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Strong law of large numbers for a class of first passage time in the first order autoregression $(AR\,(1))$ scheme

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Abstract. *In the paper we study strong law of large numbers for the family of the first passage time of random walk described by first order autoregression process* $(AR\,(1))$

Keywords. random walks, autoregression process of order one $(AR\,(1))$, first passage time, strong law of large numbers

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1 Introduction

Let $\xi_n;\ n \geq 1$ be a sequence of independent identically distributed random variables determined on some probability space (Ω, F, P). As is known ([1], [13]) the first order autoregression $(AR\,(1))$ scheme is determined by means of the recurrent relation of the form

$$X_n = \beta X_{n-1} + \xi_n, \quad n \geq 1.$$

for some fixed number $\beta \in R$. The initial value of X_0 is independent of the innovation ξ_n and we assume $X_0 = x \in R$.

Note that significant linear Markov models arising in applied fields of theory of random processes are described by the process $AR\,(1)$ ([4]-[7]).

Recently, there is great interest to study of boundary problems for the Markov chain. These problems are on the basis of theory of nonlinear renewal for Markov random walk.

Important boundary problems for Markov random walk described by the first order autoregression process were studied in the papers [8]-[12]. More exacity, in the papers [9], [10], [11] integral limit theorems were proved for the first passage time of the level by the processes

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$$T_n = \sum_{k=1}^{n} X_k X_{k-1},\ S_n = \sum_{k=1}^{n} X_{k-1}^2$$

and

$$\beta_n = \frac{T_n}{S_n},\ \ n \geq 1.$$

In the present note we consider a family of the first passage time

$$\tau_a = \inf\{n \geq 1 : U_n \geq a\} \tag{1.1}$$

of the level $a \geq 0$ by the process $U_n = \beta T_n - \frac{\beta^2}{2} S_n,\ \ n \geq 1$. Such families of the first passage time of the form (1.1) arise in statistical hypothesis test problems with respect to value of the parameter β ([1], [13]).

Note that the process $U_n,\ \ n \geq 1$ arises by studying boundary problems for the perturbed Markov random walk described by a nonlinear function of the process $AR(1)$ (see [1], [12]).

It is easy to be convinced that the process U_n as a function of β accepts the maximum value at the point $\beta_n^* = \frac{T_n}{S_n}$, that equals $U_{(n)}^{\max} = \frac{T_n^2}{2S_n}$. Note that a number of asymptotic properties of the processes $\beta_n^* = \frac{T_n}{S_n}$ and $U_{(n)}^{\max} = \frac{T_n^2}{2S_n}$ were studied in the papers ([1], [8]-[13]).

2 Formulation and proof of the basic result

At first we note the following known facts proved in the paper [1].

Subject to the conditions $|\beta| < 1,\ \ E\xi_1 = 0$ and $D\xi = 1$, there hold the following almost sure convergences

$$\frac{T_n}{n} \xrightarrow{a.s.} \frac{\beta}{1-\beta^2} = \lambda_2, \quad \text{as } n \to \infty \tag{2.1}$$

and

$$\frac{S_n}{n} \xrightarrow{a.s.} \frac{\beta}{1-\beta^2} = \lambda_2, \quad \text{as } n \to \infty. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\frac{U_n}{n} \xrightarrow{a.s.} \beta\lambda_1 - \frac{\beta^2}{2}\lambda_2$$

or

$$\frac{U_n}{n} \xrightarrow{a.s.} \frac{\beta^2}{2\left(1-\beta^2\right)} = \lambda_3, \quad \text{as } n \to \infty \tag{2.3}$$

It is easy to see that the family of the first passage time is of the form

$$\tau_a = \inf\left\{n \geq 1 : ng\left(\frac{T_n}{n}, \frac{S_n}{n}\right) \geq a\right\}, \tag{2.4}$$

where $g(x, y) = \beta x - \frac{\beta^2}{2} y$.

Note that such families of the stopping time (2.4) are the object of study when solving the problems of theory of Markov renewal ([1], [2]). It holds

Theorem 2.1 *Let the following conditions be fulfilled,*

$$0 < |\beta| < 1, \ \ E\xi_1 = 0 \ \textit{and} \ D\xi_1 = 1.$$

Then

$$\frac{\tau_a}{a} \xrightarrow{a.s.} \frac{1}{\lambda_3} \ \textit{as}\, \alpha \to \infty \tag{2.5}$$

Relation (2.5) *is called the strong law of large numbers for family the stopping times* (1.1).

Proof. From the convergence (2.3) it follows that

$$P\left(\sup_n U_n = \infty\right) = 1. \tag{2.6}$$

By definition of the first exit time τ_a we have

$$\{\tau_a \leq n\} = \left\{\sup_{1\leq k\leq n} U_k \geq a\right\}. \tag{2.7}$$

From (2.6) and (2.7) we get

$$P\left(\tau_a < \infty\right) = 1$$

for all $a \geq 0$.

The last equality shows that for each $a \geq 0$ the value τ_a is a random eigen value.

From definition of the value τ_a it is seen that τ_a as a function of a increases and there exists the limit

$$P\left(\tau_\infty = \lim_{a\to\infty} \tau_a = \infty\right) = 1$$

with probability one.

Indeed, for every $n \geq 1$ we have

$$P\left(\tau_\infty > n\right) = P\left(\lim_{a\to\infty} \tau_a > n\right) = \lim_{a\to\infty} P\left(t_a > n\right).$$

Hence, from (2.7) we find

$$P\left(\tau_\infty > n\right) = \lim_{a\to\infty} P\left(\sup_{1\leq k\leq n} Z_k < a\right) = 1$$

for all $n \geq 1$.

Consequently,

$$P\left(\tau_\infty = \infty\right) = 1. \tag{2.8}$$

This means that $\tau_a \xrightarrow{a.s.} \infty$ as $a \to \infty$.

In what follows, we introduce the following denotation

$$C = C\left(a\right) = \{\omega : \tau_a < \infty\};$$

$$A_n = A_n\left(a\right) = \{\omega : \tau_a = n\};$$

$$B = \left\{\omega : \frac{Z_n}{n} \to \lambda_3, \ n \to \infty\right\};$$

and

$$D = \left\{\omega : \frac{Z_{\tau_a}}{\tau_a} \to \lambda_3, \ a \to \infty\right\}.$$

It is clear that

$$C = \sum_{n=1}^{\infty} A_n$$

and from (2.8) we have $P(C) = 1$. It is easy to see that

$$P(D) = P(DC) = \sum_{n=1}^{\infty} P(DA_n) = \sum_{n=1}^{\infty} P(BA_n). \tag{2.9}$$

Relation (2.3) shows that $P(B) = 1$. Therefore we have $P(BA_n) = P(A_n)$. Then from equality (2.9) we get that

$$p(D) = \sum_{n=1}^{\infty} P(A_n) = P(C) = 1.$$

This equality means that

$$\frac{U_{\tau_a}}{\tau_a} \xrightarrow{a.s.} \lambda_3 \quad \text{as } a \to \infty. \tag{2.10}$$

From definition of the first passege time τ_a it follows that it holds the following bilateral inequality

$$\frac{U_{\tau_a - 1}}{\tau_a} < \frac{a}{\tau_a} \le \frac{Z_{\tau_a}}{\tau_a}. \tag{2.11}$$

The statement of the theorem follows from (2.10) and (2.11).

Remark 2.1 Note that the proof of the theorem shows that its statement remains valid for the initial value of the process X_0 satisfying the condition $EX_0^2 < \infty$ as the relations (2.1) and (2.2) are valid for the case $EX_0^2 < \infty$ as well (see [1], [13]).

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