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Basis properties of the system of eigenfunctions of a fourth order eigenvalue problem with spectral parameter in the boundary conditions

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Abstract. *In this paper we consider the eigenvalue problem for fourth order ordinary differential equation that describes the bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end of which is fixed rigidly, the right end is fixed elastically and on this end the inertial mass is concentrated. We investigate the location of eigenvalues on the real axis, the structure of root spaces and oscillation properties of eigenfunctions and their derivatives, we study the basis properties in the space* L_p, $1 < p < \infty$, *of the subsystems of eigenfunctions of this problem.*

Keywords. fourth order eigenvalue problem, the bending vibrations of a homogeneous rod, location of eigenvalues, oscillation of eigenfunctions, basis property of eigenfunctions

Mathematics Subject Classification (2010): 34B08, 34B24, 34C10, 34K11, 34L10, 47B50

1 Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x), \;\; 0 < x < 1, \tag{1.1}$$

$$y(0) = y'(0) = 0, \tag{1.2}$$

$$y''(1) - (a_1\lambda + b_1)\, y'(1) = 0 \tag{1.3}$$

$$Ty(1) - a_2\lambda y(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive and absolutely continuous function on $[0, l]$, a_1 and a_2 are real constants.

The problem (1.1)-(1.4) arises when variables are separated in the dynamical boundary value problem describing bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end of which is fixed rigidly, the right end is fixed elastically and on this end the inertial mass is concentrated (see [7, Ch. 8, § 5]).

The problem (1.1)-(1.4) was considered in [2, 8, 9] for $a_1 = 0$, in [1] for $a_2 = 0$, and in [3] for $a_1 > 0$, $a_2 < 0$ and $b_1 = 0$. In these papers the oscillation properties of eigenfunctions (and their derivatives) were investigated. Moreover, the basis properties of

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the system of eigenfunctions in $L_p(0,1),\ 1<p<\infty$ also studied, necessary and sufficient conditions for the basicity of subsystems of eigenfunctions is obtained.

Note that the signs of the parameters a_1 and a_2 play an important role. In the case $a_1>0$ and $a_2<0$, then problem (1.1)-(1.4), can be treated as a spectral problem for a self-adjoint operator in the Hilbert space $H=L_2(0,1)\oplus\mathbb{C}^2$ (see [3, 10]).

Throughout that follows we shall assume that the following conditions are satisfied:

$$a_1>0,\ a_2<0 \ \text{ and } \ b_1\in\mathbb{R}. \tag{1.5}$$

In this paper we investigate the location of the eigenvalues on the real axis, the structure of the root subspaces, the oscillation properties of the eigenfunctions and their derivatives and we study the basis properties in $L_p(0,1),\ 1<p<\infty$, of the subsystems of eigenfunctions of problem (1.1)-(1.4).

2 Preliminaries

Consider the boundary condition

$$y'(1)\cos\gamma - y''(1)\sin\gamma = 0, \tag{2.1}$$

where $\gamma\in[0,\frac{\pi}{2}]$.

Together the boundary value problem (1.1)-(1.4) we shall consider the spectral problem (1.1), (1.2), (2.1), (1.4). This problem has been considered in [8], where is proved the following result.

Theorem 2.1 [8, Theorem 5.1] *The eigenvalues of the boundary value problem* (1.1), (1.2), (2.1), (1.4) *are real, simple and form an infinitely increasing sequence* $\{\lambda_k(\gamma)\}_{k=1}^{\infty}$ *such that* $\lambda_k(\gamma)>0$ *for all* $k\in\mathbb{N}$. *Moreover, the eigenfunction* $u_k^{(\gamma)}(x)$ *corresponding to the eigenvalue* $\lambda_k(\gamma)$ *has* $k-1$ *simple zeros in the interval* $(0,1)$.

By virtue of [3, Theorem 3.1] for each fixed $\lambda\in\mathbb{C}$ there exists a unique (up to a constant factor) nontrivial solution $y(x,\lambda)$ of problem (1.1), (1.2), (1.4). The solution $y(x,\lambda)$ for each fixed $x\in[0,1]$ is an entire function of λ.

Let $A_k=(\lambda_{k-1}(0),\lambda_k(0))\,,\ k\in\mathbb{N}$, where $\lambda_0(0)=-\infty$.

It is clear that the eigenvalues $\lambda_k(0)$ and $\lambda_k(\pi/2),\ k\in\mathbb{N}$, of the boundary value problem (1.1), (1.2), (2.1), (1.4) for $\gamma=0$ and $\gamma=\pi/2$ are zeros of the entire functions $y'(1,\lambda)$ and $y''(1,\lambda)$, respectively. We observe that the function

$$F(\lambda)=y''(1,\lambda)/y'(1,\lambda)$$

is will defined for

$$\lambda\in A\equiv\left(\bigcup_{k=1}^{\infty}A_k\right)\bigcup(\mathbb{C}\backslash\mathbb{R}),$$

and is meromorphic function of finite order, $\lambda_k(\pi/2)$ and $\lambda_k(0),\ k\in\mathbb{N}$, are the zeros and poles of this function, respectively.

Lemma 2.1 [3, Lemma 3.3, formula (3.5)]. *The following relations hold:*

$$\frac{dF(\lambda)}{d\lambda}=-\frac{1}{y'^2(1,\lambda)}\left\{\int_0^1 y^2(x,\lambda)dx-a_2y^2(1,\lambda)\right\},\ \lambda\in A. \tag{2.2}$$

$$\lim_{\lambda\to-\infty}F(\lambda)=+\infty\,. \tag{2.3}$$

Remark 2.1 It follows by (2.2) and (2.3) that $y'(1,\lambda)\, y''(1,\lambda) > 0$ for $\lambda \in (-\infty, \lambda_1(\pi/2))$. Hence we have $F(0) = \frac{y''(1,0)}{y'(1,0)} > 0$.

By $\tau(\lambda)$ and $s(\lambda)$ we denote the number of zeros in the interval $(0,1)$ of functions $y(x,\lambda)$ and $y'(x,\lambda)$, respectively.

Theorem 2.2 [3, Theorem 3.2]. *If* $\lambda \in (0, \lambda_1(0)]$, *then* $\tau(\lambda) = s(\lambda) = 0$, *if* $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$ *and* $k \geq 2$, *then* $\tau(\lambda) = k-2$ *or* $\tau(\lambda) = k-1$, *if* $\lambda \in [\lambda_k(\pi/2), \lambda_k(0)]$ *and* $k \geq 2$, *then* $\tau(\lambda) = k-1$, *if* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$ *and* $k \geq 2$, *then* $s(\lambda) = k-1$.

3 Oscillatory properties of eigenfunctions of problem (1.1)-(1.4)

Lemma 3.1 *All eigenvalues of the boundary value problem* (1.1)-(1.4) *are real and simple and form an at most countable set without finite limit point.*

The proof of this lemma is similar to that of [3, Lemmas 4.1 and 4.2].
By virtue of Property 1 in [6] and formula (2.2), we have

$$\lambda_1\left(\frac{\pi}{2}\right) < \lambda_1(0) < \lambda_2\left(\frac{\pi}{2}\right) < \lambda_2(0) < \ \dots \ . \tag{3.1}$$

It is easy to see that the eigenvalues of problem (1.1)-(1.4) are the roots of the equation

$$y''(1,\lambda) - (a_1\lambda + b_1)\, y'(1,\lambda) = 0. \tag{3.2}$$

Remark 3.1 If λ is an eigenvalue of problem (1.1)-(1.4) then it follows from (3.1) that $y'(1,\lambda) \neq 0$.

By Remark 3.1 each root (with regard of multiplicities) of equation (3.1) is a root of the equation

$$F(\lambda) = a_1\lambda + b_1 \tag{3.3}$$

as well.

Remark 3.2 If $y(x,\lambda)$ is a solution of problem (1.1), (1.2), (1.4) for $\lambda < 0$, then it follows by [6, Theorem 5.4] that the zeros of this function contained in the interval $(0,1)$ are simple.

In view of [3, Corollary 3.1], as $\lambda < 0$ varies, the functions $y(x,\lambda)$ and $y'(x,\lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0,1]$ only through the endpoint $x = 0$. If these zeros pass through the point $x = 0$, then $x = 0$ would be a triple zero of function $y(x,\lambda)$, i.e. $y(0,\lambda) = y'(0,\lambda) = y''(0,\lambda) = 0$.
Let $\lambda < 0$ and μ is a real eigenvalue of the following boundary value problem

$$\begin{gathered} \ell(y)(x) = \lambda y(x), \ x \in (0,1), \\ y(0) = y'(0) = y''(0) = Ty(1) - a_2\lambda y(1) = 0. \end{gathered} \tag{3.4}$$

The oscillation index of this eigenvalue is the difference between the number of zeros of the solution $y(x,\lambda)$ of the problem (1.1), (1.2), (1.4) for $\lambda = \mu - 0$ belonging to the interval $(0,1)$ and the number of the same zeros for $\lambda = \mu + 0$ (see [5]). From this definition, it directly follows that the number of zeros of the function $y(x,\lambda)$ belonging to the interval (0, 1) is equal to the sum of the oscillation indices of all eigenvalues of problem (3.4) belonging to the interval $(\lambda, 0)$.

Lemma 3.2 *Then there exists $\zeta < 0$ such that the eigenvalues $\mu_k,\ k = 1, 2, \ldots,$ of problem* (3.4) *lying on the ray $(-\infty, \zeta)$ and enumerated in the decreasing order are simple, admit the asymptote*

$$\mu_k = -4\left(k\pi + \frac{\pi}{2}\right)^4 + o(k^4)$$

and have oscillation index 1.

Proof. The proof of this lemma is similar to that of [5, Theorem 4.1] with the use of asymptotic formula (2.3).

Let $\lambda < 0$ and $i(\mu_k)$ be the oscillation index of the eigenvalue $\mu_k,\ k \in \mathbb{N}$, of problem (3.4). Then, by condition (1.2), it follows from the above consideration that

$$s(\lambda) = \tau(\lambda) = \sum_{\mu_k \in (\lambda, 0)} i(\mu_k). \tag{3.5}$$

Let N_1 be a positive integer such that $\lambda_{N_1-1}(\pi/2) < -\frac{b_1}{a_1} \le \lambda_{N_1}(\pi/2)$ and let $\lambda_0(\pi/2) = -\infty$.

Theorem 3.1 *The eigenvalues of the boundary value problem* (1.1)-(1.4) *form an infinitely increasing sequence $\lambda_1,\ \lambda_2,\ \ldots, \lambda_k,\ \ldots$; moreover, $\lambda_k > 0$ for $k \ge 2$. The corresponding eigenfunctions $y_k(x),\ k = 1, 2, \ldots$ and their derivatives have the following oscillation properties:*

i) if $N_1 = 1$, then the functions $y_1(x)$ and $y_1'(x)$ have no zeros in the case $\lambda_1 \ge 0$, have $\sum\limits_{\mu_k \in (\lambda_1, 0)} i(\mu_k)$ simple zeros in the interval $(0,1)$ in the case $\lambda_1 < 0$, while the function $y_k(x)$ for $k \ge 2$ has either $k-2$ or $k-1$ simple zeros and the function $y_k'(x)$ has $k-1$ simple zeros in the interval $(0,1)$;

ii) if $N_1 > 1$, then the function $y_k(x)$ for $k < N_1$ has exactly $k-1$ simple zeros, for $k \ge N_1$ has either $k-2$ or $k-1$ simple zeros in the interval $(0,1)$, the function $y_k'(x)$ has $k-1$ simple zeros in the interval $(0,1)$.

Proof. Recall [see (3.3)] that the eigenvalues of problem (1.1)-(1.4) are the roots of the equation $F(\lambda) = a_1\lambda + b_1$. Since $a_2 < 0$ it follows from Lemma 2.1 that $F(\lambda) = \frac{y''(1,\lambda)}{y'(1,\lambda)}$ is a continuous decreasing function in the interval $A_k = (\lambda_{k-1}(0), \lambda_k(0))\,,\ k \in \mathbb{N}$. Taking into account of the relations (2.2), (2.3) and the representation (3.1), we have

$$\lim_{\lambda \to \lambda_{k-1}(0)+0} F(\lambda) = +\infty, \quad \lim_{\lambda \to \lambda_k(0)-0} F(\lambda) = -\infty\,.$$

Hence the function $F(\lambda)$ assumes each value in $(-\infty, +\infty)$ at a unique point in the interval $A_k,\ k \in \mathbb{N}$. By (1.5) the function $G(\lambda) = a_1\lambda + b_1$ is strictly increasing in the interval $(-\infty, +\infty)$.

It follows from the preceding considerations that in the interval $(\lambda_{k-1}(0), \lambda_k(0))$ the equation

$$F(\lambda) = G(\lambda)$$

has unique solution $\lambda = \lambda_k$, i.e. (1.3) holds. Consequently, λ_k is the k-th eigenvalue of the boundary value problem (1.1)-(1.4) and $y_k(x) = y(x, \lambda_k)$ is the corresponding eigenfunction.

It follows by Lemma 2.1 and Remark 2.1 that $F(\lambda) > 0$ for $\lambda \in (\lambda_{k-1}(0), \lambda_k(\pi/2))$, $F(\lambda_k(\pi/2)) = 0$ and $F(\lambda) < 0$ for $\lambda \in (\lambda_k(\pi/2), \lambda_k(0)),\ k = 1,\ 2,\ \ldots$. Since $a_1 > 0$ it follows that if $\lambda < -\frac{b_1}{a_1}$ then $G(\lambda) > 0$, if $\lambda > -\frac{b_1}{a_1}$ then $G(\lambda) < 0$; moreover, $G(-\frac{b_1}{a_1}) = 0$. Hence we have the following relations

$$\lambda_k \in (\lambda_k(\pi/2), \lambda_k(0)] \ \text{ for } \ k < N_1, \ \ \lambda_k \in (\lambda_{k-1}(0), \lambda_k(\pi/2)] \ \text{ for } \ k \ge N_1. \tag{3.6}$$

Moreover,

$$\lambda_1 < 0 \text{ for } b_1 > F(0),\ \lambda_1 = 0 \text{ for } b_1 = F(0) \text{ and } \lambda_1 > 0 \text{ for } b_1 < F(0). \tag{3.7}$$

Now the assertions i) and ii) of the theorem follows from Theorem 2.2 taking into account relations (3.8) and (3.9). The proof of Theorem 3.1 is complete.

Theorem 3.2 *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k - 3/2)\,\pi + O\,(1/k)\,, \tag{3.8}$$

$$\begin{aligned} y_k(x) = \sin(k-3/2)\pi x - \cos(k-3/2)\pi x + e^{-(k-3/2)\pi x} + \\ + (-1)^k e^{-(k-3/2)\pi(1-x)} + O(\,1/k), \end{aligned} \tag{3.9}$$

where relation (3.9) *holds uniformly for* $x \in [0, 1]$.

Proof. The proof of this theorem is similar to that of [8, Theorem 6.1].

4 Basis properties of the subsystems of eigenfunctions of the boundary value problem (1.1)-(1.4)

As is known (see [10]), the problem (1.1)-(1.4) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with inner product

$$(\hat{u}, \hat{v}) = (\{u, m, k\}, \{v, s, t\}) = \int_0^1 u(x)\overline{v(x)}\,dx + \ |a_1|^{-1} m\bar{s} \ + \ |a_2|^{-1} k\bar{t}, \tag{4.1}$$

where

$$L\hat{y} = L\{u, m, k\} = \{(Ty(x))'\,,\, y''(1) - b_1 y'(1),\, Ty(1)\},$$

is an operator with the domain

$$\begin{aligned} D(L) = \{\{y\ (x),\, m,\, k\}\ :\ y \in W_2^4(0,1),\ (Ty(x))' \in L_2(0,1), \\ y(0) = y'(0) = 0,\ m = a_1 y'(1),\ k = a_2 y(1)\} \end{aligned}$$

dense everywhere in H [10]. Obviously, the operator L is well defined in H. The eigenvalue problem (1.1)-(1.4) has the form

$$L\hat{y} = \lambda \hat{y},\ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_n, $n \in \mathbb{N}$, of the operator L and problem (1.1)-(1.4) coincide, and between the eigenfunctions, there is a one-to-one correspondence

$$y_n(x) \leftrightarrow \ \{y_n(x),\, m_n,\, k_n\},\ \ m_n = a_1 y_n'(1),\ k_n = a_2 y_n\,(1).$$

Theorem 4.1 *L is a self-adjoint operator in H. The system of eigenvectors $\{y_n(x),\ m_n,\ k_n\}$ of the operator L forms a Riesz basis (after normalization) in the space H.*

The proof of this theorem is similar to that of [3, Theorem 4.1].

We denote:

$$\delta_n = (\hat{y}_n, \hat{y}_n)\,.$$

By the conditions (1.5) and by (4.1) we have

$$\delta_n = \ ||y_n||^2_{L_2} + \, a_1^{-1} m_n^2 - a_2^{-1} k_n^2 > 0. \tag{4.2}$$

Hence, system of eigenvectors $\{\hat{v}_n\}_{n=1}^{\infty},\ \hat{v}_n = \delta_n^{-\frac{1}{2}} \hat{y}_n$, of operator L forms an orthonormal basis (i.e. Riesz basis) in H.

Let r and l ($r \neq l$) be arbitrary fixed natural numbers and

$$\tilde{\Delta}_{r,\,l} = \begin{vmatrix} a_1\delta_r^{-1/2} y_r'(1) & a_1\delta_l^{-1/2} y_l'(1) \\ a_2\delta_r^{-1/2} y_r(1) & a_2\delta_l^{-1/2} y_l(1) \end{vmatrix} = a_1 a_2 \delta_r^{-1}\delta_l^{-1} \begin{vmatrix} y_r'(1) & y_l'(1) \\ y_r(1) & y_l(1) \end{vmatrix}, \tag{4.3}$$

$$\Delta_{r,\,l} = \begin{vmatrix} y_r'(1) & y_l'(1) \\ y_r(1) & y_l(1) \end{vmatrix}. \tag{4.4}$$

Using (1.5) and (4.3), from (4.4) we get

$$\tilde{\Delta}_{r,\,l} \neq 0 \Leftrightarrow \Delta_{r,\,l} \neq 0. \tag{4.5}$$

Theorem 4.2 *If $\Delta_{r,\,l} \neq 0$, then the system of eigenfunctions $\{y_n(x)\}_{n=1,\,n\neq r,\,l}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$; if $\Delta_{r,\,l} = 0$, then this system is incomplete and nonminimal in the space $L_2(0,1)$.*

The proof of Theorem 4.2 for $p = 2$ is similar to that of [4, Theorem 4.1], by using Theorem 4.1 and relation (4.5), for $p \in (1, +\infty)\backslash\{2\}$ is similar to that of [9, Theorem 5.1] (see also [8, Theorem 8.1]) by using Theorem 4.2 (formulas (3.8)-(3.9)).

Using Theorem 4.2, we can obtain sufficient conditions for the system of eigenfunctions $\{y_n(x)\}_{n=1,\,n\neq r,\,l}^{\infty}$ of problem (1.1)-(1.4) to form a basis in $L_p(0,1), 1 < p < \infty$.

Remark 4.1 It follows from the asymptotic formulas (3.8) and [8, formula (6.1)] that there exist the positive integer $\tilde{k} \geq 2$ such that for $k \geq \tilde{k}$ the function $y_k(x)$ has exactly $k-2$ simple zeros in $(0,1)$. It is obvious that $\tilde{k} > N_1$.

Theorem 4.3 *Let $b_1 < 0, N_1 > 1,\ r < N_1$ and $l \geq \tilde{k}$. Then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,\,l}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$.*

Proof. Let $b_1 < 0$ and $N_1 > 1$. By Theorem 4.2, to prove of this theorem it suffices to show that $\Delta_{r,\,l} \neq 0$ if $r < N_1$ and $l \geq \tilde{k}$.

By (4.4) and Remark 3.1 we obtain

$$\Delta_{r,\,l} = y_r'(1) y_l'(1) \begin{vmatrix} \frac{y_r(1)}{y_1'(1)} & \frac{y_l(1)}{y_l'(1)} \\ 1 & 1 \end{vmatrix} = \ y_r'(1) y_l'(1) \left\{ \frac{y_r(1)}{y_r'(1)} - \frac{y_l(1)}{y_l'(1)} \right\}. \tag{4.6}$$

It follows from Theorem 3.1 and (1.2) that

$$\frac{y_k(1)}{y_k'(1)} > 0 \text{ for } k < N_1,\ \frac{y_k(1)}{y_k'(1)} < 0 \text{ for } l \geq \tilde{k}. \tag{4.7}$$

Using relations (4.7) from (4.6) we obtain $\Delta_{r,\,l} \neq 0$ for $r < N_1$ and $l \geq \tilde{k}$. The proof of this theorem is complete.

Theorem 4.4 *Let $b_1 > 0, r = 1$ and $l \geq \tilde{k}$. Then the system of eigenfunctions $\{y_k(x)\}_{k=1,\, k\neq r,\, l}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$.*

The proof of this theorem is similar to that of Theorem 4.3 with considering the relation $\frac{y_1(1)}{y_1'(1)} > 0$.

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