overline{G(1/\overline{z})} = z^{n+s}(\overline{p(1/\overline{z})} + \delta z^{-s} M) \\ &= q(z) + \delta z^n M. \end{aligned}$$

Therefore inequality (3.15) become

$$\begin{aligned} &|[p(Rz) + \Lambda_{\alpha,\beta,s,k}\, p(z)] + \delta\, [R^s + \Lambda_{\alpha,\beta,s,k}]\, z^s M| \\ &\quad \leq \left| [q(Rz) + \Lambda_{\alpha,\beta,s,k}\, q(z)] + \overline{\delta}\, [R^n + \Lambda_{\alpha,\beta,s,k}]\, z^n M \right|. \end{aligned} \tag{3.16}$$

Choosing argument δ such that

$$\begin{aligned}&\left|[q(Rz)+\Lambda_{\alpha,\beta,s,k}q(z)]+\overline{\delta}\,[R^n+\Lambda_{\alpha,\beta,s,k}]\,z^nM\right|\\&\quad=|\delta|\,|R^n+\Lambda_{\alpha,\beta,s,k}|\,|z|^nM-|q(Rz)+\Lambda_{\alpha,\beta,s,k}q(z)|\end{aligned}$$

and by applying Theorem 3.2, we rewrite the inequality (3.16) as

$$\begin{aligned}&|p(Rz)+\Lambda_{\alpha,\beta,s,k}p(z)|-|\delta|\,|R^s+\Lambda_{\alpha,\beta,s,k}|\,|z|^sM\\&\leq|\delta|\,|R^n+\Lambda_{\alpha,\beta,s,k}|\,|z|^nM-|q(Rz)+\Lambda_{\alpha,\beta,s,k}q(z)|\end{aligned}$$

for $|z|\geq k, |\alpha|\leq 1, |\beta|\leq 1$ and $R>1$. Equivalently

$$\begin{aligned}&|p(Rz)+\Lambda_{\alpha,\beta,s,k}p(z)|+|q(Rz)+\Lambda_{\alpha,\beta,s,k}q(z)|\\&\quad\leq|\delta|\,[|R^n+\Lambda_{\alpha,\beta,s,k}|\,|z|^n+|R^s+\Lambda_{\alpha,\beta,s,k}|\,|z|^s]\,M.\end{aligned}$$

Now letting $\delta\to 1$, we get the desired inequality (3.14).

From Lemma 2.4, we get for $p\in P_{n,s}$, having $n-s$ zeros in $|z|\geq k\geq 1$ and $|\alpha|\leq 1, |\beta|\leq 1$

$$\begin{aligned}2\,|p(Rz)+\Lambda_{\alpha,\beta,s,k}p(z)|&\leq|p(Rz)+\Lambda_{\alpha,\beta,s,k}p(z)|\\&\quad+|q(Rz)+\Lambda_{\alpha,\beta,s,k}q(z)|\end{aligned}\tag{3.17}$$

for $|z|\geq k$ and $R\geq 1$. Now using inequality (3.14) in above inequality (3.17), we have following application of Theorem 3.3. Which was as sharp as Theorem 3.2.

Corollary 3.6 *If $p\in P_{n,s}$ is a n^{th} degree polynomial and having $n-s$ zeros in $|z|\geq k\geq 1$, then for every $|\alpha|\leq 1, |\beta|\leq 1$ and $R\geq 1$*

$$|p(Rz)+\Lambda_{\alpha,\beta,s,k}p(z)|\leq\frac{1}{2}\,[|R^n+\Lambda_{\alpha,\beta,s,k}|\,|z|^n+|R^s+\Lambda_{\alpha,\beta,s,k}|\,|z|^s]\,M,\tag{3.18}$$

for $|z|\geq k$. Where $M=\max_{|z|=1}|p(z)|$ and $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7). The result is best possible and equality in (3.18) holds for $p(z)=z^n+z^s$.

If we consider $\beta=0$ in inequality (3.14) and Corollary 3.3.1, then we get the following respective generalizations.

Corollary 3.7 *If $p\in P_{n,s}$, then for every α with $|\alpha|\leq 1$ and $R\geq 1$*

$$\begin{aligned}&|p(Rz)-\alpha p(z)|+|q(Rz)-\alpha q(z)|\\&\qquad\leq[|R^n-\alpha||z|^n+|R^s-\alpha||z|^s]\max_{|z|=1}|p(z)|\end{aligned}\tag{3.19}$$

for $|z|\geq 1$. Where $q(z)$ is define in Theorem 3.3.

Corollary 3.8 *If $p\in P_{n,s}$ and having $n-s$ zeros in $|z|\geq k\geq 1$, then for every $|\alpha|\leq 1, |\beta|\leq 1$ and $R\geq 1$*

$$|p(Rz)-\alpha p(z)|\leq\frac{1}{2}\,[|R^n-\alpha|\,|z|^n+|R^s-\alpha|\,|z|^s]\max_{|z|=1}|p(z)|,\tag{3.20}$$

for $|z|\geq k$. The result is best possible and equality in (3.20) holds for $p(z)=z^n+z^s$.

We have the following application of inequality (1.7) on the class of polynomial $p\in P_{n,s}$ by taking $k=1$ in Corollary 3.3.1.

Corollary 3.9 *If $p \in P_{n,s}$ is a n^{th} degree polynomial and having $n-s$ zeros in $|z| \geq 1$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,1}\, p(z)| \leq \frac{1}{2}\left[|R^n + \Lambda_{\alpha,\beta,s,1}|\, |z|^n + |R^s + \Lambda_{\alpha,\beta,s,1}|\, |z|^s\right] M, \quad (3.21)$$

for $|z| \geq 1$. Where $M = \max_{|z|=1} |p(z)|$ and $\Lambda_{\alpha,\beta,s,1}$ is define in Corollary 3.3.1. The result is best possible and equality in (3.21) holds for $p(z) = z^n + z^s$.

Remark 3.2 On taking $s = 0$ in Corollary 3.3.4, we have a result due Aziz and Rather [4]. We also have some auxiliary result by putting $\alpha = 1$ in Theorem 3.3 and inequality (3.21) respectively.

Finally, we prove an extension and generalization of inequality (1.9) by using Theorem 3.1 for the polynomial $p \in P_{n,s}$ of degree n. Basically, we prove that

Theorem 3.4 *If $p \in P_{n,s}$ and having $n-s$ zeros in $|z| \geq k, k \geq 1$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$\begin{aligned} &|p(Rz) + \Lambda_{\alpha,\beta,s,k} p(z)| \\ &\quad \leq \frac{1}{2}\left\{|R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n + |R^s + \Lambda_{\alpha,\beta,s,k}|\, |z|^s\right\} \max_{|z|=1} |p(z)| \\ &\quad -\frac{1}{2}\left\{|R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n - |R^s + \Lambda_{\alpha,\beta,s,k}|\, |z|^s\right\} \min_{|z|=k} |p(z)| \end{aligned} \quad (3.22)$$

for $|z| \geq k$. Where $\Lambda_{\alpha,\beta,s,k}$ is define in (2.7). The result is sharp and equality holds for $p(z) = z^n + z^s$.

Proof. For $R = 1$, there is nothing to prove. So, we assume $R > 1$. By assumption $p(z) \neq 0$ in $|z| < k, k \geq 1$ except an $s-$fold zeros at origin and $m = \min_{|z|=k} |p(z)|$. Since $m \leq |p(z)|$ for all $|z| = 1$, therefore for a given complex number δ with $|\delta| < 1$, it follows from Rouche's Theorem that the polynomial $F(z) = p(z) - \delta m z^s$ has no zeros in $|z| < k$ except an $s-$fold zeros at origin. Define

$$G(z) = z^{n+s}\overline{F(1/\overline{z})} = z^{n+s}\overline{p(1/\overline{z})} - \overline{\delta} m z^n = q(z) - \overline{\delta} m z^n,$$

then all the zeros of $G(z)$ lies in $|z| \leq k$ with an $s-$fold zeros at origin and for $|z| = 1, |G(z)| = |F(z)|$. Now using Lemma 2.4, we have for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| = 1$

$$|F(Rz) + \Lambda_{\alpha,\beta,s,k} F(z)| \leq |G(Rz) + \Lambda_{\alpha,\beta,s,k} G(z)|\, .$$

Wchich impiles that

$$\begin{aligned} &|p(Rz) + \Lambda_{\alpha,\beta,s,k} p(z) - \delta\left[R^s + \Lambda_{\alpha,\beta,s,k}\right] m z^s| \\ &\quad \leq \left|q(Rz)\Lambda_{\alpha,\beta,s,k}\, q(z) - \overline{\delta}\left[R^n + \Lambda_{\alpha,\beta,s,k}\right] m z^n\right|, \end{aligned} \quad (3.23)$$

for $|z| \geq k$.
Since $q(z)$ has all its $n-s$ zeros in $|z| \leq k$ and for $|z| = k,\ m = \min_{|z|=k} |p(z)| = k^{-n} \min_{|z|=k} |q(z)|$, therefore from Theorem 3.1, we have for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| = k$

$$\begin{aligned} |q(Rz) + \Lambda_{\alpha,\beta,s,k} q(z)| &\geq k^{-n}\, |R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n \min_{|z|=k} |q(z)| \\ &= |R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n m. \end{aligned} \quad (3.24)$$

Then with suitable choice of argument of δ in (3.23) and making $\delta \to 1$, we get

$$|p(Rz) + \Lambda_{\alpha,\beta,s,k}\, p(z)| - |R^s + \Lambda_{\alpha,\beta,s,k}|\, m|z|^s m \\ \leq |q(Rz) + \Lambda_{\alpha,\beta,s,k} q(z)| - |R^n + \Lambda_{\alpha,\beta,s,k}|\, |z|^n m, \tag{3.25}$$

for $|z| \geq k$. Now using inequality (3.14) of Theorem 3.3 in above inequality, Theorem 3.4 has been followed.

Several other interesting result has been followed from Theorem 3.4. First we have an extension of inequality (1.9) due to Dewan and Hans [7] by taking $k = 1$ in Theorem 3.4.

Corollary 3.10 *If $p \in P_{n,s}$ and having $n - s$ zeros in $|z| \geq 1$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{\alpha,\beta,s,1}\, p(z)| \leq \frac{1}{2}\Bigg[\{|R^n + \Lambda_{\alpha,\beta,s,1}| + |R^s + \Lambda_{\alpha,\beta,s,1}|\} \max_{|z|=1} |p(z)| \\ - \{|R^n + \Lambda_{\alpha,\beta,s,1}| - |R^s + \Lambda_{\alpha,\beta,s,1}|\} \min_{|z|=1} |p(z)|\Bigg] \tag{3.26}$$

for $|z| \geq 1$. Where $\Lambda_{\alpha,\beta,s,1}$ is define in Corollary 3.1.1. The result is sharp and equality holds for $p(z) = z^n + z^s$.

The inequality (1.6) has been extended and generalized in following manner by taking $\alpha = 0$ in inequality (3.22) of Theorem 3.4.

Corollary 3.11 *If $p \in P_{n,s}$ and having $n - s$ zeros in $|z| \geq k, k \geq 1$, then for every $|\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) + \Lambda_{0,\beta,s,1} p(z)| \leq \frac{1}{2}\left[\{|R^n + \Lambda_{0,\beta,s,1}|\, |z|^n + |R^s + \Lambda_{0,\beta,s,1}|\, |z|^s\} M \\ - \{|R^n + \Lambda_{0,\beta,s,1}|\, |z|^n - |R^s + \Lambda_{0,\beta,s,1}|\, |z|^s\} m\right] \tag{3.27}$$

for $|z| \geq k$. Where $M = \max_{|z|=1} |p(z)|, m = \min_{|z|=k} |p(z)|$ and $\Lambda_{0,\beta,s,1}$ is define in Corollary 3.2.1. Equality in (3.27) holds for $p(z) = z^n + z^s$.

If $\beta = 0$ in Theorem 3.4, we have following result, which is a extension of a result due to Dewan and Hans [7] on the class of polynomial $p(z) = z^s \left\{\sum_{\nu=0}^{n-s} a_\nu z^\nu\right\}, 0 \leq s < n$.

Corollary 3.12 *If $p \in P_{n,s}$ and having $n - s$ zeros in $|z| \geq k, k \geq 1$, then for every $|\alpha| \leq 1, |\beta| \leq 1$ and $R \geq 1$*

$$|p(Rz) - \alpha p(z)| \leq \frac{1}{2}\Bigg[\{|R^n - \alpha|\, |z|^n + |R^s - \alpha|\, |z|^s\} \max_{|z|=1} |p(z)| \\ - \{|R^n - \alpha|\, |z|^n - |R^s - \alpha|\, |z|^s\} \min_{|z|=k} |p(z)|\Bigg] \tag{3.28}$$

for $|z| \geq k$. The result is sharp and equality holds for $p(z) = z^n + z^s$.

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