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On boundedness of the Riesz potential generated by Gegenbauer differential operator on Morrey spaces

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Abstract. *We consider the generalized shift operator, associated with the Gegenbauer differential operator* $G_\lambda = (x^2-1)^{\frac{1}{2}-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}$*. The maximal operator* M_G *(G-maximal operator) and the Riesz potential* I_G^α *(G-Riesz potential), associated with the generalized shift operator are investigated. At first, we prove that the G-maximal operator* M_G *is bounded from the G-Morrey space* $L_{p,\lambda,\gamma}$ *to* $L_{p,\lambda,\gamma}$ *for all* $1 < p < \infty$*, and is bounded from the G-Morrey space* $L_{1,\lambda,\gamma}$ *to the weak G- Morrey space* $WL_{q,\lambda,\gamma}$ *for* $0 \le \alpha < 2\lambda+1$*. Moreover we prove that the G- Riezs potential* I_G^α $0<\alpha<2\gamma+1$ *is bounded from the G-Morrey space* $L_{p,\lambda,\gamma}$ *to* $L_{q,\lambda,\gamma}$ *if and only if* $\frac{\alpha}{2\lambda+1-\gamma} = \frac{1}{p}-\frac{1}{q}$, $1<p<\frac{2\lambda+1-\gamma}{\alpha}$.

Also we prove that the G-Riesz potential I_G^α *is bounded from the G-Morrey space* $L_{1,\lambda,\gamma}$ *to the weak G-Morrey space* $WL_{q,\lambda,\gamma}$ *if and only if* $\frac{\alpha}{2\lambda+1-\gamma} = 1-\frac{1}{q}$

Keywords. G-Riesz potential, G-maximal function, G-Morrey space.

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1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r. Let $f \in L_1^{loc}(\mathbb{R}^n)$. The maximal operator M and the Riesz potential I^α are defined by

$$Mf(x) = \sup_{t>0}\frac{1}{|B(x,t)|}\int\limits_{B(x,t)}|f(y)|dy,$$

$$I^\alpha f(x) = \int\limits_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-\alpha}},\ 0<\alpha<n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

The operators M and I^α play important role in real and harmonic analysis.

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In the theory of partial differential equations Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play important role. They were introduced by C. Morrey in 1938 [9] and defined as follows: For $0 \le \lambda \le n$, $1 \le p < \infty$, $f \in L_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{L_{p,\lambda}} \equiv \|f\|_{L_{p,\lambda(\mathbb{R}^n)}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $L_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $L_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Also by $WL_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda(\mathbb{R}^n)}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

F. Chiarenza and M. Frasca [2] studied the boundedness of the maximal operator M in Morrey spaces $L_{p,\lambda}$. Their results can be summarized as follows:

Theorem A. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p < \infty$.

1) *If* $1 < p < \infty$, *then* H *is bounded from* $L_{p,\lambda}$ to $L_{p,\lambda}$,

2) *If* $p = 1$, *then* M *is bounded* $L_{1,\lambda}$ to $WL_{1,\lambda}$.

The classical result by Hardy-Littlewood-Sobolev stats that $1 < p < q < \infty$, then I^α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = \frac{n}{p} - \frac{n}{q}$ and for $p = 1 < q < \infty$ is bounded from $L_{1,\lambda}$ to $WL_{1,\lambda}$ if and only if $\alpha = n - \frac{n}{q}$. D.R.Adams [1] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement.

Theorem B. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p < \frac{n-\lambda}{\alpha}$.

1) *If* $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of* I^α *from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$.

2) *If* $p = 1$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of* I^α *from* $L_{1,\lambda}(\mathbb{R}^n)$ *to* $WL_{q,\lambda}(\mathbb{R}^n)$.

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of theorem B reduced to the abovementioned result by Hardy-Littlewood -Sobolev.

In the paper [5] authors consider the generalized shift operator, generated by the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\gamma_i}{x_i}\frac{\partial_i}{\partial x_i}, \qquad \gamma_1 > 0, ..., \gamma_k > 0$$

in terms of which the B-maximal operator and the B-Riesz potential are investigated in the B-Morrey spaces.

They obtain necessary and sufficient conditions for the operator I_γ^α to be bounded from B-Morrey space $L_{p,\lambda,\gamma}$ to $L_{q,\lambda,\gamma}$ or to weak B-Morrey space $WL_{q,\lambda,\gamma}$.

The results obtained in the paper are analogue of [5].

2 Definitions and auxiliary results

Analogy by [5] we introduce the following notation.

Definition 2.1 *Let* $1 \le p < \infty$, $0 \le \gamma \le 1+2\lambda$. *We denote by* $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ *Gegenbauer-Morrey (G-Morrey space) associated with the Gegenbauer differential operator* G_λ *as the* *set of locally integrable functions* $f(x)$, $x \in \mathbb{R}_+ = [0,\infty)$, *with the finite norm*

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f(cht)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}},$$

where $H(x,r) = (x-r, x+r) \cap \mathbb{R}_+$, *i.e.*

$$H(x,r) = \begin{cases} (0, x+r), 0 \le x < r, \\ (x-r, x+r), r \le x < \infty. \end{cases}$$

Definition 2.2 *Let* $1 \le p < \infty$, $0 \le \gamma \le 1 + 2\lambda$. *We denote by* $WL_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ *the weak* $L_{p,\lambda,\gamma}$ *space defined as the set of locally integrable functions* $f(x)$, $x \in \mathbb{R}_+$ *with the finite norm*

$$\|f\|_{WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)} = \sup_{r>0} r \sup_{x\in\mathbb{R}_+, t>0} \left(t^{-\gamma} |\{y \in H(x,t) : |f(chy)| > r\}|_\gamma \right)^{\frac{1}{p}}.$$

Notice that

$$WL_{p,\lambda}(\mathbb{R}_+, G) = WL_{p,\lambda,0}(\mathbb{R}_+, G),$$

$$L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+, G)$$

and $\|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}}$.

In [7] the Gegenbauer maximal function is defined as follows:

$$M_G^* f(chx) = \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int\limits_0^r A_{cht} |f(chx)| sh^{2\lambda} t dt,$$

$$A_{cht} f(chx) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int\limits_0^\pi f(chxcht - shxsht \cos\varphi)(\sin\varphi)^{2\lambda-1} d\varphi$$

is the Gegenbauer generalized shift function,

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} |f(cht)| sh^{2\lambda} t dt,$$

where

$$|H(0,r)|_\lambda = \int\limits_0^r sh^{2\lambda} t dt, \quad |H(x,r)|_\lambda = \int\limits_{H(x,r)} sh^{2\lambda} t dt,$$

moreover

$$M_G^* f(chx) \lesssim M_G f(chx). \tag{2.1}$$

Here $A \lesssim B$ denotes that exists the constant $c > 0$ such that $0 < A \le cB$, moreover C can depend on some parameters.

Symbol $A \approx B$ denote that $A \lesssim B$ and $B \lesssim A$.

The following result are analogue of the according inequality to [4].

Theorem 2.1 *For every nonnegative function* $g \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$ *for every* $1 \leq p < \infty$ *and* $0 < t < \infty$ *the inequality*

$$\int\limits_{H(x,r)} (M_G f(chy))^p \, g(chy) sh^{2\lambda} y dy \leq \int\limits_{H(x,r)} |f(chy)|^p M_G g(chy) sh^{2\lambda} y dy$$

is true.

Proof. By Hölder's inequality, we have

$$(M_G f(chy))^p = \Big(\sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int\limits_{H(y,r)} |f(cht)| sh^{2\lambda} t dt \Big)^p$$

$$\leq \sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \Big(\int\limits_{H(y,r)} |f(cht)|^p \, sh^{2\lambda} t dt \Big) \Big(\int\limits_{H(y,r)} sh^{2\lambda} t dt \Big)^{\frac{p}{p'}}$$

$$= \sup_{r>0} \frac{|H(y,r)|_\lambda^{\frac{p}{p'}}}{|H(y,r)|_\lambda^{p}} \int\limits_{H(y,r)} |f(cht)|^p \, sh^{2\lambda} t dt$$

$$= \sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int\limits_{H(y,r)} |f(cht)|^p \, sh^{2\lambda} t dt,$$

where $p' = p/(p-1)$. From this we get

$$\int\limits_{H(x,r)} (M_G f(chy))^p \, g(chy) sh^{2\lambda} y dy$$

$$\leq \int\limits_{H(x,r)} \Big(\sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int\limits_{H(y,r)} |f(cht)|^p \, sh^{2\lambda} t dt \Big) g(chy) sh^{2\lambda} y dy$$

$$= \int\limits_{H(x,r)} |f(chy)|^p \Big(\sup_{r>0} \frac{1}{|H(y,r)|_\lambda} \int\limits_{H(y,r)} g(cht) sh^{2\lambda} t dt \Big) sh^{2\lambda} y dy$$

$$= \int\limits_{H(x,r)} |f(chy)|^p \, M_G g(chy) sh^{2\lambda} y dy. \qquad \square$$

Theorem 2.2 *The following Chebyshev type inequality*

$$|\{y \in H(x,r) : M_G f(chy) > \alpha\}|_\lambda \leq \frac{1}{\alpha} \int\limits_{H(x,r)} M_G f(chy) sh^{2\lambda} y dy$$

for all $\alpha > 0$ *and* $t > 0$ *is valid.*

Proof. Since

$$M_G f(chy) \geq \alpha \mathcal{X}_{\{M_G f(chy)>\alpha\}}(chy),$$

then

$$\int\limits_{H(x,r)} M_G f(chy) sh^{2\lambda} y dy \geq \alpha \int\limits_{H(x,r)} \mathcal{X}_{\{M_G f(chy)>\alpha\}}(chy) sh^{2\lambda} y dy$$
$$= \alpha \left|\{y \in H(x,r) : M_G f(chy) > \alpha\}\right|_\lambda .$$

The assertion of the theorem, follows from the last inequality.

Further we will need some auxiliary assertions.

Lemma 2.1 [7] *For $0 < \lambda < \frac{1}{2}$ the following relations*

$$|H(0,r)|_\lambda \approx \begin{cases} (sh\frac{r}{2})^{2\lambda+1}, 0 < r < 2, \\ (sh\frac{r}{2})^{4\lambda}, \quad 2 \leq r < \infty \end{cases}$$

is true.

Lemma 2.2 *For $r > 0$, $0 < \lambda < \frac{1}{2}$ and $x \in [0,\infty)$ the following relations*

$$M_G \mathcal{X}_{H(x,r)}(chy) \approx \begin{cases} \left(\frac{sh\frac{r}{2}}{sh\frac{|x-y|+r}{2}}\right)^{2\lambda+1}, 0 < x+r < 2, \\ \left(\frac{sh\frac{r}{2}}{sh\frac{|x-y|+r}{2}}\right)^{4\lambda}, 2 \leq x+r < \infty \end{cases}$$

is true.

Proof. Let $0 \leq x < r$. Then

$$M_G \mathcal{X}_{H(0,r)}(chx) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \mathcal{X}_{H(0,r)}(cht) sh^{2\lambda} t dt$$

$$= \sup_{r>0} \frac{|(0,r) \cap (0,x+r)|_\lambda}{|(0,x+r)|_\lambda} = \sup_{r>0} \frac{\int\limits_0^r sh^{2\lambda} t dt}{\int\limits_0^{x+r} sh^{2\lambda} t dt}.$$

We estimate $|H(x,r)|_\lambda$. Let $0 \leq x < r$ and $0 < x+r < 2$. Then

$$|H(x,r)|_\lambda = \int\limits_0^{x+r} sh^{2\lambda} t dt \geq \int\limits_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt$$
$$\geq \frac{x+r}{2}\left(sh\frac{x+r}{2}\right)^{2\lambda} \geq e^{-1}\left(sh\frac{x+r}{2}\right)^{2\lambda+1}, \tag{2.2}$$

since $sht \leq et$ for $0 \leq t \leq 1$. On the other hand,

$$|H(x,r)|_\lambda = \int_0^{x+r} sh^{2\lambda} t dt = 2^{2\lambda} \int_0^{x+r} \left(sh\frac{t}{2} ch\frac{t}{2} \right)^{2\lambda} dt$$

$$= 2^{2\lambda+1} \int_0^{x+r} \left(sh\frac{t}{2} \right)^{2\lambda} \left(ch\frac{t}{2} \right)^{2\lambda-1} d(sh\frac{t}{2})$$

$$\leq 2^{2\lambda+1} \int_0^{x+r} \left(sh\frac{t}{2} \right)^{2\lambda} d(sh\frac{t}{2}) \leq \frac{2^{\lambda+1}}{2\lambda+1} \left(sh\frac{x+r}{2} \right)^{2\lambda+1}. \tag{2.3}$$

From (2.2) and (2.3) for $0 \leq x < r$ and $0 < x + r < 2$ as follows

$$|H(x,r)|_\lambda \approx \left(sh\frac{x+r}{2} \right)^{2\lambda+1}. \tag{2.4}$$

Now let $0 \leq x < r$ and $2 \leq x + r < \infty$. Then

$$|H(x,r)|_\lambda = \int_0^{x+r} sh^{2\lambda} t dt \geq \int_{\frac{x+r}{2}}^{x+r} \frac{sh^{2\lambda} t d(sht)}{cht}$$

$$\geq \int_{\frac{x+r}{2}}^{x+r} sh^{2\lambda-1} t d(sht) = \frac{1}{4\lambda} \left(sh^{2\lambda}(x+r) - sh^{2\lambda}\frac{x+r}{2} \right)$$

$$\geq \frac{1}{4\lambda} \left(sh^{2\lambda}(x+r) - 4^{-\lambda} sh^{2\lambda}(x+r) \right) = \frac{1}{4\lambda} \left(1 - 4^{-\lambda} \right) sh^{2\lambda}(x+r)$$

$$= \frac{4^\lambda - 1}{4\lambda \cdot 4^\lambda} \left(2sh\frac{x+r}{2} ch\frac{x+r}{2} \right)^{2\lambda} \geq \frac{4^\lambda - 1}{4\lambda} \left(sh\frac{x+r}{2} \right)^{4\lambda}. \tag{2.5}$$

On the other hand

$$|H(x,r)|_\lambda = \int_0^{x+r} sh^{2\lambda} t dt = 2^{2\lambda+1} \int_0^{x+r} \frac{\left(sh\frac{t}{2} \right)^{2\lambda} d(sh\frac{t}{2})}{\left(ch\frac{t}{2} \right)^{1-2\lambda}}$$

$$\leq 2^{2\lambda+1} \int_0^{x+r} \left(sh\frac{t}{2} \right)^{4\lambda-1} d(sh\frac{t}{2}) = \frac{2^{2\lambda+1}}{4\lambda} \left(sh\frac{x+r}{2} \right)^{4\lambda} \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$|H(x,r)|_\lambda \approx \left(sh\frac{x+r}{2} \right)^{4\lambda}. \tag{2.7}$$

by $0 \leq x < r$ and $2 \leq x + r < \infty$

Combining (2.4) and (2.7) we obtain

$$|H(x,r)|_\lambda \approx \begin{cases} \left(sh\frac{x+r}{2} \right)^{2\lambda+1}, 0 < x + r < 2, \\ \left(sh\frac{x+r}{2} \right)^{4\lambda}, 2 \leq x + r < \infty, \end{cases} \tag{2.8}$$

by $0 \le x < r$.

Now we consider the case then $r \le x < \infty$. Then $H(x,r) = (x-r, x+r)$. Let $0 < x+r < 2$. From (2.2) we have

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda} t dt \ge \int\limits_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \ge e^{-1}\left(sh\frac{x+r}{2}\right)^{2\lambda+1}. \tag{2.9}$$

On the other hand from (2.3) we obtain

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda} t dt \le \int\limits_{0}^{x+r} sh^{2\lambda} t dt \le \frac{2^{\lambda+1}}{2\lambda+1}\left(sh\frac{x+r}{2}\right)^{2\lambda+1}. \tag{2.10}$$

Remain consider the case, then $2 \le x+r < \infty$. By (2.5) we have

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda} t dt \ge \int\limits_{\frac{x+r}{2}}^{x+r} sh^{2\lambda} t dt \ge \frac{4^\lambda - 1}{4\lambda}\left(sh^{2\lambda}\frac{x+r}{2}\right)^{4\lambda}. \tag{2.11}$$

On the other hand by (2.6) we get

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda} t dt \le \int\limits_{0}^{x+r} sh^{2\lambda} t dt \le \frac{2^{\lambda+1}}{4\lambda}\left(sh\frac{x+r}{2}\right)^{4\lambda}. \tag{2.12}$$

Combining (2.9)-(2.12) by $r \le x < \infty$, we obtain

$$|H(x,r)|_\lambda \approx \begin{cases} \left(sh\frac{x+r}{2}\right)^{2\lambda+1}, \ 0 < x+r < 2, \\ \left(sh\frac{x+r}{2}\right)^{4\lambda}, \ 2 \le x+r < \infty, \end{cases} \tag{2.13}$$

Now from (2.8) and (2.13) we have

$$|H(x,r)|_\lambda \approx \begin{cases} \left(sh\frac{x+r}{2}\right)^{2\lambda+1}, 0 < x+r < 2, \\ \left(sh\frac{x+r}{2}\right)^{4\lambda}, \ 2 \le x+r < \infty, \end{cases} \tag{2.14}$$

by $0 \le x < \infty$ and $0 < r < \infty$. Now let $r \le x < \infty$. Then

$$M_G \mathcal{X}_{H(0,r)}(chx) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \mathcal{X}_{H(0,r)}(cht) sh^{2\lambda} t dt$$

$$= \sup_{r>0} \frac{|(0,r) \cap (x-r, x+r)|_\lambda}{|(x-r,x+r)|_\lambda} = \sup_{r>0} \frac{\int_{x-r}^{r} sh^{2\lambda} t dt}{\int_{x-r}^{x+r} sh^{2\lambda} t dt}. \tag{2.15}$$

Let $0 < r < x \le 2$, then

$$\int\limits_{x-r}^{r} sh^{2\lambda} t dt \ge \int\limits_{\frac{r}{2}}^{r} sh^{2\lambda} t dt \ge \frac{r}{2} sh^{2\lambda}\frac{r}{2} \ge \frac{1}{e}\left(sh\frac{r}{2}\right)^{2\lambda+1}. \tag{2.16}$$

On the other hand

$$\int_{x-r}^{r} sh^{2\lambda}tdt \leq \int_{0}^{r} sh^{2\lambda}tdt = 2^{2\lambda+1}\int_{0}^{r} \frac{\left(sh\frac{t}{2}\right)^{2\lambda} d(sh\frac{t}{2})}{\left(ch\frac{t}{2}\right)^{1-2\lambda}}$$

$$\leq 2^{2\lambda+1}\int_{0}^{r}\left(sh\frac{t}{2}\right)^{2\lambda} d(sh\frac{t}{2}) = \frac{2^{\lambda+1}}{2\lambda+1}\left(sh\frac{r}{2}\right)^{2\lambda+1}. \tag{2.17}$$

Thus from (2.16) and (2.17) we obtain that

$$\int_{x-r}^{r} sh^{2\lambda}tdt \approx \left(sh\frac{r}{2}\right)^{2\lambda+1}, \tag{2.18}$$

where $0 < r < x \leq 2$.

Now let $2 \leq r < x < \infty$, then

$$\int_{x-r}^{r} sh^{2\lambda}tdt \geq \int_{\frac{r}{2}}^{r} sh^{2\lambda}tdt = 2^{2\lambda+1}\int_{\frac{r}{2}}^{r} \frac{\left(sh\frac{t}{2}\right)^{2\lambda} d(sh\frac{t}{2})}{\left(ch\frac{t}{2}\right)^{1-2\lambda}}$$

$$\geq \frac{2^{2\lambda+1}}{2\lambda+1}\left(ch\frac{r}{2}\right)^{2\lambda-1}\left[\left(sh\frac{r}{2}\right)^{2\lambda+1} - \left(sh\frac{r}{4}\right)^{2\lambda+1}\right]$$

$$\geq \frac{2^{\lambda+1}}{2\lambda+1}\left(2sh\frac{r}{2}\right)^{2\lambda-1}\left[\left(sh\frac{r}{2}\right)^{2\lambda+1} - \left(2^{-(2\lambda+1)}sh\frac{r}{2}\right)^{2\lambda+1}\right]$$

$$= \frac{2^{4\lambda}(2^{\lambda+1}-1)}{(2\lambda+1)2^{2\lambda+1}}\left(sh\frac{r}{2}\right)^{2\lambda+1}\left(sh\frac{r}{2}\right)^{2\lambda-1} \tag{2.19}$$

$$= \frac{2^{2\lambda-1}(2^{2\lambda+1}-1)}{2\lambda+1}\left(sh\frac{r}{2}\right)^{4\lambda} \geq \frac{2^{2\lambda-1}}{2\lambda+1}\left(sh\frac{r}{2}\right)^{4\lambda}.$$

On the other hand

$$\int_{x-r}^{r} sh^{2\lambda}tdt \leq 2^{2\lambda+1}\int_{0}^{r} \frac{\left(sh\frac{t}{2}\right)^{2\lambda} d(sh\frac{t}{2})}{\left(ch\frac{t}{2}\right)^{1-2\lambda}}$$

$$\leq 2^{\lambda+1}\int_{0}^{r}\left(sh\frac{t}{2}\right)^{4\lambda-1} d(sh\frac{t}{2}) = \frac{2^{2\lambda-1}}{\lambda}\left(sh\frac{r}{2}\right)^{4\lambda}. \tag{2.20}$$

Combining (2.19) and (2.20) we have

$$\int_{x-r}^{r} sh^{2\lambda}tdt \approx \left(sh\frac{r}{2}\right)^{4\lambda}, \tag{2.21}$$

where $2 \leq r < x < \infty$.

Now from (2.18) and (2.21) we obtain

$$\int_{x-r}^{r} sh^{2\lambda}tdt \approx \begin{cases} \left(sh\frac{r}{2}\right)^{2\lambda+1}, 0 < r < x \leq 2, \\ \left(sh\frac{r}{2}\right)^{4\lambda}, 2 \leq r < x < \infty. \end{cases} \tag{2.22}$$

Further from (2.14) and (2.22) it follows that

$$M_G\mathcal{X}_{H(0,r)}(chx) \approx \begin{cases} \left(\frac{sh\frac{r}{2}}{sh\frac{x+r}{2}}\right)^{2\lambda+1}, 0 < x+r < 2, \\ \left(\frac{sh\frac{r}{2}}{sh\frac{x+r}{2}}\right)^{4\lambda}, 2 \le x+r < \infty. \end{cases} \tag{2.23}$$

Further

$$M_G\mathcal{X}_{H(0,t)}(chx) = \sup_{r>0} \frac{1}{|H(x,r)|_\lambda} \int\limits_{H(x,r)} \mathcal{X}_{H(0,t)}(chu) sh^{2\lambda} u du.$$

From this we have

$$M_G\mathcal{X}_{H(0,t)}(ch(y-z)) = \sup_{r>0} \frac{1}{|H(y-z,r)|_\lambda} \int\limits_{H(y-z,r)} \mathcal{X}_{H(0,t)}(chu) sh^{2\lambda} u du$$

$$= \sup_{r>0} \frac{1}{|H(y-z,r)|_\lambda} \int\limits_{H(y,r)} \mathcal{X}_{H(0,t)}(ch(v-z)) sh^{2\lambda}(v-z) dv$$

$$= \sup_{r>0} \frac{1}{|H(y-z,r)|_\lambda} \int\limits_{H(y,r)} \mathcal{X}_{H(z,t)}(chv) sh^{2\lambda} v dv = M_G\mathcal{X}_{H(z,t)}(chy).$$

Thus

$$\mathcal{X}_{H(0,t)}(ch(y-z)) = \begin{cases} 1, y-z \in H(0,t), \\ 0, y-z \notin H(0,t), \end{cases}$$

$$\Leftrightarrow \mathcal{X}_{H(z,t)}(chy) = \begin{cases} 1, y \in H(z,t), \\ 0, y \notin H(z,t). \end{cases} \tag{2.24}$$

The assertion of Lemma 2.2 it follows from (2.23) and (2.24)

Theorem 2.3 *1) If $f \in L_{1,\lambda,\gamma}(\mathbb{R}_+, G)$, $0 \le \gamma < 1+2\lambda$, then $M_G f \in WL_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ and*

$$\|M_G f\|_{WL_{1,\lambda,\gamma}} \lesssim \|f\|_{L_{1,\lambda,\gamma}}.$$

2) If $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$, $1 < p < \infty$, $0 \le \gamma < 1+2\lambda$, then $M_G f \in WL_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and

$$\|M_G f\|_{L_{p,\lambda,\gamma}} \lesssim \|f\|_{L_{p,\lambda,\gamma}}.$$

Proof. 1) By definition

$$\|M_G f\|_{WL_{1,\lambda,\gamma}(\mathbb{R}_+,G)} = \sup_{r>0} r \sup_{x\in\mathbb{R}_+, t>0} t^{-\gamma} \left|\{y \in H(x,t) : M_G f(chy) > r\}\right|_\lambda.$$

Now using the Theorem 2.2 and also the Theorem 2.1 by $p = 1$ and $g(chy) \equiv 1$ we get

$$\|M_G f\|_{WL_{1,\lambda,\gamma}(\mathbb{R}_+,G)} \le \sup_{x\in\mathbb{R}_+, t>0} t^{-\gamma} \int\limits_{H(x,r)} |f(chy)|\, sh^{2\lambda} y dy = \|f\|_{L_{1,\lambda,\gamma}}.$$

2) Using the theorem by $p > 1$ and $g(chy) \equiv 1$, second assertion of the theorem we obtain.

3 The Hardy - Littlewood - Sobolev inequality for the G-Riesz potential in Morrey spaces

We consider the G-Riesz potential (see [7])

$$I_G^\alpha f(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr \right) A_{cht} f(chx) sh^{2\lambda} t dt,$$

where $h_r(cht) = \int_1^\infty e^{-\gamma(\gamma+2\lambda)r} P_\gamma^\lambda(cht)(\gamma^2-1)^{\lambda-\frac{1}{2}} d\gamma$, and $A_{cht} f(chx) \equiv A_{cht}^\lambda f(chx)$.

Theorem 3.1 *Let* $0 < \alpha < 2\lambda+1$, $0 < \gamma < 2\lambda+1-\alpha$ *and* $1 \le p < \frac{2\lambda+1-\gamma}{\alpha}$.

1) If $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$*, then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$ *is necessary and sufficient for the boundedness of* I_G^α *from* $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ *to* $L_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

2) If $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$*, then condition* $1 - \frac{1}{p} = \frac{\alpha}{2\lambda+1-\gamma}$ *is necessary and sufficient for the boundedness of* I_G^α *from* $L_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ *and* $WL_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

Proof. 1) Sufficiency. Let $0 < \alpha < 1+2\lambda$, $0 < \gamma < 2\lambda+1-\alpha$, $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$. For I_G^α the following estimate holds (see [7], Corollary 3.1)

$$\begin{aligned} |I_G^\alpha f(chx)| &\lesssim \int_0^\infty A_{cht} (shx)^{\alpha-2\lambda-1} |f(cht)| sh^{2\lambda} t dt \\ &= \left(\int_0^r + \int_r^\infty \right) A_{cht} |f(chx)| (shx)^{\alpha-2\lambda-1} sh^{2\lambda} t dt \\ &= A_1(x,r) + A_2(x,r). \end{aligned} \tag{3.1}$$

We estimate $A_1(x,r)$. Let $0 < r < 2$. Then taking into account (2.1) we get

$$A_1(x,r) \lesssim \int_0^r \frac{A_{cht} |f(chx)| sh^{2\lambda} t dt}{(sht)^{2\lambda+1-\alpha}}$$

$$\lesssim \sum_{k=0}^\infty \int_{2^{-(k+1)}r}^{2^{-k}r} \frac{A_{cht} |f(chx)| sh^{2\lambda} t dt}{(sht)^{2\lambda+1-\alpha}}$$

$$\lesssim \sum_{k=0}^\infty \left(sh \frac{r}{2^{k+1}} \right)^\alpha \left(sh \frac{r}{2^{k+1}} \right)^{-2\lambda-1} \int_0^{2^{-k}r} A_{cht} |f(chx)| sh^{2\lambda} t dt$$

$$\lesssim (shr)^\alpha M_G^* f(chx) \left(\sum_{k=0}^\infty 2^{-(k+1)\alpha} \right) \lesssim (shr)^\alpha M_G f(chx). \tag{3.2}$$

Let $2 \leq r < \infty$ and $0 < \alpha < 4\lambda$. From the proof of Corollary 3.1 on [7] it follows that

$$A_1(x,r) = \int_0^r \frac{A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt}{(cht)^{2\lambda+1-\alpha}} \leq \int_0^r \frac{A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt}{(cht)^{4\lambda-\alpha}}$$
$$\leq \int_0^r \frac{A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt}{(sht)^{4\lambda-\alpha}} \leq \sum_{k=0}^{\infty} \int_{2^{-(k+1)}r}^{2^{-k}r} \frac{A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt}{(sht)^{4\lambda-\alpha}}$$
$$\lesssim \sum_{k=0}^{\infty} \left(sh\frac{r}{2^{k+1}}\right)^{\alpha} \left(sh\frac{r}{2^{k+1}}\right)^{-4\lambda} \int_0^{2^{-k}r} A_{cht}|f(chx)|sh^{2\lambda}tdt$$
$$\lesssim M_G f(chx) \sum_{k=0}^{\infty} \left(sh\frac{r}{2^{k+1}}\right) \lesssim (shr)^\alpha M_G f(chx), 0 < \alpha < 4\lambda. \tag{3.3}$$

Now let $4\lambda \leq \alpha$. From the proof of Corollary 3.1 on [7] it follows that $|I^\alpha_G f(chx)| \lesssim 1$, so we have

$$|A_1(x,r)| \lesssim \int_0^r A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt$$
$$= \frac{(sh\frac{r}{2})^{4\lambda}}{(sh\frac{r}{2})^{4\lambda}} \int_0^r A^\lambda_{cht}|f(chx)|sh^{2\lambda}tdt$$
$$\leq (sh\frac{r}{2})^{4\lambda} M^*_G f(chx) \leq (shr)^\alpha M_G f(chx), \;\; 4\lambda \leq \alpha < 2\lambda + 1. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we have

$$A_1(x,r) \lesssim (shr)^\alpha M_G f(chx), \;\; 0 < \alpha < 2\lambda + 1, \;\; 0 < r < \infty. \tag{3.5}$$

Further by Hölder's inequality

$$A_{cht}|f(chx)| = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^\pi |f(chxcht - shxsht\cos\varphi)|(\sin\varphi)^{2\lambda-1}d\varphi$$
$$\leq \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \left(\int_0^\pi |f(chxcht - shxsht\cos\varphi)|^p(\sin\varphi)^{2\lambda-1}d\varphi\right)^{\frac{1}{p}} \left(\int_0^\pi (\sin\varphi)^{2\lambda-1}\right)^{\frac{1}{p'}}$$
$$\leq \left(\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\right)^{\frac{1}{p}} \left(\int_0^\pi |f(chxcht - shxsht\cos\varphi)|^p(\sin\varphi)^{2\lambda-1}d\varphi\right)^{\frac{1}{p}}.$$

From this it follows that

$$(A_{cht}|f(chx)|)^p \leq A_{cht}|f(chx)|^p.$$

But then (2.1) it follows that

$$\frac{1}{|H(0,r)|_\lambda}\int_0^r \left(A_{cht}|f(chx)|\right)^p sh^{2\lambda}tdt \lesssim \frac{1}{|H(0,r)|_\lambda}\int_0^r A_{cht}|f(chx)|^p sh^{2\lambda}tdt$$
$$\lesssim \frac{1}{|H(x,r)|_\lambda}\int_{H(x,r)} |f(chx)|^p sh^{2\lambda}tdt.$$

Now taking into account of Lemma 2.1 and relation (2.14), we obtain

$$\int_0^r \left(A_{cht}|f(chx)|\right)^p sh^{2\lambda}tdt \lesssim \frac{|H(0,r)|_\lambda}{|H(x,r)|_\lambda}\int_{H(x,r)} |f(chx)|^p sh^{2\lambda}tdt$$
$$\lesssim \int_{H(x,r)} |f(chx)|^p sh^{2\lambda}tdt,$$

from this it follows that

$$\|A_{cht}f\|_{L_{p,\lambda,\gamma}} \lesssim \|f\|_{L_{p,\lambda,\gamma}}.$$

We estimate $A_2(x,r)$. By Hölder's inequality

$$A_2(x,r) \le \left(\int_r^\infty \left(A_{cht}|f(chx)|\right)^p (sht)^{-\beta} sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$
$$\times \left(\int_r^\infty (sht)^{(\frac{\beta}{p}+\alpha-2\lambda-1)p'} sh^{2\lambda}tdt\right)^{\frac{1}{p'}} = A_{21}\cdot A_{22}. \qquad (3.6)$$

Let $\gamma < \beta < 2\lambda + 1 - p\alpha$. For A_{21} we have

$$A_{21} \le \left(\sum_{j=0}^\infty \int_{2^j r}^{2^{j+1}r} \left(A_{cht}|f(chx)|\right)^p (sht)^{-\beta} sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$
$$\lesssim \left(\sum_{j=0}^\infty \left(sh2^j r\right)^{\gamma-\beta}\left(sh2^j r\right)^{-\gamma} \int_{2^j r}^{2^{j+1}r} \left(A_{cht}|f(chx)|\right)^p sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$
$$\lesssim \left(\sum_{j=0}^\infty \left(sh2^j r\right)^{\gamma-\beta}\left(2^j r\right)^{-\gamma} \int_{2^j r}^{2^{j+1}r} \left(A_{cht}|f(chx)|\right)^p sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$
$$\lesssim (shr)^{\frac{\gamma-\beta}{p}} \|A_{cht}f\|_{L_{p,\lambda,\gamma}} \left(\sum_{j=0}^\infty 2^{(\gamma-\beta)j}\right)^{\frac{1}{p}} \lesssim (shr)^{\frac{\gamma-\beta}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \qquad (3.7)$$

And for A_{22} we have

$$A_{22} \le \left(\int_r^\infty (sht)^{(\frac{\beta}{p}+\alpha-2\lambda-1)p'} sh^{2\lambda} tdt \right)^{\frac{1}{p'}} \lesssim \left(\int_r^\infty (sht)^{(\frac{\beta}{p}+\alpha-2\lambda-1)p'+2\lambda} d(sht) \right)^{\frac{1}{p'}}$$

$$\lesssim (shr)^{\frac{\beta}{p}+\alpha-2\lambda-1+\frac{2\lambda+1}{p'}} = (shr)^{\frac{\beta}{p}+\alpha-2\lambda-1+(2\lambda+1)(1-\frac{1}{p})} \lesssim (shr)^{\frac{\beta}{p}+\alpha-\frac{2\lambda+1}{p}}. \quad (3.8)$$

Taking into account (3.7) and (3.8) in (3.6) we obtain

$$A_2(x,r) \lesssim (shr)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \quad (3.9)$$

Now, from (3.5), (3.8) and (3.1) we get

$$|I_G^\alpha f(x)| \lesssim \Big((shr)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}} + (shr)^\alpha M_G f(chx) \Big). \quad (3.10)$$

The right-hand attains its minimum at

$$shr = \left(\frac{2\lambda+1-\gamma}{p\alpha} - 1 \right)^{\frac{p}{2\lambda+1-\gamma}} \left(\frac{\|f\|_{L_{p,\lambda,\gamma}}}{M_G f(chx)} \right)^{\frac{p}{2\lambda+1-\gamma}}. \quad (3.11)$$

Now, taking into account (3.11) and (3.10), we have

$$|I_G^\alpha f(chx)| \lesssim \left(\frac{M_G f(chx)}{\|f\|_{L_{p,\lambda,\gamma}}} \right)^{1-\frac{p\alpha}{2\lambda+1-\gamma}} \|f\|_{L_{p,\lambda,\gamma}}$$

$$= (M_G f(chx))^{1-\frac{p\alpha}{2\lambda+1-\gamma}} \|f\|_{L_{p,\lambda,\gamma}}^{\frac{p\alpha}{2\lambda+1-\gamma}}. \quad (3.12)$$

Now from (3.12) by Theorem 2.3 we have

$$\int_{H(x,r)} |I_G^\alpha f(chx)|^q \, sh^{2\lambda} tdt$$

$$\lesssim \|f\|_{L_{p,\lambda,\gamma}}^{\frac{\alpha pq}{2\lambda+1-\gamma}} \left(\int_{H(x,r)} (M_G f(chx))^{q-\frac{\alpha pq}{2\lambda+1-\gamma}} \, sh^{2\lambda} tdt \right)$$

$$\lesssim r^\gamma \|f\|_{L_{p,\lambda,\gamma}}^{\frac{\alpha pq}{2\lambda+1-\gamma}} \|f\|_{L_{p,\lambda,\gamma}}^{q-\frac{\alpha pq}{2\lambda+1-\gamma}} = r^\gamma \|f\|_{L_{p,\lambda,\gamma}}^q, \quad (3.13)$$

where $0 < r < 2$.

From (3.13), we obtain

$$\|I_G^\alpha f\|_{q,\lambda,\gamma} \lesssim \|f\|_{L_{p,\lambda,\gamma}},$$

i.e. I_G^α is bounded from $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $L_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

Necessity. Let $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, $f \in L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ and I_G^α be bounded from $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ to $L_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

Moreover assume that $f(x) > 0$ is increasing. The dilation function $f_t(chx)$ we define as follows:

$$\begin{cases} f(ch(tht)x) \le f_t(chx) \le f(ch(ctht)x), 0 < t < 1, \\ f(ch(tht)x) \le f_t(chx) \le f(ch(sht)x), 1 \le t < \infty. \end{cases} \quad (3.14)$$

Then for $1 \le t < \infty$ we have

$$\|f_t\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(x,r)} |f_t(chy)|^p\, sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\ge \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(x,r)} |f(ch(tht)y)|^p\, sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$[(tht)y = u, y = (ctht)u, dy = (ctht)du]$$

$$= (ctht)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(xtht, rtht)} |f(chu)|^p\, sh^{2\lambda}(ctht)udu\right)^{\frac{1}{p}}$$

$$\ge (ctht)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(xtht, rtht)} |f(chu)|^p\, sh^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$= (ctht)^{\frac{2\lambda+1}{p}}\left(\sup_{r>0}\frac{rtht}{r}\right)^{\frac{2\lambda-1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (ctht)^{\frac{2\lambda+1-\gamma}{p}}\|f\|_{L_{p,\lambda,\gamma}}$$

$$= \left(\frac{cht}{sht}\right)^{\frac{2\lambda+1-\gamma}{p}}\|f\|_{L_{p,\lambda,\gamma}} \ge \frac{(cht)^{\alpha}}{(sht)^{\frac{2\lambda+1-\gamma}{p}}}\|f\|_{L_{p,\lambda,\gamma}}$$

$$\ge (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}}\|f\|_{L_{p,\lambda,\gamma}}. \tag{3.15}$$

On the other hand,

$$\|f_t\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(x,r)} |f_t(chy)|^p\, sh^{2\lambda} y dy\right)^{\frac{1}{p}}$$

$$\le \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(x,r)} |f(ch(sht)y)|^p\, sh^{2\lambda} y dy\right)^{\frac{1}{p}} \quad [(sht)y = u, dy = \frac{du}{sht}]$$

$$= (sht)^{-\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(xsht, rsht)} |f(chu)|^p\, sh^{2\lambda}\frac{u}{sht}du\right)^{\frac{1}{p}}$$

$$\le (sht)^{-\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0}\left(r^{-\gamma}\int\limits_{H(xsht, rsht)} |f(chu)|^p\, sh^{2\lambda}udu\right)^{\frac{1}{p}}$$

$$= (sht)^{-\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rsht}{r}\right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}}$$

$$= (sht)^{-\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}} \le (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.16}$$

Consider the case when $0 < t < 1$. From (3.14) we have

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f_t(chy)|^p \, sh^{2\lambda} y dy\right)^{\frac{1}{p}} \\
&\le \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f(ch(ctht)y)|^p \, sh^{2\lambda} y dy\right)^{\frac{1}{p}} \\
&[(ctht)y = u, dy = (tht)du] \\
&= (tht)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xctht, rctht)} |f(chu)|^p \, sh^{2\lambda}(tht)udu\right)^{\frac{1}{p}} \\
&\le (tht)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xctht, rctht)} |f(chu)|^p \, sh^{2\lambda} udu\right)^{\frac{1}{p}} \\
&= (tht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rctht}{r}\right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (tht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= \left(\frac{sht}{cht}\right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \lesssim \frac{1}{(cht)^{\frac{2\lambda+1-\gamma}{p}-\alpha}} \|f\|_{L_{p,\lambda,\gamma}} \\
&\le (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}.
\end{aligned} \tag{3.17}$$

On the other hand,

$$\begin{aligned}
\|f_t\|_{L_{p,\lambda,\gamma}} &= \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f_t(chy)|^p \, sh^{2\lambda} y dy\right)^{\frac{1}{p}} \\
&\ge \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f(ch(tht)y)|^p \, sh^{2\lambda} y dy\right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&[(tht)y = u, dy = (ctht)du] \\
&= (ctht)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xtht, rtht)} |f(chu)|^p \, sh^{2\lambda}(ctht)udu \right)^{\frac{1}{p}} \\
&\geq (ctht)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xtht, rtht)} |f(chu)|^p \, sh^{2\lambda}udu \right)^{\frac{1}{p}} \\
&= (ctht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rtht}{r} \right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (ctht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \\
&= \left(\frac{cht}{sht} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \geq \frac{(cht)^\alpha}{(sht)^{\frac{2\lambda+1-\gamma}{p}}} \|f\|_{L_{p,\lambda,\gamma}} \\
&\geq (sht)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}} . \qquad (3.18)
\end{aligned}$$

From (3.15)-(3.18) it follows

$$\|f_t\|_{L_{p,\lambda,\gamma}} \approx (t)^{\alpha + \frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}} , \qquad (3.19)$$

where $0 < t < \infty$.

By the definition of (G-Riesz) potential we can write

$$I_G^\alpha f_t(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int\limits_0^\infty \left(\int\limits_0^\infty u^{\frac{\alpha}{2}-1} h_u(chv)du \right) A_{chv} f_t(chx) sh^{2\lambda}vdv.$$

Using (3.14) we have

$$\begin{aligned}
\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} &\leq \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f_t(chy)|^q \, sh^{2\lambda}ydy \right)^{\frac{1}{q}} \\
&\leq \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f(ch(ctht)y)|^q \, sh^{2\lambda}ydy \right)^{\frac{1}{q}}
\end{aligned}$$

$$[(ctht)y = z, dy = (tht)dz]$$

$$= (tht)^{\frac{1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xctht, rctht)} |I_G^\alpha f(chz)|^q \, sh^{2\lambda}(tht)zdz \right)^{\frac{1}{q}}$$

$$\leq (tht)^{\frac{2\lambda+1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xctht, rctht)} |I_G^\alpha f(chz)|^q \, sh^{2\lambda}(tht)zdz \right)^{\frac{1}{q}}$$

$$= (tht)^{\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{rctht}{r} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = (tht)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}$$

$$= \frac{1}{(ctht)^{\frac{2\lambda+1-\gamma}{q}}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} \lesssim (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, \tag{3.20}$$

where $0 < t < 1$.

On the other hand,

$$\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f_t(chy)|^q \, sh^{2\lambda}ydy \right)^{\frac{1}{q}}$$

$$\geq \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f(ch(tht)y)|^q \, sh^{2\lambda}ydy \right)^{\frac{1}{q}} \quad [(tht)y = z, dy = (ctht)dz]$$

$$\geq (ctht)^{\frac{1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xtht, rtht)} |I_G^\alpha f(chz)|^q \, sh^{2\lambda}(ctht)zdz \right)^{\frac{1}{q}}$$

$$\geq (ctht)^{\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{rtht}{r} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = \frac{(cht)^{\frac{2\lambda+1-\gamma}{q}}}{(sht)^{\frac{2\lambda+1-\gamma}{q}}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}$$

$$\geq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}. \tag{3.21}$$

Combining (3.20) and (3.21) we obtain that

$$\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, 0 < t < 1. \tag{3.22}$$

Consider the case, when $1 \leq t < \infty$. By (3.14), we have

$$\|f_t\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f_t(chy)|^p \, sh^{2\lambda}ydy \right)^{\frac{1}{p}}$$

$$\leq \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |f(ch(ctht)y)|^p \, sh^{2\lambda}ydy \right)^{\frac{1}{p}}$$

$$[(ctht)y = u, dy = (tht)du]$$

$$= (tht)^{\frac{1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xctht, rctht)} |f(chu)|^p \, sh^{2\lambda}(tht)udu \right)^{\frac{1}{p}}$$

$$\leq (tht)^{\frac{2\lambda+1}{p}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xctht, rctht)} |f(chu)|^p \, sh^{2\lambda}udu \right)^{\frac{1}{p}}$$

$$= (tht)^{\frac{2\lambda+1}{p}} \left(\sup_{r>0} \frac{rctht}{r} \right)^{\frac{\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} = (tht)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}}$$

$$= \left(\frac{sht}{cht} \right)^{\frac{2\lambda+1-\gamma}{p}} \|f\|_{L_{p,\lambda,\gamma}} \lesssim \frac{1}{(cht)^{\frac{2\lambda+1-\gamma}{p}-\alpha}} \|f\|_{L_{p,\lambda,\gamma}}$$

$$\leq (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}} \|f\|_{L_{p,\lambda,\gamma}}. \tag{3.23}$$

On the other hand,

$$\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} \leq \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(x,r)} |I_G^\alpha f(ch(sht)y)|^q \, sh^{2\lambda}ydy \right)^{\frac{1}{q}}$$

$$[(sht)y = z, dy = \frac{dz}{sht}]$$

$$= (sht)^{-\frac{1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{-\gamma} \int\limits_{H(xsht, rsht)} |I_G^\alpha f(chz)|^q \, sh^{2\lambda} \left(\frac{z}{sht} \right) dz \right)^{\frac{1}{q}}$$

$$= (sht)^{-\frac{2\lambda+1}{q}} \sup_{x\in\mathbb{R}_+, r>0} \left(r^{\gamma} \int\limits_{H(xsht, rsht)} |I_G^\alpha f(chz)|^q \, sh^{2\lambda}zdz \right)^{\frac{1}{q}}$$

$$= (sht)^{-\frac{2\lambda+1}{q}} \left(\sup_{r>0} \frac{rsht}{r} \right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}} = (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}. \tag{3.24}$$

From (3.23) and (3.24), we have

$$\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, 1 \leq t < \infty. \tag{3.25}$$

Finally, by (3.22), (3.23) and (3.25) we arrive at

$$\|I_G^\alpha f_t\|_{L_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{L_{q,\lambda,\gamma}}, 0 < t < \infty, \tag{3.26}$$

for all $0 < t < \infty$.

Since I_G^{α} is bounded from $L_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ to $L_{q,\lambda,\gamma}(\mathbb{R}_+,G)$, i.e.

$$\|I_G^{\alpha}f\|_{L_{q,\lambda,\gamma}} \lesssim \|f_t\|_{L_{p,\lambda,\gamma}},$$

then taking into account (3.26) and (3.19), we obtain

$$\begin{aligned}\|I_G^{\alpha}f\|_{L_{q,\lambda,\gamma}} &\approx (sht)^{\frac{2\lambda+1-\gamma}{q}}\,\|I_G^{\alpha}f_t\|_{L_{q,\lambda,\gamma}}\\ &\le (sht)^{\frac{2\lambda+1-\gamma}{q}}\,\|f_t\|_{L_{q,\lambda,\gamma}} \le (sht)^{\alpha+\frac{\gamma-2\lambda-1}{p}}\,(sht)^{\frac{2\lambda+1-\gamma}{q}}\,\|f_t\|_{L_{q,\lambda,\gamma}}\\ &\le (sht)^{\alpha+(\gamma-2\lambda-1)(\frac{1}{p}-\frac{1}{q})}\,\|f\|_{L_{p,\lambda,\gamma}}\\ &= (sht)^{\alpha+(2\lambda+1)(\frac{1}{q}-\frac{1}{p})}(sht)^{\gamma(\frac{1}{p}-\frac{1}{q})}\,\|f\|_{L_{p,\lambda,\gamma}}\\ &\lesssim \|f\|_{L_{p,\lambda,\gamma}}\begin{cases}(sht)^{\alpha+(2\lambda+1)(\frac{1}{p}-\frac{1}{q})}, & 0<t<1,\\ (sht)^{\alpha+(2\lambda+1-\gamma)(\frac{1}{q}-\frac{1}{p})}, & 1\le t<\infty.\end{cases}\end{aligned}$$

If $\frac{1}{p}<\frac{1}{q}+\frac{\alpha}{2\lambda+1}$, then at $t\to 0$ we have $\|I_G^{\alpha}f\|_{L_{q,\lambda,\gamma}}=0$ for all $f\in L_{p,\lambda,\gamma}(\mathbb{R}_+)$. If $\frac{1}{p}>\frac{1}{q}+\frac{\alpha}{2\lambda+1-\gamma}$, then at $t\to\infty$ $\|I_G^{\alpha}f\|_{L_{q,\lambda,\gamma}}=0$. for all $f\in L_{p,\lambda,\gamma}$.

Therefore $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1-\gamma}$.

Sufficiency. Let $f\in L_{1,\lambda,\gamma}(G)$. Then

$$\left|\left\{y\in H(0,r):|I_G^{\alpha}f(chy)|>2\beta\right\}\right|_{\lambda}$$

$$\le\left|\left\{y\in H(0,r):A_1(y,r)>\beta\right\}\right|_{\lambda}+\left|\left\{y\in H(x,r):A_2(y,r)>\beta\right\}\right|_{\lambda}.$$

From (3.1) we have

$$\begin{aligned}A_2(y,r) &\le \int_r^{\infty}\frac{A_{cht}|f(cht)|sh^{2\lambda}tdt}{(sht)^{2\lambda+1-\alpha}} \le \sum_{j=0}^{\infty}\int_{2^j r}^{2^{j+1}r}\frac{A_{cht}|f(cht)|sh^{2\lambda}tdt}{(sht)^{2\lambda+1-\alpha}}\\ &\le \|A_{cht}f\|_{L_{1,\lambda,\gamma}}\sum_{j=0}^{\infty}\frac{\left(2^j r\right)^{\gamma}}{(sh2^j r)^{2\lambda+1-\alpha}}\\ &\le r^{\gamma}(shr)^{\alpha-2\lambda-1}\,\|f\|_{L_{1,\lambda,\gamma}}\sum_{j=0}^{\infty}2^{(\gamma+\alpha-2\lambda-1)j}\\ &\lesssim (shr)^{\gamma+\alpha-2\lambda-1}\|f\|_{L_{1,\lambda,\gamma}}.\end{aligned}\tag{3.27}$$

Denoting

$$\beta=(shr)^{\gamma+\alpha-2\lambda-1}\|f\|_{L_{1,\lambda,\gamma}}$$

from (3.27) we obtain that $|A_2(y,r)|\le\beta$ and, consequently, $|y\in H(0,r):A_2(y,r)>\beta|_{\gamma}=0$.

Taking into account inequality (3.5), we obtain

$$\begin{aligned}|\{y\in H(x,r):A_1(y,r)>\beta\}|_{\lambda} &\le \left|\left\{y\in H(x,r):M_Gf(chy)>\frac{\beta}{c\cdot sh^{\alpha}r}\right\}\right|_{\lambda}\\ &\le \frac{sh^{\alpha}r}{\beta}\,\|f\|_{L_{1,\lambda,\gamma}}.\end{aligned}$$

Thus we have

$$|\{y\in H(x,r):|I_G^\alpha f(chy)|>2\beta\}|_\lambda \leq \frac{sh^\alpha r}{\beta}\|f\|_{L_{1,\lambda,\gamma}} = (shr)^{\gamma-2\lambda-1}$$

$$= \left(\frac{1}{\beta}\|f\|_{L_{1,\lambda,\gamma}}\right)^{\frac{2\lambda+1-\gamma}{2\lambda+1-\alpha-\gamma}} = \left(\frac{1}{\beta}\|f\|_{L_{1,\lambda,\gamma}}\right)^q.$$

Necessity. We preliminarily establish estimates for $\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}$. From (3.24) at $0<t<1$ we have

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \geq \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0}\left(u^{-\gamma}\int\limits_{\{y\in H(x,u):\left|I_G^\alpha f(ch(tht)y)\right|>r\}} sh^{2\lambda}y dy\right)^{\frac{1}{q}}$$

$$[(tht)y=z, dy=(ctht)dz]$$

$$=(ctht)^{\frac{1}{q}}\sup_{r>0} r\sup_{x\in\mathbb{R}_+,u>0}\left(u^{-\gamma}\int\limits_{\{y\in H(xtht,utht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}(ctht)zdz\right)^{\frac{1}{q}}$$

$$=(ctht)^{\frac{1}{q}}\sup_{r>0} r\sup_{u>0}\left(\frac{utht}{u}\right)^{\frac{\gamma}{q}}$$

$$\times\sup_{x\in\mathbb{R}_+,u>0}\left((utht)^{-\gamma}\int\limits_{\{y\in H(xtht,utht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}(ctht)zdz\right)^{\frac{1}{q}}$$

$$\geq (ctht)^{\frac{2\lambda+1}{q}}(tht)^{\frac{\gamma}{q}}\sup_{r>0} r\sup_{x\in\mathbb{R}_+,u>0}\left((utht)^{1-\gamma-2\lambda}\int\limits_{\{y\in H(xtht,utht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}zdz\right)^{\frac{1}{q}}$$

$$=(tht)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \geq (sht)^{\frac{\gamma-2\lambda-1}{q}}\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.28}$$

On the other hand,

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \leq \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0}\left(u^{-\gamma}\int\limits_{\{y\in H(x,u):\left|I_G^\alpha f(ch(ctht)y)\right|>r\}} sh^{2\lambda}y dy\right)^{\frac{1}{q}}$$

$$[(ctht)y=z, dy=(tht)dz]$$

$$= (tht)^{\frac{1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \left(u^{-\gamma} \int\limits_{\{y\in H(xctht,uctht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}(tht)zdz \right)^{\frac{1}{q}}$$

$$\leq (tht)^{\frac{2\lambda+1}{q}} \sup_{u>0} \left(\frac{uctht}{u}\right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}$$

$$= (tht)^{\frac{2\lambda+1-\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \leq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} . \quad (3.29)$$

From (3.28) and (3.29) we obtain that for $0 < t < 1$

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} . \quad (3.30)$$

Consider the case when $1 \leq t < \infty$. From (3.14) we have

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \geq \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \Big(u^{-\gamma} \int\limits_{\{y\in H(x,u):\left|I_G^\alpha f(ch(tht)y)\right|>r\}} sh^{2\lambda}ydy \Big)^{\frac{1}{q}}$$

$$[(tht)y = z, dy = (ctht)dz]$$

$$= (ctht)^{\frac{1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \Big(u^{-\gamma} \int\limits_{\{y\in H(xtht,utht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}(ctht)zdz \Big)^{\frac{1}{q}}$$

$$= (ctht)^{\frac{2\lambda+1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \Big(u^{-\gamma} \int\limits_{\{y\in H(xtht,utht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}zdz \Big)^{\frac{1}{q}}$$

$$= (ctht)^{\frac{2\lambda+1}{q}} \sup_{u>0} \left(\frac{utht}{u}\right)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}$$

$$= (tht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} \geq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} . \quad (3.31)$$

On the other hand

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \leq \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \Big(u^{-\gamma} \int\limits_{\{y\in H(x,u):\left|I_G^\alpha f(ch(sht)y)\right|>r\}} sh^{2\lambda}ydy \Big)^{\frac{1}{q}}$$

$$[(sht)y = z, dy = (\frac{1}{sht})dz]$$

$$= (sht)^{\frac{1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \Big(u^{-\gamma} \int\limits_{\{y\in H(xsht,usht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}\Big(\frac{1}{sht}\Big)zdz \Big)^{\frac{1}{q}}$$

$$\leq (sht)^{-\frac{2\lambda+1}{q}} \sup_{r>0} r \sup_{x\in\mathbb{R}_+,u>0} \Big(u^{-\gamma} \int\limits_{\{y\in H(xsht,usht):\left|I_G^\alpha f(chz)\right|>r\}} sh^{2\lambda}zdz \Big)^{\frac{1}{q}}$$

$$= (sht)^{-\frac{2\lambda+1}{q}} \sup_{u>0} \Big(\frac{usht}{u}\Big)^{\frac{\gamma}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}$$

$$\leq (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} . \quad (3.32)$$

From (3.31) and (3.32) we obtain that for $1 \le t < \infty$

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.33}$$

Finally from (3.30) and (3.33) we obtain that for every $0 < t < \infty$

$$\|I_G^\alpha f_t\|_{WL_{q,\lambda,\gamma}} \approx (sht)^{\frac{\gamma-2\lambda-1}{q}} \|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}}. \tag{3.34}$$

From the boundedness of I_G^α from $L_{1,\lambda,\gamma}(\mathbb{R}_+)$ to $WL_{q,\lambda,\gamma}(\mathbb{R}_+)$ on using (3.19) and (3.34), we have

$$\begin{aligned}\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} &\lesssim (sht)^{\frac{2\lambda+1-\gamma}{q}} (sht)^{\alpha+\gamma-2\lambda-1} \|f\|_{L_{1,\lambda,\gamma}} = (sht)^{\alpha-(2\lambda+1-\gamma)(1-\frac{1}{q})} \|f\|_{L_{1,\lambda,\gamma}} \\ &= (sht)^{\alpha-(2\lambda+1)(1-\frac{1}{q})} (sht)^{\gamma(1-\frac{1}{q})} \|f\|_{L_{1,\lambda,\gamma}} \\ &\lesssim \|f\|_{L_{1,\lambda,\gamma}} \begin{cases} (sht)^{\alpha-(2\lambda+1)(1-\frac{1}{q})}, \ 0 < t < 1, \\ (sht)^{\alpha+(\gamma-2\lambda-1)(1-\frac{1}{q})}, \ 1 \le t < \infty. \end{cases}\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{2\lambda+1-\gamma}$, then tending $t \to 0$ $\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} = 0$ for all $f \in L_{1,\lambda,\gamma}(\mathbb{R}_+, G)$. If $1 > \frac{1}{q} + \frac{\alpha}{2\lambda+1-\gamma}$, then tending $t \to \infty$ we obtain that $\|I_G^\alpha f\|_{WL_{q,\lambda,\gamma}} = 0$ for all $f \in L_{1,\lambda,\gamma}(\mathbb{R}_+, G)$.

Thus $1 - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$.

Theorem is proved.

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