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Solvability of equation of Prandtl-von Mises type, theorems of embedding

Kamal N. Soltanov · **Mahmud A. Ahmadov**

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Abstract. *In this article we study the mixed problem for the Prandtl-von Mises type equation. Here we proved the existence theorem and investigate the behavior of solutions of the posed problem. Furthermore, we investigate also of some metric classes of the functional spaces corresponded to this problem and their properties.*

Keywords. Prandtl-von Mises type equation, implicit degenerating, solvability, behavior of solutions, functions spaces, embedding theorems

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1 Introduction

Consider the following problem

$$\frac{\partial u}{\partial t} - |u|^{\rho}\, \Delta u + b_0\, |u|^{\mu+1} = h\,(t,x)\,, \quad (t,x) \in Q_T \equiv (0,T) \times \Omega, \tag{1.1}$$

$$u\,(0,x) = 0, \quad x \in \Omega \subset R^n, \quad n \geq 1, \tag{1.2}$$

$$u\,(t,x)\,|_{\,\Gamma} = 0, \quad \Gamma \equiv [0,T] \times \partial\Omega,\ T > 0, \tag{1.3}$$

here Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega$ (for example, $\partial\Omega \in C^1$), $\Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is a Laplacian, $\rho > 0, \mu \geq 0$, $b_0 \in R^1$ are numbers, $h\,(t,x)$ is a given function.

The equation (1.1) describes the behavior of a flow on a boundary layer, and this equation when $\rho = 1$ is also called Prandtl-von Mises type equation (see, [14,23,8]). The equation (1.1) was investigated in various conditions by Friedman and Mcleaod ([7]), Lukhaus and Dal Passo ([12]), Tsutsumı and Ishıwata ([22]), Walter ([23]), Wiegner ([24]), Winkler ([25]) etc. (see and references therein). In these works were studied the solvability of the considered problems and the behavior of their solutions under various conditions. It should

K.N. Soltanov
Department of Mathematics, Faculty of Sciences, Hacettepe University Beytepe, Ankara, TR-06800, TURKEY
E-mail: soltanov@hacettepe.edu.tr ; sultan_kamal@hotmail.com

M.A. Ahmadov
Department of Mathematics, Westfield State College, 577 Western Ave., Westfield, MA, 01085 , USA
E-mail: mahmadov@wsc.ma.edu

be noted that in these works the investigation of the problem was studied under complementary conditions: such as any possible solution is a positive function in the studied domain. But the solution of the problem, in general, is an unknown function which can be vanished on some subset of the studied domain. In works [16,17,20] unlike the mentioned works the solvability of the considered problem was investigated without additional conditions on a solution, but only in the one-dimensional case, and are obtained the functional spaces where the solutions belong [16,20] (see also references in [18]). Moreover in [16,17,20] were studied properties of these spaces and their relations with Sobolev spaces etc.

This we consider the problem (1.1) - (1.3) in n-dimensional case and prove the existence theorem for the problem (1.1) - (1.3), and also we study the behavior of its solution. Furthermore here we investigate the class of spaces procreated by the considered problem and study their properties. In particular, we obtain some smoothness results of solutions, which follow from the obtained here embedding theorems.

The investigation of the boundary-value problems often lead to study of the functional spaces related to the considered problems directly. More precisely, the mentioned spaces appear as domains of operators procreated by the studied boundary-value problems. It is possible that Sobolev spaces and their some generalizations appeared for such aims.

Unlike linear cases, in the nonlinear cases of the boundary-value problems sometimes the corresponding classes of the functions can be subsets of some linear spaces, which do not possess of the linear structures. Therefore it is need to investigate of these classes separately. This problem that we study here is a problem of such type. Consequently here in the beginning we investigate the appropriate classes of the functions.

So, after formulation of the problem we define the necessary function classes, then we state the existence theorem for the considered problem that is investigated in the section 2. Here for the proof of the existence theorem is used the general existence theorem for the evolution equation in Banach space that is proved in Ap. 6.2 (this theorem is similar to the theorem from [S3]). In the section 3, we investigate a metric classes of the functional spaces and prove the embedding and compactness theorems for them, which are nesessary for the proof of the main theorem 2.1. In the section 4 is proved the solvability theorem 2.1 for the considered problem, later in the section 5 is studied the behavior of the solutions of the problem (1.1)-(1.3). In the section 6: Ap. 6.1 we give some properties of the metric spaces (that is called as $pn-$spaces), and in Ap. 6.2 we prove the solvability theorem for the nonlinear operator equation then using this theorem, in Ap. 6.3 we prove the general theorem 2.2 given in the section 2.

2 Existence Theorem

Define the following function space:

$$\mathbf{P}_{1,p,q}\left(Q_T\right) \equiv W^{1,q}\left(0,T;L^q\left(\Omega\right)\right) \cap L^p\left(0,T;\overset{0}{S}_{\Delta,\rho,2}\left(\Omega\right)\right), \tag{2.1}$$

where $p,q \geq 1$, $\rho \geq 0$ are some numbers, $W^{1,q}\left(0,T;L^q\left(\Omega\right)\right)$ is Sobolev space, and $\overset{0}{S}_{\Delta,\rho,2}\left(\Omega\right)$ define as follows: for functions $u : \Omega \longrightarrow R^1$

$$\overset{0}{S}_{\Delta,\rho,2}\left(\Omega\right) \equiv \left\{ u \in L^1\left(\Omega\right) \left| [u]^{\rho+2}_{S_{\Delta,\rho,2}} \equiv \int_{\Omega} |u|^{\rho} \left|\Delta u\right|^2 dx < +\infty,\ u\left(x\right) \mid_{\partial\Omega} = 0 \right. \right\}. \tag{2.2}$$

Thus for functions $u : Q_T \longrightarrow R^1$ we get

$$L_p\left(0,T;\overset{0}{S}_{\Delta,\rho,2}(\Omega)\right) \equiv \left\{ u \in L^1(Q_T) \,\middle|\, [u]^p_{L\left(S_{\Delta,\rho,2}\right)} \equiv \int_0^T [u]^p_{S_{\Delta,\rho,2}}\, dt < +\infty, \right.$$

$$\left. u(t,x) \,\big|_{\,[0,T]\times\partial\Omega} = 0 \right\}. \tag{2.3}$$

The solution of the problem we will understand as follows:

Definition 2.1 *A function $u(t,x) \in \mathbf{P}_{1,p,q}(Q_T)$ is said to be the solution of the problem (1.1) - (1.3) if the following equation*

$$\int_0^T \int_\Omega \left(\frac{\partial u}{\partial t} - |u|^\rho \Delta u + b_0 |u|^{\mu+1} \right) v dx dt = \int_0^T \int_\Omega h v dx dt$$

holds for each function $v \in L^p(Q_T)$.

Our main result on solvability of the problem (1.1) - (1.3) is

Theorem 2.1 *Let us is fulfilled either of conditions: if $0 < \rho \leq 2$ then*

$$\min\left\{0, \frac{\rho}{2} - 1\right\} \leq \mu < \rho; \text{ if } 2 < \rho \text{ then } \frac{\rho}{2} - 1 \leq \mu < \rho,$$

and $b_0 \in R^1$. Then, for any $h \in L^2\left(0,T;W_0^{1,2}(\Omega)\right)$ the problem (1.1) - (1.3) is solvable in $\mathbf{P}(Q) \equiv \mathbf{P}_{1,p,q}(Q_T) \cap \{u(t,x) \mid u(0,x) = 0\}$, where $p = \rho + 2$, $q = p' = \frac{\rho+2}{\rho+1}$.

The proof is based on a general result given below (Theorem 2.2).

Let X and Y be Banach spaces with duals X^* and Y^* respectively, Y be a reflexive Banach space, $\mathcal{M}_0 \subseteq X$ be a weakly complete "reflexive" $pn-$space (see, Appendix A or [18,20]), $X_0 \subseteq \mathcal{M}_0 \cap Y$ be a separable vector topological space such that $\overline{X_0}^{\mathcal{M}_0} \equiv \mathcal{M}_0$, $\overline{X_0}^{Y} \equiv Y$. Consider the following problem:

$$\frac{dx}{dt} + f(t, x(t)) = y(t), \quad y \in L^{p_1}(0,T;Y); \quad x(0) = 0. \tag{2.4}$$

Let the following conditions be fulfilled:

i) $f : \underset{0}{\mathbf{P}}_{1,p_0,p_1}(0,T;\mathcal{M}_0,Y) \to L^{p_1}(0,T;Y)$ is a weakly compact mapping[1], where

$$\underset{0}{\mathbf{P}}_{1,p_0,p_1}(0,T;\mathcal{M}_0,Y) \equiv L^{p_0}(0,T;\mathcal{M}_0) \cap W^{1,p_1}(0,T;Y) \cap \{x(t) \mid x(0) = 0\},$$

$$1 < \max\{p_1, p_1'\} \leq p_0 < \infty, \; p_1' = \frac{p_1}{p_1 - 1};$$

(*ii*) there is a linear continuous operator $L : W^{s,p_2}(0,T;X_0) \to W^{s,p_2}(0,T;Y^*)$, $s \geq 0$, $p_2 \geq 1$ such that L commutes with $\frac{d}{dt}$ and the conjugate operator L^* has $ker(L^*) = \{0\}$;

[1] This condition is explained in the similar condition 1 of Appendix B.

(iii) there exist a continuous function $\varphi : R^1_+ \cup \{0\} \longrightarrow R^1$ and numbers $\tau_0 \geq 0$ and $\tau_1 > 0$ such that $\varphi(r)$ is nondecreasing for $\tau \geq \tau_0$, $\varphi(\tau_1) > 0$ and operators f and L satisfy the following inequality for any $x \in L^{p_0}(0,T;X_0)$

$$\int_0^T \langle f(t, x(t)), Lx(t)\rangle dt \geq \varphi\left([x]_{L^{p_0}(\mathcal{M}_0)}\right)[x]_{L^{p_0}(\mathcal{M}_0)};$$

(iv) there exist a linear bounded operator $L_0 : X_0 \to Y$ and constants $C_0 > 0$, C_1, $C_2 \geq 0$, $\nu > 1$ such that the inequalities

$$\int_0^T \langle \xi(t), L\xi(t)\rangle dt \geq C_0 \|L_0\xi\|^{\nu}_{L^{p_1}(0,T;Y)} - C_2,$$

$$\int_0^t \langle \frac{dx}{d\tau}, Lx(\tau)\rangle d\tau \geq C_1 \|L_0 x\|^{\nu}_Y(t) - C_2, \quad a.e.\ t \in (0,T]$$

hold for any $x \in W^{1,p_0}(0,T;X_0)$ and $\xi \in L^{p_0}(0,T;X_0)$.

Theorem 2.2 *Assume that conditions (i) - (iv) are fulfilled. Then the Cauchy problem (2.4) is solvable in* $\underset{0}{\mathbf{P}}_{1,p_0,p_1}(0,T;\mathcal{M}_0,Y)$ *in the following sense*

$$\int_0^T \left\langle \frac{dx}{dt} + f(t,x(t)), y^*(t)\right\rangle dt = \int_0^T \langle y(t), y^*(t)\rangle\, dt, \quad \forall y^* \in L^{p_1'}(0,T;Y^*),$$

for any $y \in G \subseteq L^{p_1}(0,T;Y)$*, where* $G \equiv \underset{r\geq\tau_1}{\cup} G_r$ *:*

$$G_r \equiv \left\{ y \in L^{p_1}(0,T;Y) \,\middle|\, \int_0^T |\langle y(t), Lx(t)\rangle|\ dt \leq \int_0^T \langle f(t,x(t)), Lx(t)\rangle dt - c, \right.$$
$$\left. \forall\, x \in L^{p_0}(0,T;X_0),\ [x]_{L^{p_0}(0,T;\mathcal{M}_0)} = r \right\},\ C_2 < c < \infty.$$

The proof of this theorem is presented in Appendix C (one can also refer to proofs of the similar theorems in [18,20]). The next proposition follows immediately from the theorem 2.2.

Corollary 2.1 *Under assumptions of Theorem 2.2 the problem (2.4) is solvable in* $\underset{0}{\mathbf{P}}_{1,p_0,p_1}(0,T;\mathcal{M}_0,Y)$ *for any* $y \in L^{p_1}(0,T;Y)$ *satisfying the condition: there is* $r > 0$ *such that the inequality*

$$\|y\|_{L^{p_1}(0,T;Y)} \leq \varphi\left([x]_{L^{p_0}(0,T;\mathcal{M}_0)}\right)$$

holds for any $x \in L^{p_0}(0,T;X_0)$ *with* $[x]_{L^{p_0}(\mathcal{M}_0)} \geq r$*. Furthermore, if* $\varphi(\tau) \nearrow \infty$ *as* $\tau \nearrow \infty$ *then the problem (2.4) is solvable in* $\underset{0}{\mathbf{P}}_{1,p_0,p_1}(0,T;\mathcal{M}_0,Y)$ *for any* $y \in L^{p_1}(0,T;Y)$ *satisfying the inequality*

$$\sup\left\{ \frac{1}{[x]_{L^{p_0}(0,T;\mathcal{M}_0)}} \int_0^T \langle y(t), Lx(t)\rangle\ dt \mid x \in L^{p_0}(0,T;X_0) \right\} < \infty.$$

3 Preliminary Results

In this section we introduce and investigate properties of a class of nonlinear function spaces (pn-spaces) that are connected to the considered problem directly. This investigation will be necessary for the application of the theorem 2.2 (and the corollary 2.1) to the considered problem.

Consider the following classes of functions $u : \Omega \longrightarrow R$

$$S_{1,\alpha,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \;\middle|\; [u]_{S_1}^{\alpha+\beta} \equiv \int_{\Omega} \left[|u|^{\alpha+\beta} + |u|^{\alpha} |\nabla u|^{\beta} \right] dx < \infty, \right\}, \quad (3.1)$$

$$S_{\Delta,\alpha,\beta}(\Omega) \equiv \left\{ u \in L^1(\Omega) \;\middle|\; [u]_{S_\Delta}^{\alpha+\beta} \equiv [u]_{S_1}^{\alpha_1+\beta_1} + \int_{\Omega} |u|^{\alpha} |\Delta u|^{\beta} dx < \infty, \right\}, \quad (3.2)$$

where $\alpha \geq 0$, $\frac{\alpha_1}{\beta_1} > -1$, $\beta, \beta_1 \geq 1$ and $\alpha_1 + \beta_1 = \alpha + \beta$ (we will note that under these conditions if $\beta_1 \leq \beta$ then $S_{1,\alpha,\beta}(\Omega) \subseteq S_{1,\alpha_1,\beta_1}(\Omega)$; these classes are metric spaces (see, Ap. A)).

Here and hereafter we assume $\beta > 1$. Further, we consider the case $\frac{\alpha}{\beta} > -1$, $\beta > 1$, $\alpha > \beta - 1$, as well.

Also, consider the following spaces of functions $u : Q_T \longrightarrow R$

$$L^p(0,T; S_{1,\alpha,\beta}(\Omega)) \equiv \left\{ u \in L^1(\Omega) \;\middle|\; [u]_{L(S_1)}^{p} \equiv \int_0^T [u]_{S_1}^{p} dt < \infty, \right\}, \quad (3.3)$$

$$P_{p_0,p_1}(0,T; S_{\Delta,\alpha,\beta}(\Omega); X) \equiv W^{1,p_0}(0,T;X) \cap L^{p_1}(0,T; S_{\Delta,\alpha,\beta}(\Omega)), \quad (3.4)$$

where $p, p_0, p_1, \beta > 1$, $\alpha \geq 0$ and X is a Banach space. Particularly, X can be choosen in such a way that $L^{p_0}(\Omega) \subseteq X$ for some $p_0 \geq 1$.

The space $L^{p_1}(0,T; S_{\Delta,\alpha,\beta}(\Omega))$ is defined as $L^p(0,T; S_{1,\alpha,\beta}(\Omega))$ by using (3.2) instead of (3.1).

Proposition 3.1 *The equivalency*

$$\mathcal{M}_{\eta,W^1_\beta(\Omega)} \equiv \left\{ u \in L^1(\Omega) \;\middle|\; \eta(u) \in W^{1,\beta}(\Omega),\ \eta(u) \equiv |u|^{\frac{\alpha}{\beta}} u \right\} \equiv S_{1,\alpha,\beta}(\Omega)$$

holds, if $\alpha \geq 0$, $\beta \geq 1$ *and* $\eta(\theta) \equiv |\theta|^{\frac{\alpha}{\beta}} \theta$, $\theta \in R$.

This express relations between $W^{1,\beta}(\Omega)$ and $S_{1,\alpha,\beta}(\Omega)$ follows immediately from defination (3.1). Indeed, it is enough to note that $\eta(u) \equiv |u|^{\frac{\alpha}{\beta}} u = v \Longleftrightarrow u = |v|^{\frac{-\alpha}{\alpha+\beta}} v \equiv \eta^{-1}(v)$.

Taking into account the last equivalency and definition (3.2) of the space $S_{\Delta,\alpha,\beta}(\Omega)$ we get

$$S_{\Delta,\alpha,\beta}(\Omega) \equiv \mathcal{M}_{\eta,W^1_{\beta_1}(\Omega)} \cap \left\{ u \;\middle|\; |u|^{\frac{\alpha}{\beta}} \Delta u \in L^{\beta}(\Omega) \right\}. \quad (3.5)$$

Now we are going to express the relation between the Sobolev space $W^{2,\beta}(\Omega)$ and $S_{\Delta,\alpha,\beta}(\Omega)$, for this we use a few auxiliary results.

The following equality will be used in our discussion. Let's put $\eta(u) \equiv |u|^{\frac{\alpha}{\beta}} u \equiv v$. Then

$$\Delta v \equiv (\Delta \circ \eta)(u) \equiv \Delta\eta(u) \equiv \Delta\left(|u|^{\frac{\alpha}{\beta}} u\right) = \nabla \cdot \left(\frac{\alpha+\beta}{\beta} |u|^{\frac{\alpha}{\beta}} \nabla u\right)$$
$$= \frac{\alpha+\beta}{\beta} |u|^{\frac{\alpha}{\beta}} \Delta u + \frac{\alpha(\alpha+\beta)}{\beta^2} |u|^{\frac{\alpha}{\beta}-2} u |\nabla u|^2. \tag{3.6}$$

Proposition 3.2 *Let $\alpha > -1, \beta \geq \beta_0 \geq 0$, $\beta \geq 1$ be some numbers, $\beta_0 + \beta \geq 2$ and $\Omega \subset R^n$, $n \geq 1$, be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Then the inequality*

$$\int_\Omega |u|^\alpha |\nabla u|^{\beta_0+\beta} dx$$
$$\leq c(\varepsilon) \sum_{i=1}^n \int_\Omega |u|^{\alpha+\beta_0} \left|D_i^2 u\right|^\beta dx + \varepsilon\kappa(\beta-\beta_0) \int_\Omega |u|^{\alpha+\beta_0+\beta} dx \tag{3.7}$$

holds for any $u \in C^2(\Omega) \cap C_0^1\left(\overline{\Omega}\right)$, where $\varepsilon > 0$ is some number, $\kappa(s) = 1$ if $s > 0$, and $\kappa(s) = 0$ if $s = 0$.

Proof. We have

$$\int_\Omega |u|^\alpha |\nabla u|^{\beta_0+\beta} dx \leq c\sum_{i=1}^n \int_\Omega |u|^\alpha |D_i u|^{\beta_0+\beta} dx$$
$$= -c_1 \sum_{i=1}^n \int_\Omega |u|^\alpha u |D_i u|^{\beta_0+\beta-2} D_j^2 u \, dx$$

Rewriting here the expression under the integral in the following form

$$\left(|u|^{\alpha-\frac{\alpha+\beta_0}{\beta}-\alpha\frac{\beta_0+\beta-2}{\beta_0+\beta}} u\right)\left(|u|^{\alpha\frac{\beta_0+\beta-2}{\beta_0+\beta}} |D_i u|^{\beta+\beta_0-2}\right)\left(|u|^{\frac{\alpha+\beta_0}{\beta}} D_i^2 u\right) \text{ if } \beta > \beta_0$$

and

$$\left(|u|^{\frac{\alpha}{\beta'}} u |D_i u|^{2\beta-2}\right)\left(|u|^{\frac{\alpha}{\beta}} D_i^2 u\right) \text{ if } \beta = \beta_0$$

and applying Young's inequality with appropriate exponents we obtain

$$\int_\Omega |u|^\alpha |\nabla u|^{\beta_0+\beta} dx \leq \varepsilon_0 \sum_{i=1}^n \int_\Omega \left[\kappa(\beta-\beta_0)|u|^{\alpha+\beta_0+\beta} + |u|^\alpha |D_i u|^{\beta_0+\beta}\right] dx$$
$$+c(\varepsilon_0) \sum_{i=1}^n \int_\Omega |u|^{\alpha+\beta_0} \left|D_i^2 u\right|^\beta dx$$
$$\leq \kappa(\beta-\beta_0) \int_\Omega |u|^{\alpha+\beta_0+\beta} dx + \varepsilon_1 \int_\Omega |u|^\alpha |\nabla u|^{\beta_0+\beta} dx$$
$$+c(\varepsilon_0) \sum_{i=1}^n \int_\Omega |u|^{\alpha+\beta_0} \left|D_i^2 u\right|^\beta dx, \tag{3.8}$$

here $\varepsilon_0 > 0$ is an arbitrarily small number. The second term in (3.8) is obtained by using the equivalency

$$\int\limits_{\Omega} |u|^{\alpha} \sum_{i=1}^{n} |D_i u|^{\beta_0+\beta} \, dx \leq \int\limits_{\Omega} |u|^{\alpha} |\nabla u|^{\beta_0+\beta} \, dx \leq n \int\limits_{\Omega} |u|^{\alpha} \sum_{i=1}^{n} |D_i u|^{\beta_0+\beta} \, dx. \quad (3.9)$$

Note that the first term of (3.8) vanishes if $\beta = \beta_0$.

Remark 3.1 It is not difficult to see that if $\alpha + \beta_0 + \beta > 1$, $\alpha > -1$, $\beta_0 \geq 0$, $\beta_1 \geq 1$ then

$$\int\limits_{\Omega} |u|^{\alpha+\beta_0+\beta} \, dx \leq c \int\limits_{\Omega} |u|^{\alpha+\beta_0} |\nabla u|^{\beta} \, dx \quad (3.10)$$

and if $1 \leq \alpha_0 + \beta_0 \leq \alpha_1 + \beta_1$, $1 \leq \beta_0 \leq \beta_1$, $\alpha_0\beta_1 \geq \alpha_1\beta_0$ then

$$\int\limits_{\Omega} |u|^{\alpha_0} |\nabla u|^{\beta_0} \, dx \leq c \int\limits_{\Omega} |u|^{\alpha_1} |\nabla u|^{\beta_1} \, dx + c_1 \quad (3.11)$$

hold for any $u \in C_0^1(\Omega)$, where

$$c = c(\alpha, \beta_0, \beta, mes\ \Omega) > 0, \ c_1 = c_1(\alpha_0, \beta_0, \alpha_1, \beta_1, mes\ \Omega) \geq 0.$$

Moreover, if $\alpha_0 + \beta_0 = \alpha_1 + \beta_1$ then $c_1 = 0$ (see, for example, [18] - [20]).

Proposition 3.3 *Let $\alpha > -1$, $\beta \geq 1$ be some numbers, $\alpha + \beta \geq 2$ and $\Omega \subset R^n$, $n \geq 1$, be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Then the inequality*

$$\int\limits_{\Omega} |u|^{\alpha+\beta} \, dx \leq c \int\limits_{\Omega} |u|^{\alpha} |\Delta u|^{\beta} \, dx. \quad (3.12)$$

holds for any $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$, where $c = c(\alpha, \beta, mes\ \Omega) > 0$.

Proof. Rewriting $\alpha + \beta = \alpha + \beta - 2 + 2 = \alpha + \beta_0 + \beta_1$ and applying the inequality (3.10) we get

$$\int\limits_{\Omega} |u|^{\alpha+\beta} \, dx \equiv \int\limits_{\Omega} |u|^{\alpha+(\beta-2)+2} \, dx \leq c \int\limits_{\Omega} |u|^{\alpha+\beta-2} |\nabla u|^{2} \, dx. \quad (3.13)$$

Using the Green's formula to the right hand side and estimating we obtain

$$\int\limits_{\Omega} |u|^{\alpha+\beta-2} |\nabla u|^{2} \, dx \leq c \int\limits_{\Omega} |u|^{\alpha+\beta-1} |\Delta u| \, dx$$

$$= c \int\limits_{\Omega} |u|^{\alpha+\beta-1-\frac{\alpha}{\beta}} |u|^{\frac{\alpha}{\beta}} |\Delta u| \, dx.$$

Hence applying the Young's inequality with exponents $\left(\beta, \frac{\beta}{\beta-1}\right)$ and arbitrary $\varepsilon > 0$ gives

$$\leq c(\varepsilon) \int\limits_{\Omega} |u|^{\alpha} |\Delta u|^{\beta} \, dx + \varepsilon \int\limits_{\Omega} |u|^{\alpha+\beta} \, dx. \quad (3.14)$$

The inequality (3.12) follows from (3.13) taking (3.14) into consideration and making ε sufficiently small.

The following result is a special case of the main inequality (3.22)[2]

Lemma 3.1 *Let $\alpha > -1$, $\beta > \frac{n}{n-1}$ be some numbers, $\Omega \subset R^n$, $n \geq 2$, be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Then the inequality*

$$\int\limits_{\Omega} |u|^{\alpha} |\nabla u|^{2\beta} \, dx \leq c_1 \int\limits_{\Omega} |u|^{\alpha+\beta} |\Delta u|^{\beta} \, dx + c_2 \int\limits_{\Omega} |u|^{\alpha+2\beta} \, dx \tag{3.15}$$

holds for any $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$, where $c = c(\alpha, \beta) > 0$.

Proof. The proof is based on the boundedness in the Lebesgue space $L^p(\Omega)$ of the local Hardy-Littlewood maximal function

$$M_{\Omega} w(x) = \sup_{0<r<dist(x,\partial\Omega)} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} w(y) dy,$$

$$|B_r(x)| = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n,$$

when $1 < p < +\infty$ (see [15]), the local spherical maximal function

$$(A_r w)(x) = \sup_{0<r<dist(x,\partial\Omega)} \int\limits_{S_1(0)} w(x+ry) \, dS(y); \; S_r(x) = \partial B_r(x),$$

when $p > \frac{n}{n-1}$, $n \geq 2$ (see [15]), and on $L^p(\Omega)-$convergency of averages of a function to the function itself

$$\lim_{r \searrow 0} \int\limits_{\Omega} \left| \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} w(y) \, dy - w(x) \right|^p dx = 0. \tag{3.16}$$

Let's put $w(x) \equiv |u(x)|^{\rho} |\nabla u(x)|^2$ for a function $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$. Then, under the conditions of Propositon 3.2 and boundedness of the local Hardy-Littlewood maximal function, for $\rho = \frac{\alpha}{\beta}$ we have

$$\int\limits_{\Omega} \left(\frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |u|^{\rho} |\nabla u|^2 \, dy \right)^{\beta} dx \leq c \int\limits_{\Omega} \left(|u|^{\rho} |\nabla u|^2 \right)^{\beta} dx. \tag{3.17}$$

Moreover, it is obvious that

$$\frac{1}{|B_r(x)|} \int\limits_{S_r(x)} |u|^{\rho} u \frac{\partial u}{\partial \nu} \, dS(y)$$

$$= \frac{1}{|B_1(0)| \, r^n} \int\limits_{S_1(0)} |u(x+r\eta)|^{\rho} u(x+r\eta) \left(\nabla u(x+r\eta) \cdot \nu\right) r \, r^{n-1} dS(\eta),$$

[2] For $n = 1$ the similar results to results of this section was proved in the earlier works (see, for example, [S1, S5]). Therefore, it is enough to consider just dimension $n \geq 2$.

Therefore, from the boundedness of a local spherical maximal function, we have

$$\int\limits_{\Omega}\left|\frac{1}{|S_1(0)|}\int\limits_{S_1(0)}|u(x+r\eta)|^{\rho}u(x+r\eta)\left(\nabla u\left(x+r\eta\right)\cdot\nu\right)ds\left(\eta\right)\right|^{\beta}dx$$

$$\leq c\int\limits_{\Omega}\left(|u(x)|^{\rho+1}|\nabla u\left(x\right)|\right)^{\beta}dx, \tag{3.18}$$

where the positive constant c does not depend on the function $u\left(x\right)$.

According to (3.6) we have

$$\nabla\cdot\left(\ |u|^{\rho}\ u\nabla u\right)=|u|^{\rho}\varDelta u+(\rho+1)\ \ |u|^{\rho}\ \ |\nabla u|^{2}.$$

Taking the integral of both sides of this equality on $B_r\left(x\right)$for $x\in\Omega$, and $0<r<dist\left(x,\partial\Omega\right)$ we obtain

$$\frac{\rho+1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}|\nabla u|^{2}\ dy$$

$$=\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}\nabla\cdot\left(\ |u|^{\rho}u\ \nabla u\right)dy-\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}u\varDelta u dy$$

or

$$\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}\ \ |\nabla u|^{2}dy$$

$$=\frac{1}{(\rho+1)}\left\{-\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}u\varDelta u dy+\frac{1}{|B_r\left(x\right)|}\int\limits_{S_r(x)}|u|^{\rho}u\frac{\partial u}{\partial\nu}\,dS\left(y\right)\right\}. \tag{3.19}$$

Using (3.19), the left part of (3.15) is estimated in the following way

$$\int\limits_{\Omega}\left[|u|^{\rho}|\nabla u|^{2}\right]^{\beta}dx\leq c\int\limits_{\Omega}\left||u|^{\rho}|\nabla u|^{2}-\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}|\nabla u|^{2}\ dy\right|^{\beta}dx$$

$$+c\int\limits_{\Omega}\left|\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}|\nabla u|^{2}\ dy\right|^{\beta}dx=I_1(r)+I_2(r). \tag{3.20}$$

According to (3.16) we have $\lim\limits_{r\to0}I_1(r)=0$. Therefore, it is enough to show that $I_2(r)$ can be estimated uniformly with respect to the r. Taking into account (3.17), (3.18) and (3.19) in $I_2(r)$ we get

$$I_2(r)=c\int\limits_{\Omega}\left|\frac{1}{|B_r\left(x\right)|}\int\limits_{B_r(x)}|u|^{\rho}|\nabla u|^{2}\ dy\right|^{\beta}dx$$

$$= c\int\limits_{\Omega}\left|-\frac{1}{|B_r(x)|}\int\limits_{B_r(x)}|u|^{\rho}u\Delta u dy+\frac{1}{|B_r(x)|}\int\limits_{S_r(x)}|u|^{\rho}u\frac{\partial u}{\partial\nu}\,dS(y)\right|^{\beta}dx$$

$$\leq c_1\int\limits_{\Omega}\left|\frac{1}{|B_r(x)|}\int\limits_{B_r(x)}|u|^{\rho}u\Delta u dy\right|^{\beta}dx+c_1\int\limits_{\Omega}\left|\frac{1}{|B_r(x)|}\int\limits_{S_r(x)}|u|^{\rho}u\frac{\partial u}{\partial\nu}\,dS(y)\right|^{\beta}dx$$

$$\leq c_1\int\limits_{\Omega}||u|^{\rho}u\Delta u|^{\beta}dx+c_1\int\limits_{\Omega}||u|^{\rho}u\nabla u|^{\beta}dx$$

$$\leq c_1\int\limits_{\Omega}||u|^{\rho}u\Delta u|^{\beta}dx+\varepsilon\int\limits_{\Omega}\left||u|^{\rho}|\nabla u|^2\right|^{\beta}dx+c_2(\varepsilon)\int\limits_{\Omega}|u|^{(\rho+2)\beta}dx.$$

Consequently

$$I_2(r)\leq c_1\int\limits_{\Omega}|u|^{\alpha+\beta}|\Delta u|^{\beta}dx+\varepsilon\int\limits_{\Omega}|u|^{\alpha}|\nabla u|^{2\beta}dx+c_2(\varepsilon)\int\limits_{\Omega}|u|^{\alpha+2\beta}dx. \tag{3.21}$$

where c_1,c_2 are positive quantities that are not dependent on r. Choosing sufficiently small $\varepsilon>0$, such that $\varepsilon<1$, then substituting the right side of (3.21) into (3.20) and passing to the limit by $r\searrow 0$ in the obtained inequality we get the desired inequality (3.15).

From Proposition 3.3 and Lemma 3.1 follows

Corollary 3.1 *Under the conditions of Lemma 3.1 the inequality*

$$\int\limits_{\Omega}|u|^{\alpha}|\nabla u|^{2\beta}dx\leq c\int\limits_{\Omega}|u|^{\alpha+\beta}|\Delta u|^{\beta}dx \tag{3.22}$$

holds with $c=c(\alpha,\beta)$ *that is not dependent on* u.

Our next goal is the study of the relation between spaces $W^{2,\beta}(\Omega)$ and $S_{\Delta,\alpha,\beta}(\Omega)$. We start with definition of the Sobolev space of second order:

$$W^{2,\beta}(\Omega)\equiv\left\{u\in L^1(\Omega)\ \middle|\ u,D_iu,D_iD_ju\in L^{\beta}(\Omega),\ \ i,j=\overline{1,n}\right\}.$$

It is well known ([4]) that

$$W^{2,\beta}(\Omega)\equiv\left\{u\in L_1(\Omega)\ \middle|u,D_i^2u\in L^{\beta}(\Omega),\ i=\overline{1,n}\right\}.$$

Moreover, for sufficiently smooth domains ([1,4]) holds

$$W^{2,\beta}(\Omega)\cap W_0^{1,\beta}(\Omega)\equiv\left\{u\ \middle|\ \Delta u\in L^{\beta}(\Omega),\ u\,|_{\partial\Omega}=0\right\}.$$

We will define the following classes of functions

$$\mathcal{M}_{\Delta\circ\eta,L^{\beta}(\Omega)}\equiv\left\{u\ \middle|\ \Delta\circ\ \eta(u)\in L^{\beta}(\Omega),\ \eta(u)\equiv|u|^{\frac{\alpha}{\beta}}u\right\}. \tag{3.23}$$

Now, we will study the relation between $W^{2,\beta}(\Omega)\cap W_0^{1,\beta}(\Omega)$ and

$$\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega)\equiv\left\{u(x)\middle|\,|u|^{\frac{\alpha}{\beta}}\ \Delta u\in L^{\beta}(\Omega)\right\}\cap\{u\mid u\,|_{\partial\Omega}=0\}.$$

In beginning we will prove the following statement.

Lemma 3.2 *Let* $\alpha \geq 0$, $\alpha_1 > -1$, $\beta_1 \geq \beta \geq \frac{\beta_1}{2} \geq 1$, $\beta > \frac{n}{n-1}$ *be some numbers,* $\alpha + \beta = \alpha_1 + \beta_1$, *(if* $\beta_1 = 2\beta$ *then* $\alpha > \beta - 1$*) and* $\Omega \subset \Re^n$, $n \geq 1$, *be a bounded domain with sufficiently smooth boundary* $\partial\Omega$. *Then, the following expression*

$$\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) = S_{\Delta,\alpha,\beta}(\Omega) \cap \{u \mid u\,|_{\partial\Omega} = 0\} \subseteq \mathcal{M}_{\Delta\circ\eta, L^\beta(\Omega)} \cap \{u \mid u\,|_{\partial\Omega} = 0\}$$

holds.

Proof. Let $u \in \overset{0}{S}_{\Delta,\alpha,\beta}(\Omega)$ be an arbitrary function. Then, according to definition (2.2), Remark 1 and Corollary 3.1 $\eta_1(u) \equiv |u(x)|^{\frac{\alpha_1}{\beta_1}} u(x) \in W_0^{1,\beta_1}(\Omega)$ as far as $u\,|_{\partial\Omega} = 0 \iff \eta_1(u)\,|_{\partial\Omega} = 0$, and $|u|^{\frac{\alpha}{\beta}}\ \Delta u \in L^\beta(\Omega)$. Moreover, if $\alpha_1 > 0$ then $u\,|_{\partial\Omega} = 0 \Longrightarrow \frac{\partial}{\partial n}\eta_1(u)\,|_{\partial\Omega} = 0$.

According to (3.22) and (3.11) the inequality

$$\int_\Omega |u|^{\alpha_1} |\nabla u|^{\beta_1} dx \leq c \int_\Omega |u|^{\alpha} |\Delta u|^{\beta} dx$$

holds under the conditions of Lemma, where $c = c(\alpha, \beta, \alpha_1, \beta_1) > 0$ that is not dependent on u.

Taking into account this and Corollary 3.1 we conclude

$$\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) \equiv S_{\Delta,\alpha,\beta}(\Omega) \cap \{u \mid u\,|_{\partial\Omega} = 0\}. \tag{3.24}$$

On the other hand, the definition (3.23) implies that $u \in \mathcal{M}_{\Delta\circ\eta, L^\beta(\Omega)}$ is equivalent to $\Delta v \equiv \Delta\eta(u) \in L^\beta(\Omega)$. Indeed, using (3.6) and estimating $L^\beta(\Omega)$ of Δv we get

$$\|\Delta v\|_{L_\beta(\Omega)}^\beta = \int_\Omega \left| |u|^{\frac{\alpha}{\beta}}\ \Delta u + \frac{\alpha}{\beta}\ |u|^{\frac{\alpha}{\beta}-2}\ u\ |\nabla u|^2 \right|^\beta dx$$

$$\leq c \left\{ \int_\Omega \left| |u|^{\frac{\alpha}{\beta}}\ \Delta u \right|^\beta dx + \int_\Omega \left|\ |u|^{\frac{\alpha}{\beta}-1}\ \ |\nabla u|^2 \right|^\beta dx \right\}.$$

Thus using the inequality (3.22) and the equivalence (3.24) we get

$$\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) \subseteq \mathcal{M}_{\Delta\circ\eta, L^\beta(\Omega)} \cap \{u \mid u\,|_{\partial\Omega} = 0\}.$$

Corollary 3.2 *Under the conditions of Lemma 3.2 the implication*

$$u \in \overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) \Longrightarrow v \equiv \eta(u) \in W^{2,\beta}(\Omega) \cap W_0^{1,\beta}(\Omega)$$

holds.

Proof. If $v(x) \equiv \eta(u) \equiv |u(x)|^{\frac{\alpha}{\beta}} u(x)$ then $\nabla v(x) \equiv \left(\frac{\alpha}{\beta} + 1\right) |u(x)|^{\frac{\alpha}{\beta}} \nabla u(x)$ and (3.6) takes place for Δv.

According to the proof above, the inclusion $v \equiv \eta(u) \in W^{2,\beta}(\Omega) \cap W_0^{1,\beta}(\Omega)$ is equivalent to $\Delta v = \Delta \circ \eta(u) \in L^\beta(\Omega)$, as far as $u\,|_{\partial\Omega} = 0 \iff \eta(u)\,|_{\partial\Omega} = 0$ (moreover, if $\alpha > 0$ then $u\,|_{\partial\Omega} = 0 \Longrightarrow \frac{\partial}{\partial n}\eta(u)\,|_{\partial\Omega} = 0$). This implies that $\eta(u) \in W^{2,\beta}(\Omega) \cap W_0^{1,\beta}(\Omega)$ is equivalent to $u \in \mathcal{M}_{\Delta\circ\eta, L^\beta(\Omega)} \cap \{u \mid u\,|_{\partial\Omega} = 0\}$. Therefore, taking into account the lemma we conclude the desired inclusion.

Notation If parameters $\alpha, \alpha_1 \geq 0, \beta, \beta_1, p, p_0, p_1 \geq 1$ satisfy certain conditions then can be obtained some relations between spaces $S_{\Delta,\alpha,\beta}(\Omega)$, $S_{1,\alpha_1,\beta_1}(\Omega)$, $L^p(\Omega)$, $P_{p_0,p_1}\Big(0,T;X;S_{\Delta,\alpha,\beta}(\Omega)\Big)$, $L^p(0,T;S_{1,\alpha_1,\beta_1}(\Omega))$ according to their definitions. More precisely can be proved results on inclusion and compactness for them with use the way that is similar to our earlier works [17–20]. Here, we are presenting some of such type of results.

Theorem 3.1 *Let $\alpha, \alpha_1 \geq 0$, $\beta > \frac{n}{n-1}, \beta_1 \geq 1$ be such numbers that $\frac{\alpha_1+\beta_1}{\alpha+\beta} \leq 1$, $2\alpha_1\beta \geq \beta_1(\alpha-\beta)$ and $\beta_1 \leq 2\beta$, $\alpha > \beta - 1$. Then $\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) \subset S_{1,\alpha_1,\beta_1}(\Omega)$.*

The proof follows from the inequality

$$\int\limits_{\Omega} |u|^{\alpha_1} |\nabla u|^{\beta_1} dx \leq c(\varepsilon) \int\limits_{\Omega} |u|^{\alpha} |\Delta u|^{\beta} dx + \varepsilon \left(\int\limits_{\Omega} |u|^s dx \right)^{\frac{\alpha+\beta}{s}},$$

where $s = s(\alpha, \alpha_1, \beta, \beta_1) \leq \alpha + \beta$, that can be derived by using the inequalities (3.10), (3.11), (3.15) and (3.22) (for details refer to [18,19]).

Remark 3.2 Note that it is not difficult to verify that if $\frac{\alpha+\beta}{\alpha_1+\beta_1} \geq 1$, $\beta \geq \beta_1$ and $\frac{n(\alpha_1+\beta_1)}{n-\beta_1} \geq p$, $n > \beta_1$ then the following inclusions

$$S_{1,\alpha,\beta}(\Omega) \subseteq S_{1,\alpha_1,\beta_1}(\Omega) \subset L^p(\Omega), \quad S_{\Delta,\alpha,\beta}(\Omega) \subseteq S_{\Delta,\alpha_1,\beta_1}(\Omega)$$

take place. Moreover, by arguments similar to the above one can show relations between the considered and Sobolev spaces, and also with use the Embedding Theorems for Sobolev spaces one can show that the inclusion $S_{1,\alpha,\beta}(\Omega) \subset L^p(\Omega)$ and consequently, inclusion $\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) \subset L^p(\Omega)$ are compact (for detail one can refer to [18–20]).

Corollary 3.3 *Assume that conditions of Theorem 3.1 are fulfilled. Then, the following inclusions*

$$P_{p_0,p_1}(0,T;S_{\Delta,\alpha,\beta}(\Omega);X) \subseteq P_{\widetilde{p}_0,\widetilde{p}_1}\left(0,T;S_{\Delta,\alpha_1,\beta_1}(\Omega);\widetilde{X}\right),$$

$$P_{p_0,p_1}(0,T;S_{\Delta,\alpha,\beta}(\Omega);X) \subset L^p(0,T;S_{1,\alpha_1,\beta_1}(\Omega)),$$

hold if $X \subseteq \widetilde{X}$, and $p_0 \geq \widetilde{p}_0 \geq 1, p_1 \geq \widetilde{p}_1 \geq 1$, $p_1 \geq p \geq 1$.

Remark 3.3 If $\alpha \geq 0$, $\frac{\alpha_1}{\beta_1} > -1$, $\frac{1}{2}\beta_1 = \beta > \frac{n}{n-1}$ such numbers that $\alpha + \beta = \alpha_1 + \beta_1$, $\alpha > \beta - 1$, then

$$S_{\Delta,\alpha,\beta}(\Omega) \iff \left\{ u(x) \,\middle|\, \eta(u) \equiv |u|^{\rho} u \in W^{2,\beta}(\Omega) \;\; \rho = \frac{\alpha}{\beta} \right\},$$

i.e.

$$u \in S_{\Delta,\alpha,\beta}(\Omega) \Longrightarrow v \equiv \eta(u) \equiv |u|^{\rho} u \in W^{2,\beta}(\Omega) \Longrightarrow$$
$$u \equiv \eta^{-1}(v) \equiv |v|^{-\frac{\rho}{\rho+1}} v \in S_{\Delta,\alpha,\beta}(\Omega)$$

under the conditions that all operations make a sense (see, [13] and also [18,19]).

Furthermore, note that $S_{1,\alpha,\beta}(\Omega)$ and $S_{\Delta,\alpha,\beta}(\Omega)$ are metric spaces ([18–20]) with the corresponding metrics of the form:

$$d_{S_{1,\alpha,\beta}(\Omega)}(u;v) \equiv \|\eta(u) - \eta(v)\|_{W^{1,\beta}(\Omega)}^{(\rho+1)^{-1}}, \quad \eta(\tau) \equiv |\tau|^{\rho}\tau, \rho = \frac{\alpha}{\beta}, \alpha \geq 0, \beta > 1,$$

$$d_{S_{\Delta,\alpha,\beta}(\Omega)}(u;v) \equiv \|\eta_1(u) - \eta_1(v)\|_{W^{1,\beta_1}(\Omega)}^{(\rho_1+1)^{-1}} + \||u|^{\rho}\Delta u - |v|^{\rho}\Delta v\|_{L^{\beta}(\Omega)}^{(\rho+1)^{-1}},$$

where $\rho_1 = \frac{\alpha_1}{\beta_1}$, $\eta_1(\tau) \equiv |\tau|^{\rho_1}\tau$, and $\alpha_1 + \beta_1 = \alpha + \beta$ (see (2.1)).

Moreover, it is not difficult to see that the metrics of spaces $\overset{0}{S}_{1,\alpha,\beta}(\Omega)$, $\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega)$ have the forms:

$$d_{\overset{0}{S}_{1,\alpha,\beta}(\Omega)}(u;v) \equiv \|||u|^{\rho}\nabla u - |v|^{\rho}\nabla v|\|_{L^{\beta}(\Omega)}^{(\rho+1)^{-1}};$$

$$d_{\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega)}(u;v) \equiv \||u|^{\rho}\Delta u - |v|^{\rho}\Delta v\|_{L^{\beta}(\Omega)}^{(\rho+1)^{-1}}.$$

We will prove the following theorem, which is based on Theorem 3.1, Corollary 3.3 and Embedding Theorems for the Sobolev spaces.

Theorem 3.2 *Let* $\alpha, \alpha_1 \geq 0$, $\beta > \frac{n}{n-1}$, $\beta_1 \geq 1$ *be such numbers that* $\beta_1 < \frac{n\beta}{n-\beta}$, $\beta < n$, $\alpha_1 + \beta_1 < \frac{n(\alpha+\beta)}{n-\beta}$ *and* $\alpha\beta_1 \geq \alpha_1\beta$, $\alpha > \beta - 1$. *Then, the inclusion*

$$\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega) \subset \overset{0}{S}_{1,\alpha_1,\beta_1}(\Omega)$$

is compact.

Proof. Since $u \in \overset{0}{S}_{\Delta,\alpha,\beta}(\Omega)$, we have $\eta(u) \equiv v \in W^{2,\beta}(\Omega) \cap W_0^{1,\beta}(\Omega)$ and

$$u \in \overset{0}{S}_{1,\alpha_1,\beta_1}(\Omega) \Longleftrightarrow \eta_1(u) \equiv v \in W_0^{1,\beta_1}(\Omega)$$

($\eta_1(u) \equiv |u|^{\rho_1}u,\ \rho_1 = \frac{\alpha_1}{\beta_1}$). Moreover, as far as $W^{2,\beta}(\Omega) \subset W^{1,\beta_1}(\Omega)$ is compact for $\beta_1 < \frac{n\beta}{n-\beta}$, we get the compactness of the inclusion $\eta(G) \subset W_0^{1,\beta_1}(\Omega)$ for any bounded subset G from $\overset{0}{S}_{\Delta,\alpha,\beta}(\Omega)$.

This implies the desired statement.

Corollary 3.4 *If* $0 < \rho \leq 2$, *then* $\overset{0}{S}_{\Delta,\rho,2}(\Omega) \subset W_0^{1,p}(\Omega)$, $p = \rho + 2$.

The results stated above allow us to prove the compact embedding for the following vector spaces: $P_{1,p_0,p_1}(0,T;S_{\Delta,\alpha,\beta}(\Omega);X)$, $L^p(0,T;S_{1,\alpha_1,\beta_1}(\Omega))$ with use known results ([10,11,18,19]). We need the following

Lemma 3.3 *Let* $\alpha, \alpha_1, \alpha_2 \geq 0$, $\beta, \beta_1, \beta_2 \geq 1$, *be such numbers that* $\alpha + \beta = \alpha_1 + \beta_1 = \alpha_2 + \beta_2$, $\beta_1 < \beta < \beta_2$. *Then, for any* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that the inequality*

$$[u]_{S_{\Delta,\alpha,\beta}} \leq \varepsilon [u]_{S_{\Delta,\alpha_2,\beta_2}} + c(\varepsilon)[u]_{S_{\Delta,\alpha_1,\beta_1}},\ \forall u \in S_{\Delta,\alpha_2,\beta_2}(\Omega)$$

holds.

The proof is obvious.

Lemma 3.4 *Let* $\alpha, \alpha_0 \geq 0$, $\beta, \beta_0, p \geq 1$, $\beta > \frac{n}{n-1}$, $2 \geq \frac{(\alpha+\beta)\beta_0}{(\alpha_0+\beta_0)\beta} \geq 1$ *be such numbers that* $\beta_0 < \frac{n\beta}{n-\beta}$, $\beta < n$, $\alpha_0 + \beta_0 < \frac{n(\alpha+\beta)}{n-\beta}$ *and* $\alpha\beta_0 \geq \alpha_0\beta$, $\alpha > \beta - 1$, $p \leq \alpha + \beta$. *Then, for any* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that the inequality*

$$d_{S_{1,\alpha_0,\beta_0}}(u;v) \leq \varepsilon\left([u]_{S_{\Delta,\alpha,\beta}} + [v]_{S_{\Delta,\alpha,\beta}}\right) + c(\varepsilon)\|u - v\|_{L_p},\ \forall u, v \in S_{\Delta,\alpha,\beta}(\Omega)$$

holds.

The proof is similar to the proof of the same type results from [18, 19] and is based on the compactness of the inclusion $S_{\Delta,\alpha_2,\beta_2}(\Omega) \subset S_{1,\alpha_1,\beta_1}(\Omega) \subset L^p(\Omega)$. (We note that in [6] were proved the compactness of the inclusion $S_{1,\rho,2}(\Omega) \subset L^{\widehat{p}}(\Omega)$ and with use this were proved the inequation such type where $S_{1,\alpha_1,\beta_1}(\Omega)$ substituted with $L^{\rho+2}(\Omega)$, $\widehat{p} > 1, \rho > 0$ are some numbers.)

These lemmas allow us to get the following compactness

Theorem 3.3 *Let $S_{1,\alpha_1,\beta_1}(\Omega)$, $S_{\Delta,\alpha,\beta}(\Omega)$ and X be spaces defined above and $S_{\Delta,\alpha,\beta}(\Omega) \subset S_{1,\alpha_1,\beta_1}(\Omega)$ be compact. Let $\alpha_1 \geq 0$, $\beta, \beta_1, p, p_0, p_1 \geq 1$ be such numbers that $\alpha + \beta = p = p_1$, $\alpha\beta_1 \geq \alpha_1\beta$, $\beta > \frac{n}{n-1}$, $\alpha > \beta - 1$. Then, the inclusion $P_{1,p_0,p_1}(0,T; S_{\Delta,\alpha,\beta}(\Omega); X) \subset L^p(0,T; S_{1,\alpha_1,\beta_1}(\Omega))$ is compact.*

The proof is similar to the proof of the same type of results from [10, 17–20]. Therefore, we are not providing it here. The other compactness theorems similar to Theorem 3.2 and Lemma 3.3 can also be proved , but we are not presenting them here, as well. However, if it will be neccessary, we are going to use those theorems for the spaces $P_{1,p_0,p_1}\left(0,T; \overset{0}{S}_{\Delta,\alpha,\beta}(\Omega);\, X\right)$, $L_p\left(0,T; \overset{0}{S}_{1,\alpha_1,\beta_1}(\Omega)\right)$ under the corresponding conditions on parameters $\alpha, \alpha_1, \beta, \beta_1, p, p_0, p_1$ and for further details we will refer of the reader to our earlier works [17–20].

4 Proof of Theorem 1

Now we can lead the proof by using Theorem 2.2 (Corollary 2.1), and in order to apply it we introduce the following spaces and mappings:

$$\mathcal{M}_0 \equiv \overset{0}{S}_{\Delta,\rho,2}(\Omega), \quad X_0 \equiv W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \qquad Y \equiv L^q(\Omega),\ X \equiv L^p(\Omega),$$

$$f(u) \equiv -|u|^\rho \Delta u + b_0 |u|^{\mu+1}, \quad L \equiv -\Delta, \quad L_0 \equiv \nabla, \qquad Y^* \equiv L^p(Q),\ p = \rho + 2$$

$$\underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y) \equiv \underset{0}{\mathbf{P}}_{1,p,q}\left(0,T; \overset{0}{S}_{\Delta,\rho,2}(\Omega); L^q(\Omega)\right) \cap L^\infty\left(0,T; W_0^{1,2}(\Omega)\right)$$

where

$$\underset{0}{\mathbf{P}}_{1,p,q}\left(0,T; \overset{0}{S}_{\Delta,\rho,2}(\Omega); L^q(\Omega)\right) \equiv L^p\left(0,T; \overset{0}{S}_{\Delta,\rho,2}(\Omega)\right) \cap$$
$$W^{1,q}(0,T; L^q(\Omega)) \cap \{u(t,x) \mid u(0,x) = 0\}.$$

It is not difficult to see that

$$\langle f(u), Lu\rangle \equiv \left\langle -|u|^\rho \Delta u + b_0 |u|^{\mu+1}, -\Delta u\right\rangle = \int_\Omega |u|^\rho (\Delta u)^2 dx + \int_\Omega b_0 |u|^{\mu+1} \Delta u dx$$

for any $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $u \in L^p\left(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right)$.

Taking into account the embedding theorems from Section 3, the last equality implies that

$$\int_0^T \langle f(u), Lu\rangle dt \geq (1-\varepsilon) \int_0^T \int_\Omega |u|^\rho (\Delta u)^2 dxdt - c_1(\varepsilon)$$

$$= (1-\varepsilon) \int_0^T [u]^{\rho+2}_{\overset{0}{S}_{\Delta,\rho,2}(\Omega)} dt - c_1(\varepsilon) \equiv \varphi\left([u]_{L^{\rho+2}\left(\overset{0}{S}_{\Delta,\rho,2}\right)}\right) [u]_{L^{\rho+2}\left(\overset{0}{S}_{\Delta,\rho,2}\right)}, \quad (4.1)$$

holds under conditions of Theorem 2.1, where $c_0 > 0,\ c_1, \varepsilon \geq 0$, and ε is a sufficiently small nonnegative number.

Furthermore, it is obvious that $\int_0^t \left\langle \frac{\partial u}{\partial \tau}, Lu \right\rangle d\tau \equiv \frac{1}{2} |||\nabla u(t)|||^2_{L^2}$ for any $u \in W^{1,p}(0,T;X_0) \cap \{u(t) \mid u(0) = 0\}$ and almost any $t \in (0,T]$. Moreover,

$$\int_0^T \langle w, Lw \rangle d\tau \equiv |||\nabla w|||^2_{L^2(Q)} \quad \text{for any } \ w \in L^p(0,T;X_0),$$

where $w \equiv \frac{\partial u}{\partial \tau} \in L^p(0,T;X_0)$.

Using the generalized coercivity of pair f and $-\Delta$ on

$$L^p\left(0,T;W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right) \cap W^{1,q}(0,T;L^q(\Omega))$$

(see, 4.1) the following a priori estimations for a solution $u(t,x)$ of considered problem are obtained in a well known way:

$$[u]_{L^p\left(\overset{0}{S}_{\Delta,\rho,2}\right)} \leq c\left(|b_0|, \|h\|_{L^2(W^{1,2}(\Omega))}, \rho, \mu\right), \quad p = \rho + 2,\ q = p'$$

and

$$\|u\|_{W^{1,q}(L^q) \cap L^\infty(W^{1,2})} \leq c\left(|b_0|, \|h\|_{L^2(W^{1,2}(\Omega))}, \rho, \mu\right).$$

Thus, each possible solution $u(t,x)$ of the considered problem belongs to a bounded subset of

$$L^p\left(0,T;\overset{0}{S}_{\Delta,\rho,2}(\Omega)\right) \cap W^{1,q}(0,T;L^q(\Omega)) \cap L^\infty\left(0,T;W_0^{1,2}(\Omega)\right),$$

and, consequently, the solutions belong to a bounded subset of

$$P_{1,p,q}\left(0,T;\overset{0}{S}_{\Delta,\rho,2}(\Omega);L^q(\Omega)\right) \text{ and } L^\infty\left(0,T;W_0^{1,2}(\Omega)\right).$$

To apply Theorem 2.2 (Corollary 2.1) it remains to show that f is a weakly compact (continuous) mapping from

$$P_1(Q_T) \equiv P_{1,p,q}\left(0,T;\overset{0}{S}_{\Delta,\rho,2}(\Omega);L^q(\Omega)\right) \cap L^\infty\left(0,T;W_0^{1,2}(\Omega)\right)$$

into $L^q(Q_T)$. To this end, it is enough to use the following expressions:

$$|u|^{\rho-2}\ u\ |\nabla u|^2 = \left(|u|^{\gamma\rho}\ \nabla u\right) \cdot \left(|u|^{(1-\gamma)\rho-2}\ u\,\nabla u\right), \quad (4.2)$$

$$|u|^{\rho-2}\ u\ |\nabla u|^2 = \frac{1}{\rho(\rho+1)(1-\theta)}\ \Delta\left(|u|^\rho u\right)$$
$$-\frac{1}{\rho(\theta\rho+1)(1-\theta)}\ |u|^{(1-\theta)\rho}\ \Delta\left(|u|^{\theta\rho} u\right), \quad (4.3)$$

because of

$$|u|^{\rho}\ \Delta u = \frac{1}{\rho+1}\ \Delta\left(|u|^{\rho} u\right) - \rho\,|u|^{\rho-2}\ u\ |\nabla u|^{2}\,, \tag{4.4}$$

where γ is a number from condition 4) (see, below), if $\rho \geq 1$, and θ is such a number that $\frac{1}{2} \leq \theta < 1$ if $\rho > 0$.

Thus, according to the embedding theorems mentioned above, for solutions we have:

$$u\,(t,x) \in P_{1,p_0,q_0}\left(0,T;\overset{0}{S}_{\Delta,\alpha,\beta}\,(\Omega)\,;L^{q}\,(\Omega)\right) \text{ and } u\,(t,x) \in L^{p}\left(0,T;\overset{0}{S}_{1,\alpha_1,\beta_1}\,(\Omega)\right),$$

if the parameters $\alpha_1 \geq 0,\ \beta, \beta_1, p, p_0, p_1 \geq 1,\ \alpha > \beta - 1$ satisfy one of the following conditions: 1) $\alpha_1 = (\rho-1)\,q$, $\beta_1 = 2q$, $p = \rho+2$, $q = p' = \frac{\rho+2}{\rho+1}$; 2) $\alpha = \rho$, $\beta = 2$, $p_0 = \rho+2, q_0 = q$; 3) $\alpha = s\beta$, $\beta > 1$, $p_0 = (s+1)\,\beta$, $q_0 = \beta$, $1 \leq s \leq \frac{3\rho-2}{4}$; or 4) $\alpha = \gamma\rho\beta$, $\beta = \frac{\rho+2}{\gamma\rho+1}$, $\frac{1}{2} \leq \gamma < 1$, $p_0 = \rho+2$, $q_0 = q$.

We will use only the case (4.3) in addition $\theta = \frac{2}{3}$ if $\rho \in (0,3)$ and $\theta = \frac{1}{2}$ if $\rho \geq 3$. In particular, if one choose $\theta = \frac{2}{3}$ then it is sufficient to use the expression:

$$|u|^{\rho-2}\ u\ |\nabla u|^{2} = \frac{3}{\rho\,(\rho+1)}\ \Delta\left(|u|^{\rho} u\right) - \frac{3}{\rho}\,|u|^{\frac{\rho}{3}}\ \nabla\cdot\left(|u|^{\frac{2\rho}{3}}\,\nabla u\right),$$

because in this case the nesessary inclusions $|u|^{\frac{2\rho}{3}}\ \Delta u \in L^{\frac{2(\rho+2)}{\rho+4}}\,(Q)$ and $|u|^{\frac{\rho}{3}} \in L^{\frac{2(\rho+2)}{\rho}}\,(Q)$ take place.

Therefore, if a sequence $\{u_m\} \subset P_{1,p,q}\left(0,T;\overset{0}{S}_{\Delta,\rho,2}\,(\Omega)\,;L^{q}\,(\Omega)\right)$ converges weakly to
$u \in P_{1,p,q}\left(0,T;\overset{0}{S}_{\Delta,\rho,2}\,(\Omega)\,;L^{q}\,(\Omega)\right)$ in $P_{1,p,q}\left(0,T;\overset{0}{S}_{\Delta,\rho,2}\,(\Omega)\,;L^{q}\,(\Omega)\right)$ then, according to the embedding and compactness theorems from Section 3, first terms in (4.4) and in (4.3) converge weakly and second term in (4.4) converges weakly as in second term of (4.3) one of the factors in converges weakly and the another one converges strongly in the corresponding spaces. (Similar explanations takes place for the case if one will use the presentation (4.2).) Whence follows, that $f\,(u_m) \rightharpoonup f\,(u)$ in $L^{q}\,(Q)$. (In the above reasonings we suppose, without of notice that, at least there exist a subsequence of $\{u_m\}$ that converges in the corresponding spaces.) Consequently we obtain the weakly compactness of the mapping f.

Hence, all conditions of Corollary 2.1 are fulfilled. Applying it to the considered problem (1.1)-(1.3) we obtain the statement of Theorem 2.1.

Remark 4.1 The solvability theorem such as Theorem 2.1 for the problem (1.1)-(1.3), but with $u\,(0,x) = u_0\,(x)$ for $u_0 \in \overset{0}{S}_{\Delta,\rho,2}\,(\Omega)$ is also valid and can be proved as in [19] (or [18]).

Remark 4.2 The problem (1.1)-(1.3) can also be considered with the term $b(t,x,u)$ instead of $b_0\,|u|^{\mu+1}$. In this case, it is enough to assume fulfilling of the following conditions: The function $b\,(t,x,u)$ is the Caratheodory function on $Q \times R^1$, there exist functions $b_0\,(t,x)\,, b_1\,(t,x) \geq 0$ and number $\mu \geq 0$ such that $\min\left\{0, \frac{\rho}{2}-1\right\} \leq \mu < \rho$ and

$$|b\,(t,x,u)| \leq b_0\,(t,x)\,|u|^{\mu+1} + b_1\,(t,x)\,,$$

where

$$b_0 \in L^{\infty}\,(Q)\,,\ b_1 \in L^{2}\left(0,T;W_0^{1,2}\,(\Omega)\right) \qquad \text{if } \mu \geq \frac{\rho}{2}-1;$$

$b \in L^{\infty}\left(0,T;W^{1,\infty}\left(\Omega\right);C^{1}\left(R^{1}\right)\right)$ and

$$\left|D_{i}b\left(t,x,u\right)\right| \leq \widetilde{b}_{0}\left(t,x\right)\left|u\right|^{\mu+1}+b_{1}\left(t,x\right),\ i=\overline{1,n},$$
$$\left|b_{\xi}\left(t,x,\xi\right)\right| \leq b_{2}\left(t,x\right)\left|\xi\right|^{\mu}+b_{3}\left(t,x\right),$$

$$\widetilde{b}_{0},b_{2}\in L^{\infty}\left(Q\right),\ \widetilde{b}_{1}\in L^{2}\left(Q\right),\ b_{3}\in L^{2}\left(0,T;W_{0}^{1,2}\left(\Omega\right)\right),$$

$q=p'=\frac{p}{p-1}$, and $p=\rho+2$ if $\mu<\frac{\rho}{2}-1$.

5 On Behaviour of Solutions of Problem (1.1)-(1.3)

In this section we investigate the behavior of solutions for different $\mu \geq 0$: $\min\left\{0,\frac{\rho}{2}-1\right\} \leq \mu<\rho$ and $u\left(0,x\right)=u_{0}\left(x\right)$, and in the case $\mu=\rho$.

Theorem 5.1 *Let* $\min\left\{0,\frac{\rho}{2}-1\right\} \leq \mu<\rho$, $u_{0}\in W_{0}^{1,p}\left(\Omega\right)$, $h\in L^{\infty}\left(R_{+}^{1};L^{q}\left(\Omega\right)\right)$ *and* $\left\|h\right\|_{L^{q}}\left(t\right) \leq C_{0}$, $p=\rho+2$. *Then, the solution of the problem (1.1)-(1.3) with the initial condition* $u\left(0,x\right)=u_{0}\left(x\right)$ *satisfies the inequality*

$$\left\|u\left(t\right)\right\|_{L^{2}\left(\Omega\right)}^{2} \leq \left(\frac{C+C_{2}\left\|h\right\|_{L^{\infty}\left(L^{q}\right)}^{q}}{C_{1}}\right)^{\frac{2}{p}}+\left(C_{1}\frac{\rho}{2}t\right)^{-\frac{2}{\rho}}, \tag{5.1}$$

i.e. the solution of the problem (1.1)-(1.3) remains bounded as $t\nearrow\infty$, *where*
$C_{j}=C_{j}\left(\rho,\mu,b_{0},C_{0},\left\|u_{0}\right\|_{W_{0}^{1,p}\left(\Omega\right)},mes\ \Omega\right)$.

Proof. Consider the functional

$$\Phi\left(t\right)\equiv\Phi\left(u\left(t\right)\right)\equiv\frac{1}{2}\int_{\Omega}\left|u\left(t\right)\right|^{2}dx\equiv\frac{1}{2}\left\|u\left(t\right)\right\|_{L^{2}\left(\Omega\right)}^{2}.$$

If $u\left(t\right)$ is a solution of the problem (1.1)-(1.3), then the function $\Phi\left(t\right)$ has the property

$$\Phi'\left(t\right)=\left\langle u',u\right\rangle=\left\langle\left|u\right|^{\rho}\Delta u-b_{0}\left|u\right|^{\mu+1}+h,u\right\rangle=-\left(\rho+1\right)\left\langle\left|u\right|^{\rho}\nabla u,\nabla u\right\rangle-$$

$$-\left\langle b_{0}\left|u\right|^{\mu}u,u\right\rangle+\left\langle h,u\right\rangle \leq -\left(\rho+1\right)\left\|\left|u\right|^{\frac{\rho}{2}}\nabla u\right\|_{L^{2}}^{2}+\left|b_{0}\right|\left\|u\right\|_{L^{\mu+2}}^{\mu+2}+\left|\left\langle h,u\right\rangle\right|.$$

Applying the results of Section 3 we get

$$\Phi'\left(t\right) \leq -\left(\rho+1\right)\left\|\left|\left|u\right|^{\frac{\rho}{2}}\nabla u\right|\right\|_{L^{2}}^{2}+\left|b_{0}\right|\left\|u\right\|_{L^{\mu+2}}^{\mu+2}+\left\|u\right\|_{L^{p}}\left\|h\right\|_{L^{q}}$$

$$\leq -\left(\rho+1\right)\frac{\left(\rho+2\right)^{2}}{4}\left\|\left|\nabla\left(\left|u\right|^{\frac{\rho}{2}}u\right)\right|\right\|_{L^{2}}^{2}+2\varepsilon\left\|u\right\|_{L^{p}}^{p}+C\left(\varepsilon\right)\left(1+\left\|h\right\|_{L^{q}}^{q}\right)$$

$$\leq -C_{0}\left\|\left|\nabla\left(\left|u\right|^{\frac{\rho}{2}}u\right)\right|\right\|_{L^{2}}^{2}+C\left(\varepsilon\right)\left(1+\left\|h\right\|_{L^{q}}^{q}\right)$$

$$\leq -\widetilde{C}_{0}\left\|u\right\|_{L^{p}}^{p}+C\left(\varepsilon\right)\left(1+\left\|h\right\|_{L^{q}}^{q}\right) \leq -C_{1}\left\|u\right\|_{L^{2}}^{p}+C\left(\varepsilon\right)+k\left(\varepsilon\right)\left\|h\right\|_{L^{\infty}\left(L^{q}\right)}^{q},$$

because of $\overset{0}{S}_{1,\rho,2}\left(\Omega\right)\subset L^{p}\left(\Omega\right)\subset L^{2}\left(\Omega\right)$ with the corresponding inequalities. Hence we have

$$\Phi'\left(t\right)+C_{1}\left(\Phi\left(t\right)\right)^{\frac{p}{2}} \leq C+C_{2}\left\|h\right\|_{L^{\infty}\left(L^{q}\right)}^{q}, \tag{5.2}$$

where

$$C = C\left(\rho, \mu, b_0, C_0, \|u_0\|_{W_0^{1,p}(\Omega)}\,,\ mes\ \Omega\right),$$
$$C_1 = C_1\left(\rho, \mu, b_0, C_0, \|u_0\|_{W_0^{1,p}(\Omega)}\,,\ mes\ \Omega\right)$$

and $\Phi(0) = \|u_0\|_{L^2}^2$.

Then, applying the following form of Gronwall's lemma (Lemma 5.1) to the inequality (5.2), which was proved by Ghidaglia, with $y(t) \equiv \Phi(t)$, $\theta = C_1$, $\eta = C + C_2\|h\|_{L^q}^q$, $l = \frac{p}{2}$ we obtain the inequality (5.1).

Lemma 5.1 *([21]) Let $y(t)$ be a positive absolutely continuous function on R_+^1 which satisfies*

$$y' + \theta y^l \le \eta, \quad l > 1, \theta > 0, \eta \ge 0.$$

Then, for $t \ge 0$,

$$y(t) \le \left(\frac{\eta}{\theta}\right)^{\frac{1}{l}} + (\theta(l-1)t)^{-\frac{1}{l-1}}.$$

Now, consider the following problem:

$$\frac{\partial u}{\partial t} - |u|^\rho \Delta u - b(x)|u|^{\rho+1} = 0, \quad (t, x) \in Q, \tag{5.3}$$

$$u(0, x) = u_0(x) \ge 0, \quad u\,|_{\,\Gamma} = 0\,,\ \Gamma \equiv [0, T] \times \partial\Omega, \tag{5.4}$$

Let λ_1 be the first eigenvalue and $v_1(x)$ be the corresponding eigen-function of the problem

$$-\Delta v = \lambda v,\ \ x \in \Omega \quad v\,|_{\,\partial\Omega} = 0\,.$$

Lemma 5.2 *Let $\rho, b(x) > 0$, $u_0(x) \ge 0$ and $\mu = \rho$, moreover $u_0 \in L^{\rho+2}(\Omega)$, $\|b\|_{L^\infty(\Omega)} \le c$, $c = c(\Omega) > 0$, $\Omega \subset R^n$ be as above. Then, if $M = \frac{c(\rho+2)^2}{4(\rho+1)} < \lambda_1$ then a solution of the problem (5.3)-(5.4) remains bounded as $t \nearrow \infty$, i.e. the inequality of the type (5.1) also takes place.*

Proof. Let $\rho, b(x) > 0$ and $\mu = \rho$. Then, using the previous reasoning we get

$$\Phi'(t) = \left\langle u', u\right\rangle = -(\rho+1)\left\langle |u|^\rho \nabla u, \nabla u\right\rangle + \left\langle b(x)|u|^{\rho+1}, u\right\rangle$$
$$\le -(\rho+1)\frac{4}{(\rho+2)^2}\left\|\nabla\left(|u|^{\frac{\rho}{2}} u\right)\right\|_{L^2}^2 + \|b\|_{L^\infty}\left\||u|^{\frac{\rho}{2}} u\right\|_{L^2}^2.$$

Since $M < \lambda_1$ a solution remains bounded when $t \nearrow \infty$ as in the previous case.

Remark 5.1 Suppose $b(x) > \lambda_1$ and $u(t, x) > 0$ for $x \in \Omega$ or on a subdomain $\widetilde{\Omega} \subset\subset \Omega$. Note that this case was studied under various conditions in [7,24,22,25]. In our consideration we study the problem (5.3)-(5.4) in the following way:

If $u(t, x) > 0$ for $x \in \Omega$, then the equation (5.3) can be represented as

$$u^{-\rho}\frac{\partial u}{\partial t} - \Delta u - b(x)u = 0, \quad (t, x) \in Q.$$

Hence, we have

$$\left\langle u^{-\rho}\frac{\partial u}{\partial t}, v_1\right\rangle = \left\langle \Delta u + b(x)u, v_1\right\rangle \Longrightarrow \left\langle u^{-\rho}\frac{\partial u}{\partial t}, v_1\right\rangle = -\lambda_1\left\langle u, v_1\right\rangle + \left\langle b(x)u, v_1\right\rangle$$

or

$$\left\langle u^{-\rho}\frac{\partial u}{\partial t}, v_1\right\rangle \geq \delta \left\langle u, v_1\right\rangle,$$

$$(b(x) - \lambda_1) \geq \delta > 0 \Longrightarrow (1-\rho)^{-1}\frac{\partial}{\partial t}\left\langle u^{1-\rho}, v_1\right\rangle \geq \delta \left\langle u, v_1\right\rangle.$$

The blow-up result can be obtained from here as in [25] (see [7,24,22,25] and references therein).

6 Appendixes

6.1 Appendix A

In the beginning we would like to give a general definition of $pn-$spaces and also short informations on their properties. Let X, Y be a locally convex vector topological spaces, $B \subseteq Y$ be a Banach space and $g : D(g) \subseteq X \longrightarrow Y$. Let's introduce the following subset of X

$$\mathcal{M}_{gB} \equiv \{x \in X \mid g(x) \in B,\ \mathrm{Im} g \cap B \neq \varnothing\}.$$

Definition 6.1 *A subset $\mathcal{M} \subseteq X$ is called a $pn-$space (i.e. pseudonormed space) if S is a topological space and there is a function $[\cdot]_{\mathcal{M}} : \mathcal{M} \longrightarrow R^1_+ \equiv [0,\infty)$ (which is called $p-$norm of $\mathcal{M}$) such that*

qn) $[x]_{\mathcal{M}} \geq 0$, $\forall x \in \mathcal{M}$ and $x = 0 \Longrightarrow [x]_{\mathcal{M}} = 0$;

pn) $[x_1]_{\mathcal{M}} \neq [x_2]_{\mathcal{M}} \Longrightarrow x_1 \neq x_2$, for $x_1, x_2 \in \mathcal{M}$, and $[x]_{\mathcal{M}} = 0 \Longrightarrow x = 0$;

The following conditions are often fulfilled in the spaces $\mathcal{M}_{gB}$.

N) There exist a convex function $\nu : R^1 \longrightarrow \overline{R^1_+}$ and number $K \in (0,\infty]$ such that $[\lambda x]_{\mathcal{M}} \leq \nu(\lambda)[x]_{\mathcal{M}}$ for any $x \in \mathcal{M}$ and $\lambda \in R^1$, $|\lambda| < K$, moreover, $\lim\limits_{|\lambda| \longrightarrow \lambda_j} \frac{\nu(\lambda)}{|\lambda|} = c_j$, $j = 0, 1$ where $\lambda_0 = 0$, $\lambda_1 = K$ and $c_0 = c_1 = 1$ or $c_0 = 0$, $c_1 = \infty$, i.e. if $K = \infty$ then $\lambda x \in \mathcal{M}$ for any $x \in \mathcal{M}$ and $\lambda \in R^1$.

Let $g : D(g) \subseteq X \longrightarrow Y$ be such a mapping that $\mathcal{M}_{gB} \neq \varnothing$ and the following conditions are fulfilled

G_1) $g : D(g) \longleftrightarrow \mathrm{Im} g$ is a bijection and $g(0) = 0$;

G_2) there is a function $\nu : R^1 \longrightarrow \overline{R^1_+}$ satisfying condition N such that

$$\|g(\lambda x)\|_B \leq \nu(\lambda)\|g(x)\|_B,\ \forall x \in \mathcal{M}_{gB},\ \forall \lambda \in R^1.$$

If the mapping g satisfies conditions G_1 and G_2 then $\mathcal{M}_{gB}$ is a $pn-$space with $p-$norm defined in the following way: there is a one-to-one function $\psi : R^1_+ \longrightarrow R^1_+$, $\psi(0) = 0$, $\psi, \psi^{-1} \in C^0$ such that $[x]_{\mathcal{M}_{gB}} \equiv \psi^{-1}(\|g(x)\|_B)$. In this case $\mathcal{M}_{gB}$ is a metric space with a metric: $d_{\mathcal{M}}(x_1; x_2) \equiv \|g(x_1) - g(x_2)\|_B$. A sequence $\{x_\kappa\} \subset \mathcal{M}_{gB}$ weakly converges in $\mathcal{M}_{gB}$ iff the sequence $\{g(x_\kappa)\}$ weakly converges in B. Further, we consider just such type of $pn-$spaces.

Definition 6.2 *The $pn-$space $\mathcal{M}_{gB}$ is called weakly complete if $g(\mathcal{M}_{gB})$ is weakly closed in B. The pn-space $\mathcal{M}_{gB}$ is "reflexive" if each bounded weakly closed subset of $\mathcal{M}_{gB}$ is weakly compact in $\mathcal{M}_{gB}$.*

It is clear that if B is a reflexive Banach space and $\mathcal{M}_{gB}$ is a weakly complete $pn-$space, then $\mathcal{M}_{gB}$ is "reflexive". Moreover, if B is a separable Banach space, then $\mathcal{M}_{gB}$ is also separable.

6.2 Appendix B

Now we will prove a general solvability theorem for a nonlinear equation in the Banach spaces. So let X, Y are Banach spaces. We will consider an equation

$$f(x) = y, \quad y \in Y, \tag{6.1}$$

where $f : D(f) \subseteq X \longrightarrow Y$ is a nonlinear bounded operator, i.e. there is a continuous function $\mu : R_+ \longrightarrow R_+$ such that the inequation

$$\|f(x)\|_Y \leq \mu(\|x\|_X), \quad \forall x \in D(f),$$

and prove a general solvability theorem for it. It is clear that (6.1) is equivalent to the following functional equation:

$$\langle f(x), y^* \rangle = \langle y, y^* \rangle, \quad \forall y^* \in Y^*. \tag{6.2}$$

We consider the following conditions:

1) $f : \mathcal{M}_0 \subseteq D(f) \longrightarrow Y$ is a weakly compact mapping, i.e. for any weakly convergence sequence $\{x_m\}_{m=1}^{\infty} \subset \mathcal{M}_0$ in $\mathcal{M}_0$ (i.e. $x_m \stackrel{\mathcal{M}_0}{\rightharpoonup} x_0 \in \mathcal{M}_0$) there is a subsequence $\{x_{m_k}\}_{k=1}^{\infty} \subseteq \{x_m\}_{m=1}^{\infty}$ such that $f(x_{m_k}) \stackrel{Y}{\rightharpoonup} f(x_0)$ weakly in Y (or for a general sequence if $\mathcal{M}_0$ not is separable space) and $\mathcal{M}_0$ be a weakly complete $pn-$space;

2) there exists a mapping $g : X_0 \subseteq X \longrightarrow Y^*$ and a continuous function $\varphi : R^1_+ \longrightarrow R^1$ nondecreasing for $\tau \geq \tau_0 \geq 0$ and $\varphi(\tau_1) > 0$ for a number $\tau_1 > 0$ such that it generates a "coercive" pair in a generalized sense with f on the topological space $X_1 \subseteq X_0 \cap \mathcal{M}_0$, i.e.

$$\langle f(x), g(x) \rangle \geq \varphi([x]_{\mathcal{M}_0})[x]_{\mathcal{M}_0}, \quad \forall x \in X_1,$$

where X_1 is such a topological space that $\overline{X_1}^{X_0} \equiv X_0$ and $\overline{X_1}^{\mathcal{M}_0} \equiv \mathcal{M}_0$, and $\langle \cdot, \cdot \rangle$ is a dual form of the pair (Y, Y^*), moreover, one of the following conditions (α) or (β) holds:

(α) if $g \equiv L$ is a linear continuous operator, then X_1 is a "reflexive" space (see [S3, S4]), $X_0 \equiv X_1 \subseteq \mathcal{M}_0$ is a separable topological vector space which is dense in $\mathcal{M}_0$ and $\ker L^* = \{0\}$.

(β) if g is a bounded operator (linear or nonlinear), then Y is a reflexive separable space, $g(X_1)$ contains an everywhere dense linear manifold of Y^* and g^{-1} is weakly compact operator from Y^* to $\mathcal{M}_0$.

Theorem 6.1 *Let conditions 1 and 2 hold. Then the equation (6.1) (or (6.2)) is solvable in $\mathcal{M}_0$ for any $y \in Y$ satisfying the following inequality: there exists $r > 0$ such that*

$$\varphi([x]_{\mathcal{M}_0})[x]_{\mathcal{M}_0} \geq \langle y, g(x) \rangle, \textit{ for } \quad \forall x \in X_1 \quad \textit{with} \quad [x]_{\mathcal{M}} \geq r. \tag{6.3}$$

Proof. Assume that conditions 1 and 2 (α) are fulfilled and $y \in Y$ such that (6.3) holds. We are going to use Galerkin's approximation method. Let $\{x^k\}_{k=1}^{\infty}$ be a complete system in the (separable) space $X_1 \equiv X_0$. Then, we are looking for approximate solutions in the form $x_m = \sum_{k=1}^{m} c_{mk} x^k$, where c_{mk} are unknown coefficients, that might be determined from the system of algebraic equations

$$\Phi_k(c_m) := \left\langle f(x_m), g\left(x^k\right) \right\rangle - \left\langle y, g\left(x^k\right) \right\rangle = 0, \quad k = 1, 2, ..., m \tag{6.4}$$

with $c_m \equiv (c_{m1}, c_{m2}, ..., c_{mm})$, as g is a linear operator.

We observe that the mapping $\Phi(c_m) := (\Phi_1(c_m), \Phi_2(c_m), ..., \Phi_m(c_m))$ is continuous by virtue of condition 1. (6.4) implies the existence of such $r = r(\|y\|_Y) > 0$ that the "acute angle" condition is fulfilled for all x_m with $[x_m]_{\mathcal{M}_0} \geq r$, i.e. for any $c_m \in S_{r_1}^{R^m}(0) \subset R^m$, $r_1(r) > 0$ the inequality

$$\sum_{k=1}^{m} \langle \Phi_k(c_m), c_{mk} \rangle \equiv \left\langle f(x_m), g\left(\sum_{k=1}^{m} c_{mk} x^k\right)\right\rangle - \left\langle y, g\left(\sum_{k=1}^{m} c_{mk} x^k\right)\right\rangle$$

$$= \langle f(x_m), g(x_m)\rangle - \langle y, g(x_m)\rangle \geq 0, \quad \forall c_m \in \mathbb{R}^m, \|c_m\|_{\mathbb{R}^m} = r_1.$$

holds. The solvability of system (6.4) for each $m = 1, 2, \ldots$ follows from a well-known lemma on the "acute angle" ([10]), which is equivalent to the Brouwer's fixed-point theorem. Thus, the sequence $\{x_m \mid m \geq 1\}$ of the approximate solutions, that is contained in a bounded subset of the space $\mathcal{M}_0$. Further arguments are analogously to the arguments from [10, 19] therefore we omit them. It remains to pass to the limit in (6.4) by m and use a weak convergency of a subsequence of the sequence $\{x_m \mid m \geq 1\}$, of the weak compactness of the mapping f, and finally, of the completeness of the system $\left\{x^k\right\}_{k=1}^{\infty}$ in the space X_1.

Hence, we get the limit element $x_0 = w - \lim_{j \nearrow \infty} x_{m_j} \in \mathcal{M}_0$ that is a solution of the equation

$$\langle f(x_0), g(x)\rangle = \langle y, g(x)\rangle, \quad \forall x \in X_0, \tag{6.5}$$

or

$$\langle g^* \circ f(x_0), x\rangle = \langle g^* \circ y, x\rangle, \quad \forall x \in X_0. \tag{6.6}$$

Thus the case 2 (α) is proved.

Let takes place the second case, i.e. the conditions 1 and 2(β) are fulfilled and $y \in Y$ is such that (6.3) holds, then we will search the approximate solutions in the form

$$x_m = g^{-1}\left(\sum_{k=1}^{m} c_{mk} y_k^*\right) \equiv g^{-1}\left(y_{(m)}^*\right), \quad i.e.\ x_m = g^{-1}\left(y_{(m)}^*\right) \tag{6.7}$$

where $\{y_k^*\}_{k=1}^{\infty} \subset Y^*$ is a complete system in the (separable) space Y^* and belongs to $g(X_1)$. The unknown coefficients c_{mk} might be determined from the system of algebraic equations

$$\widetilde{\Phi}_k(c_m) := \langle f(x_m), y_k^*\rangle - \langle y, y_k^*\rangle = 0, \quad k = 1, 2, ..., m \tag{6.8}$$

with $c_m \equiv (c_{m1}, c_{m2}, ..., c_{mm})$. Taking into account this and our conditions we get

$$\langle f(x_m), y_k^*\rangle - \langle y, y_k^*\rangle = \left\langle f\left(g^{-1}\left(y_{(m)}^*\right)\right), y_k^*\right\rangle - \langle y, y_k^*\rangle = 0, \tag{6.9}$$

for $k = 1, 2, ..., m$.

As it was observed above, the mapping

$$\widetilde{\Phi}(c_m) := \left(\widetilde{\Phi}_1(c_m), \widetilde{\Phi}_2(c_m), ..., \widetilde{\Phi}_m(c_m)\right)$$

is continuous by virtue of the conditions 1 and 2(β). Also, (6.3) implies the existence of such $\widetilde{r} > 0$ that the "acute angle" condition is fulfilled for all $y_{(m)}^*$ with $\left\|y_{(m)}^*\right\|_{Y^*} \geq \widetilde{r}$, i.e. for any $c_m \in S_{\widetilde{r}_1}^{R^m}(0) \subset R^m$, $\widetilde{r}_1 \geq \widetilde{r}$ the inequality

$$\sum_{k=1}^{m} \left\langle \widetilde{\Phi}_k(c_m), c_{mk} \right\rangle \equiv \left\langle f(x_m), \sum_{k=1}^{m} c_{mk} y_k^* \right\rangle - \left\langle y, \sum_{k=1}^{m} c_{mk} y_k^* \right\rangle$$

$$= \left\langle f\left(g^{-1}\left(y^*_{(m)}\right)\right), y^*_{(m)}\right\rangle - \left\langle y, y^*_{(m)}\right\rangle = \left\langle f\left(x_m\right), g\left(x_m\right)\right\rangle - \left\langle y, g\left(x_m\right)\right\rangle \geq 0,$$

$$\forall c_m \in \mathbb{R}^m, \|c_m\|_{\mathbb{R}^m} = \widetilde{r}_1.$$

holds by virtue of our conditions. Consequently, the solvability of system (6.8) (or (6.9)) for each $m = 1, 2, \ldots$ follows from the "acute angle" lemma as above. Thus, we obtained a sequence $\left\{y^*_{(m)} \mid m \geq 1\right\}$ of the approximate solutions, that is contained in a bounded subset of Y^*. This implyies an existence of a subsequence $\left\{y^*_{(m_j)}\right\}_{j=1}^{\infty}$ that converges weakly in Y^*. Consequently, the sequence $\left\{x_{m_j}\right\}_{j=1}^{\infty} \equiv \left\{g^{-1}\left(y^*_{(m_j)}\right)\right\}_{j=1}^{\infty}$ converges weakly in the space $\mathcal{M}_0$ by virtue of the condition 2(β) (may be after passing to the subsequence). It remains to pass to the limit in (6.9) by j and use a weak convergency of the subsequence of the sequence $\left\{y^*_{(m)} \mid m \geq 1\right\}$, of the weak compactness of mappings f and g^{-1}, and of the completeness of the system $\{y^*_k\}_{k=1}^{\infty}$in the space Y^*.

Hence, we get a limit element

$$x_0 = w - \lim_{j \nearrow \infty} x_{m_j} = w - \lim_{j \nearrow \infty} g^{-1}\left(y^*_{(m_j)}\right) \in \mathcal{M}_0$$

which is the solution of the equation

$$\left\langle f\left(x_0\right), y^*\right\rangle = \left\langle y, y^*\right\rangle, \quad \forall y^* \in Y^*. \tag{6.10}$$

Q.E.D.

Remark 6.1 *It is obvious that if there exists a function* $\psi : R^1_+ \longrightarrow R^1_+$, $\psi \in C^0$*such that* $\psi(\xi) = 0 \Longleftrightarrow \xi = 0$ *and the inequality* $\psi\left(\|x_1 - x_2\|_X\right) \leq \|f(x_1) - f(x_2)\|_Y$ *is fulfilled for all* $x_1, x_2 \in \mathcal{M}_0$, *then a solution of the equation (6.2) is unique.*

Corollary 6.1 *Assume that the conditions of Theorem 7 are fulfilled and there is a continuous function* $\varphi_1 : R^1_+ \longrightarrow R^1_+$ *such that* $\|g(x)\|_{Y^*} \leq \varphi_1\left([x]_{\mathcal{M}_0}\right)$ *for any* $x \in X_0$ *and* $\varphi(\tau) \nearrow +\infty$ *and* $\frac{\varphi(\tau)\tau}{\varphi_1(\tau)} \nearrow +\infty$ *as* $\tau \nearrow +\infty$. *Then, the equation (6.2) is solvable in* $\mathcal{M}_0$ *for any* $y \in Y$.

6.3 Appendix C

Now, we are ready to provide the proof of Theorem 2.2. Let $\left\{x^k\right\}_{k=1}^{\infty}$ be a complete system in the (separable) space X_0 and $\left\{\theta^s(t)\right\}_{s=1}^{\infty}$ be a complete system in the (separable) space $L^p(0, T)$, then $\left\{\theta^s(t)\, x^k\right\}_{s,k=1}^{\infty}$ is a complete system in the separable space $L^p(0, T; X_0)$.

Proof of the Theorem 2.2 We are going to use the method of elliptic regularization (see, for example, [10][3]). Namely, first we prove the solvability of the following auxiliary elliptic problem with a small parameter $\varepsilon > 0$.

$$-\varepsilon \frac{d^2 x_\varepsilon}{dt^2} + \frac{dx_\varepsilon}{dt} + f(t, x_\varepsilon(t)) = y(t), \tag{6.11}$$

$$x_\varepsilon(0) = 0, \quad \frac{dx_\varepsilon}{dt} \mid_{t=T} = 0, \ \varepsilon > 0. \tag{6.12}$$

[3] see, also, Soltanov K. N., Sprekels J. - Nonlinear equations in nonreflexive Banach spaces and fully nonlinear equations, Advances in Mathematical Sciences and Applications, 1999, v. 9, no. 2, 939-972.

A solution of the problem (6.11)-(6.12) would be understood as an element $x_\varepsilon(t) \in \underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y)$ that satisfies the following functional equation

$$\varepsilon\int_0^T \left\langle \frac{dx_\varepsilon}{dt}, \frac{dy^*}{dt} \right\rangle dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, y^* \right\rangle dt$$

$$+\int_0^T \langle f(t,x_\varepsilon(t)), y^*\rangle\, dt = \int_0^T \langle y, y^*\rangle\, dt \tag{6.13}$$

for any $y^* \in W^{1,q'}(0,T;Y^*) \cap \{y^*(t) \mid y^*(0) = 0\}$.

Lemma 6.1 *Under the conditions of Theorem 2.2 the equation (6.13) is solvable in the space* $\underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y)$ *for any* $y \in G$*, where* G *is defined in Theorem 2.2.*

The statement of this lemma follows from Theorem 6.1 of Appendix B (see, also [18, 20]). Indeed, the mapping generated by the considered problem (6.11)-(6.12) is weakly compact from

$$\underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y) \text{ into } \left(W^{1,q'}(0,T;Y^*) \cap \{y^*(t) \mid y^*(0) = 0\}\right)^*$$

by virtue of condition (*ii*) and because of first two terms are linear bounded operators. Moreover, inequalities

$$\varepsilon\int_0^T \left\langle \frac{dx_\varepsilon}{dt}, L\frac{dx_\varepsilon}{dt} \right\rangle dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, Lx_\varepsilon \right\rangle dt$$

$$+\int_0^T \langle f(t,x_\varepsilon(t)), Lx_\varepsilon\rangle\, dt - \int_0^T \langle y, Lx_\varepsilon\rangle\, dt \geq \varepsilon C_0 \left\|\frac{dx_\varepsilon}{dt}\right\|^{\nu}_{L^q(0,T;Y)}$$

$$+\varphi\left([x_\varepsilon]_{L^p(\mathcal{M}_0)}\right)[x_\varepsilon]_{L^p(\mathcal{M}_0)} - c_1\|y\|_{L^q(Y)}[x_\varepsilon]_{L^p(\mathcal{M}_0)} - (1+\varepsilon)C_2$$

$$\geq \left[\varphi\left([x_\varepsilon]_{L^p(0,T;\mathcal{M}_0)}\right) - c_1\|y\|_{L^q(0,T;Y)}\right][x_\varepsilon]_{L^p(0,T;\mathcal{M}_0)} - c \tag{6.14}$$

are fulfilled for any

$$x_\varepsilon \in W^{1,p}(0,T;X_0) \cap \{x_\varepsilon(t) \mid x_\varepsilon(0) = 0\}.$$

It is also clear that for a sufficiently large $p-$norm of $x_\varepsilon(t)$ there is a subset of $L^q(0,T;Y)$ such that the last expression in (6.14) is greater than zero under the conditions of Theorem 2.2. The arguments stated above and conditions *iii* and *iv* show that the other conditions of the above mentioned result are fulfilled. Consequently, the equation (6.13) is solvable (see also [20]). Thus, for each $y \in L^q(0,T;Y)$ there is a function $x_\varepsilon \in \underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y)$ that satisfies the equation (6.13) for any $\forall y^* \in W^{1,q'}(0,T;Y^*)$ i.e.

$$\varepsilon\int_0^T \left\langle \frac{dx_\varepsilon}{dt}, \frac{dy^*}{dt} \right\rangle dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, y^* \right\rangle dt$$

$$+\int_0^T \langle f(t,x_\varepsilon(t)), y^*\rangle\, dt = \int_0^T \langle y, y^*\rangle\, dt, \quad \forall y^* \in W_0^{1,q'}(0,T;Y^*). \tag{6.15}$$

The equality (6.15) can be rewritten in the form

$$\varepsilon\int_0^T \left\langle \frac{dx_\varepsilon}{dt}, \frac{dy^*}{dt}\right\rangle dt = \int_0^T \left\langle y - \frac{dx_\varepsilon}{dt} - f(t,x_\varepsilon(t)), y^*\right\rangle dt$$

where $y - \frac{dx_\varepsilon}{dt} - f(t,x_\varepsilon(t))$ belongs to $L^q(0,T;Y)$ because of $y \in L^q(0,T;Y)$ and $\frac{dx_\varepsilon}{dt}$, $f(t,x_\varepsilon(t)) \in L^q(0,T;Y)$ for any $x_\varepsilon \in \underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y)$. Hence, according to our conditions, for each fixed $\varepsilon > 0$ and of boundedness of the right part of (6.14) we obtain $\varepsilon\frac{d^2x_\varepsilon}{dt^2} \in L^q(0,T;Y)$. Consequently, the boundary condition $\frac{dx_\varepsilon}{dt}\mid_{t=T}$ is defined properly. Thus, the function $x_\varepsilon(t)$ is a solution of the equation

$$-\varepsilon\int_0^T \left\langle \frac{d^2x_\varepsilon}{dt^2}, y^*\right\rangle dt + \int_0^T \left\langle \frac{dx_\varepsilon}{dt}, y^*\right\rangle dt$$

$$+\int_0^T \langle f(t,x_\varepsilon(t)), y^*\rangle\, dt = \int_0^T \langle y, y^*\rangle\, dt, \quad \forall y^* \in L^{q'}(0,T;Y^*). \tag{6.16}$$

On the other hand, considering this equation for any $y^* \in W^{1,q'}(0,T;Y^*)$ and to make a comparison with (6.15) we obtain $\frac{dx_\varepsilon}{dt}\mid_{t=T} = 0$ (by using argumentations similar to arguments from [10,6,18]).

Hence, we proved that the problem (6.11)-(6.12) is solvable in the space

$$\underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y) \cap W^{\,2,q'}(0,T;Y) \cap \left\{ x_\varepsilon(t) \;\middle|\; \frac{dx_\varepsilon}{dt}\mid_{t=T} = 0 \right\}$$

for each fixed $\varepsilon > 0$.

Now, it is necessary to pass to the limit at $\varepsilon \searrow 0$. To this end, we need the uniformly by ϵ estimation of $\frac{dx_\varepsilon}{dt}$.

Further, we consider the equation

$$-\varepsilon\int_0^T \frac{d^2}{dt^2}\left\langle x_\varepsilon, Lx^k\right\rangle \theta^s(t)\, dt + \int_0^T \frac{d}{dt}\left\langle x_\varepsilon, Lx^k\right\rangle \theta^s(t)\, dt$$

$$= \int_0^T \left\langle y(t) - f(t,x_\varepsilon(t)), Lx^k\right\rangle \theta^s(t)\, dt \equiv \int_0^T \left\langle f_0(t,x_\varepsilon(t)), Lx^k\right\rangle \theta^s(t)\, dt, \tag{6.17}$$

where $\left\{\theta^s(t)\, x^k\right\}_{s,k=1}^{\infty}$ is a complete system in $W^{1,p}(0,T;X_0)$, and consequently, $\dfrac{dx_\varepsilon}{dt}$ is a solution of the problem

$$-\varepsilon\frac{d^2x_\varepsilon}{dt^2} + \frac{dx_\varepsilon}{dt} = f_{0\varepsilon}(t), \quad t \in (0,T) \tag{6.18}$$

$$x_\varepsilon(0) = 0, \quad \frac{dx_\varepsilon}{dt}\mid_{t=T} = 0, \tag{6.19}$$

as $x_\varepsilon(t)$ belongs to a bounded subset of $L^p(0,T;\mathcal{M}_0)$ and $f_0(t,x_\varepsilon(t))$ belongs to a bounded subset of $L^q(0,T;Y)$ for $\varepsilon\searrow 0$, consequently $f_{0\varepsilon}\in L^q(0,T)$ also belongs to a bounded subset of this space for $\varepsilon\searrow 0$, under the conditions of $y(t)$.

The solution of the problem (6.18)-(6.19) satisfies

$$\frac{dx_\varepsilon(t)}{dt}=\frac{1}{\varepsilon}\int_0^{T-t} f_{0\varepsilon}(T-\tau)\ exp\left\{-\frac{T-t-\tau}{\varepsilon}\right\}d\tau.$$

Applying the generalized Minkowski's inequality and taking into account that

$$\frac{1}{\varepsilon}\int_0^{\infty} exp\left\{-\frac{\tau}{\varepsilon}\right\}d\tau=1$$

we get $\left\|\frac{dx_\varepsilon}{dt}\right\|_{L^q(0,T;Y)}\le C<\infty$ for some positive C that is independent on ε. Thus, for each $y(t)\in L^q(0,T;Y)$ the function $x_\varepsilon(t)$ belongs to a bounded subset of the space $\underset{0}{\mathbf{P}}_{1,p,q}(0,T;\mathcal{M}_0,Y)$ uniformly on ε. The "reflexivity" of $\mathcal{M}_0$ and the reflexivity of Y allow us to pass to the limit for $\varepsilon\searrow 0$ in all terms of the (6.17) except for the first one. Therefore, it remains to estimate just the first term of (6.17). We have

$$\left|-\varepsilon\int_0^T\left\langle\frac{d^2x_\varepsilon}{dt^2},y^*\right\rangle dt\right|\le\varepsilon\int_0^T\left|\left\langle\frac{dx_\varepsilon}{dt},\frac{dy^*}{dt}\right\rangle\right|dt$$

$$\le\varepsilon\left\|\frac{dx_\varepsilon}{dt}\right\|_{L^q(0,T;Y)}\left\|\frac{dy^*}{dt}\right\|_{L^{q'}(0,T;Y^*)}$$

for any $y^*\in W^{1,q'}(0,T;Y^*)\cap\{y^*(t)\mid y^*(0)=0\}$. Taking into account the estimation $\varepsilon\left\|\frac{dx_\varepsilon}{dt}\right\|^{\nu}_{L^q(0,T;Y)}\le C<\infty$ for $\nu>1$, that is valid by virtue of the a priori estimations, we get

$$\left|-\varepsilon\int_0^T\left\langle\frac{d^2x_\varepsilon}{dt^2},y^*\right\rangle dt\right|\le\varepsilon^{\frac{\nu-1}{\nu}}\widetilde{C}\left\|\frac{dy^*}{dt}\right\|_{L^{q'}(0,T;Y^*)}.$$

This means that the first term of (6.17) vanishes when $\varepsilon\searrow 0$. Thus, considering the equation (6.17) for any $\xi\in L^p(0,T;X_0)$, passing to the limit at $\varepsilon\searrow 0$ and taking into account that $\ker L^*=\{0\}$ we complete the proof of Theorem 2.2.

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