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Inverse scattering problem on the semi-axis for a system of six ordinary differential equations

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Abstract. *In this article, the inverse scattering problem of recovering the matrix coefficient of a first order system of ordinary differential equations on the half-axis from its scattering matrix is considered. For an ordinary system of six equations on a semi-axis, under joint consideration of four problems the inverse scattering problem on a first approximation was solved. The coefficients of the considered system are uniquely determined by the scattering matrix on a semi-axis.*

Keywords. Scattering problem · scattering matrix · transformation matrix · factorization

Mathematics Subject Classification (2010): 35J25, 35R30

1 Introduction

In this article, we study inverse scattering problem on a semi-axis in the case when there are four given incident waves and two scattering waves.

The theorem about existness and uniqueness of solution of inverse scattering problem on a semi-axis when there are no singular numbers is proved.

Inverse scattering problem for a system of Dirac equations on a semi-axis with a self-adjoint potential was studied in the papers of M.G.Gasymov, M.G. Gasymov and B.M. Levitan [2,3], but for a system of two ordinary equations on a semi-axis with a nonselfadjoint potential was studied in the papers of L.P. Nizhnik and F.L. Vu [7].

Inverse scattering problem for the case $n \geq 3$ on a whole axis was studied in the papers of D. Kaup [6], A.B. Shabat and V.E. Zakharov [8], R. Beals and R.R. Coifman [1], but on a semi axis was studied in the papers of N.Sh.Iskenderov and A.A. Mamedov [4], K.A. Jabbarova [5].

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2 Scattering problem

On a semi-axis $x \geq 0$ consider a system of six equations for a first order of the form:

$$-i\frac{dy_k\left(x\right)}{dx}+\sum_{j=1}^{6}c_{kj}\left(x\right)y_j(x)=\lambda\xi_k y_k(x), k=\overline{1,6}\,, \tag{2.1}$$

where $\|c_{kj}(x)\|_{k,j=1}^{6}$ is diagonal matrix with zero diagonal elements: its elements are complex-valued, measurable functions satisfying the conditions:

$$|c_{kj}(x,t)| \leq ce^{-\varepsilon x},\ c>0,\ \ \varepsilon>0 \tag{2.2}$$

λ is spectral parameter: $\xi_1 > \xi_2 > \xi_3 > \xi_4 > 0 > \xi_5 > \xi_6$.

The scattering problem on a semi-axis for a system (2.1) is in finding solution to the system (2.1) by the given incident waves $A_j e^{i\lambda\xi_j x}$ $\left(j=\overline{1,4}\right)$

$$y_j(x,\lambda)=A_j e^{i\lambda\xi_j x}+o(1),\ x\to+\infty,\ j=\overline{1,4} \tag{2.3}$$

and boundary conditions at $x=0$.The solutions of a system (2.1) we will understand absolutely-continuous functions as almost everywhere satisfying(2.1).

For the system (2.1) we will look at four problems together. Every problem is consist of finding solution to the system of equations (2.1), satisfying one of conditions:

$$1.\begin{cases} y_5^1(0,\lambda)=y_1^1(0,\lambda)+y_2^1(0,\lambda)+y_3^1(0,\lambda), \\ y_6^1(0,\lambda)=y_4^1(0,\lambda); \end{cases} \tag{2.4}$$

$$2.\begin{cases} y_5^2(0,\lambda)=y_1^2(0,\lambda)+y_2^2(0,\lambda)+y_4^2(0,\lambda), \\ y_6^2(0,\lambda)=y_3^2(0,\lambda); \end{cases} \tag{2.5}$$

$$3.\begin{cases} y_5^3(0,\lambda)=y_1^3(0,\lambda)+y_3^3(0,\lambda)+y_4^3(0,\lambda), \\ y_6^3(0,\lambda)=y_2^3(0,\lambda); \end{cases} \tag{2.6}$$

$$4.\begin{cases} y_5^4(0,\lambda)=y_2^4(0,\lambda)+y_3^4(0,\lambda)+y_4^4(0,\lambda), \\ y_6^4(0,\lambda)=y_1^4(0,\lambda). \end{cases} \tag{2.7}$$

Under joint consideration of these four conditions together is called scattering problem for the system (2.1) on a semi-axis.

Theorem 2.1 *Let coefficients $c_{kj}(x), k,j=\overline{1,6}$ of a system of equations* (2.1) *be measurable functions and satisfy the conditions* (2.2). *Then there exists a unique boundary solution of the scattering problem on a semi-axis with the given incident waves $A_j e^{i\lambda\xi_j x}$ $(j=\overline{1,4})$. Every existence of boundary solutions assumed asymptotics*

$$y_j^k(x,\lambda)=B_j^k e^{i\lambda\xi_j x}+\ o(1)\,,\ x\to+\infty\,, j=5,6\,,\ k=\ \overline{1,4,} \tag{2.8}$$

where B_j^k $(j=5,6\,,\ k=\overline{1,4})$ are found by the boundary conditions (2.4)-(2.7) *and they are expressed by A_j $(j=\overline{1,4})$, coefficients $c_{kj}(x)$ $(k,j=\overline{1,4})$ and solutions, accordingly, first, ... , fourth problems.*

Proof of this theorem is similar to one [4].

Like this, we can determine the matrix $S_k(\lambda)$ $(k=\overline{1,4})$, transformer $\{A_1,A_2,A_3,A_4\}^t$ to $\left\{B_5^k,B_6^k\right\}$ $(k=\overline{1,4})$. Here t- denotes the transposition sign. The matrix $S_k(\lambda)$ is in the form

$$S_k(\lambda)=\begin{pmatrix} S_{11}^k(\lambda) & S_{12}^k(\lambda) & S_{13}^k(\lambda) & S_{14}^k(\lambda) \\ S_{21}^k(\lambda) & S_{22}^k(\lambda) & S_{23}^k(\lambda) & S_{24}^k(\lambda) \end{pmatrix},\ k=\overline{1,4} \tag{2.9}$$

We will call this matrix function

$$S(\lambda)=(S_1(\lambda),\ ...\ ,S_4(\lambda)) \tag{2.10}$$

scattering matrix for the system (2.1) on the semi-axis.

3 The integral presentations of solutions

We can express any existing boundary solution of system (2.1) on a semi-axis by the asymptotics
$A_j e^{i\lambda\xi_j x}$ $(j=\overline{1,4})$, $B_j^k e^{i\lambda\xi_j x}$ $(j=5,6\,,\ k=\overline{1,4})$, by value of solutions at $x=0$ or $y_j\ (0,\lambda)$ $(j=\overline{1,6})$ or by some combinations of these values:

Lemma 3.1 *Let the coefficients $c_{kj}\ (x)$, $k,j=\overline{1,5}$ satisfying condition (2.2), $c_{kk}\ (x)=0$ and $Im\,\lambda=0$. Then any existing boundary solution of the system (2.1) assumes the following integral presentation:*

$$y_k^1(x,\lambda)=h_k^1(x,\lambda)+\sum_{j=1}^{6}\int_{\xi_6 x}^{\xi_1 x}A_{kj}^1(x,\tau)e^{i\lambda\tau}d\tau\,h_j^1(0,\lambda), \tag{3_1}$$

$$\begin{aligned} y_k^m(x,\lambda)=h_k^m(x,\lambda)+\int_{-\infty}^{+\infty}\sum_{j=1}^{m-2}A_{kj}^m(x,\tau)e^{i\lambda\tau}d\tau\,h_j^m(0,\lambda)+\int_{-\infty}^{\xi_{m-1}x}A_{k\,m-1}^m(x,\tau)e^{i\lambda\tau}d\tau\times \\ \times h_{m-1}^m(0,\lambda)+\sum_{j=m}^{6}\int_{-\infty}^{\xi_m x}A_{k\,j}(x,\tau)e^{i\lambda\tau}d\tau\,h_j^m(0,\lambda)\ ,\ m=\overline{2,6}, \end{aligned} \tag{3_m}$$

$$\begin{aligned} y_k^7(x,\lambda)=h_k^7(x,\lambda)+\int_{\xi_1 x}^{+\infty}A_{k1}^7(x,\tau)e^{i\lambda\tau}d\tau\,h_1^7(0,\lambda)\ +\int_{-\infty}^{+\infty}\sum_{j=2}^{5}A_{kj}^7(x,\tau)e^{i\lambda\tau}d\tau\,h_j^7(0,\lambda)+ \\ +\int_{-\infty}^{\xi_6 x}A_{k6}^7(x,\tau)e^{i\lambda\tau}d\tau\ h_6^7(0,\lambda)\ , \end{aligned} \tag{3_7}$$

$$\begin{aligned} y_k^m=h_k^m(x,\lambda)+\int_{\xi_{m-7},x}^{+\infty}A_{k1}^m(x,\tau)e^{i\lambda\tau}d\tau\ h_{m-7}^m(0,\lambda)+\int_{\xi_{m-6},x}^{+\infty}A_{k2}^m(x,\lambda)e^{i\lambda\tau}d\tau\times \\ \times h_{m-6}^m(0,\lambda)+\sum_{j=m-5}^{6}\int_{-\infty}^{+\infty}A_{kj}^m(x,\lambda)e^{i\lambda\tau}d\tau\ h_j^m(0,\lambda)\ ,\ m=\overline{8,11}\ , \end{aligned} \tag{3_m}$$

$$y_k^{12}(x,\lambda) = h_k^{12}(x,\lambda)+\sum_{j=1}^{5}\int\limits_{\xi_5 x}^{+\infty} A_{kj}^{12}(x,\lambda)e^{i\lambda\tau}d\tau\ h_j^{12}(0,\lambda)+\int\limits_{\xi_6 x}^{+\infty} A_{k6}^{12}(x,\lambda)e^{i\lambda\tau}d\tau\ h_6^{12}(0,\lambda)\ , \quad (3_{12})$$

where

$$h^k(x,\lambda) = \left\{h_1^k(x,\lambda),\ \dots\ ,\ h_6^k(x,\lambda)\right\},\ k=\overline{1,12},$$

$$h^1(x,\lambda) = \left\{y_1(0,\lambda)e^{i\lambda\xi_1 x},\ \dots, y_6(0,\lambda)e^{i\lambda\xi_6 x}\right\},$$

$$h^2(x,\lambda) = \left\{A_1 e^{i\lambda\xi_1 x}, y_2(0,\lambda)e^{i\lambda\xi_2 x},\ \dots, y_6(0,\lambda)e^{i\lambda\xi_6 x}\right\},$$

$$h^3(x,\lambda) = \left\{A_1 e^{i\lambda\xi_1 x}, A_2 e^{i\lambda\xi_2 x}, y_3(0,\lambda)e^{i\lambda\xi_3 x},\ \dots, y_6(0,\lambda)e^{i\lambda\xi_6 x}\right\},$$

$$h^4(x,\lambda) = \left\{A_1 e^{i\lambda\xi_1 x}, A_2 e^{i\lambda\xi_2 x}, A_3 e^{i\lambda\xi_3 x}, y_4(0,\lambda)e^{i\lambda\xi_4 x},\ \dots, y_6(0,\lambda)e^{i\lambda\xi_6 x}\right\},$$

$$h^5(x,\lambda) = \left\{A_1 e^{i\lambda\xi_1 x}, A_2 e^{i\lambda\xi_2 x}, A_3 e^{i\lambda\xi_3 x}, A_4 e^{i\lambda\xi_4 x}, y_5(0,\lambda)e^{i\lambda\xi_5 x}, y_6(0,\lambda)e^{i\lambda\xi_6 x}\right\},$$

$$h^6(x,\lambda) = \left\{A_1 e^{i\lambda\xi_1 x}, A_2 e^{i\lambda\xi_2 x}, A_3 e^{i\lambda\xi_3 x}, A_4 e^{i\lambda\xi_4 x}, B_5 e^{i\lambda\xi_5 x}, y_6(0,\lambda)e^{i\lambda\xi_6 x}\right\},$$

$$h^7(x,\lambda) = \left\{A_1 e^{i\lambda\xi_1 x}, A_2 e^{i\lambda\xi_2 x}, A_3 e^{i\lambda\xi_3 x}, A_4 e^{i\lambda\xi_4 x}, B_5 e^{i\lambda\xi_5 x}, B_6 e^{i\lambda\xi_6 x}\right\},$$

$$h^8(x,\lambda) = \left\{y_1(0,\lambda)e^{i\lambda\xi_1 x}, A_2 e^{i\lambda\xi_2 x}, A_3 e^{i\lambda\xi_3 x}, A_4 e^{i\lambda\xi_4 x}, B_5 e^{i\lambda\xi_5 x}, B_6 e^{i\lambda\xi_6 x}\right\},$$

$$h^9(x,\lambda) = \left\{y_1(0,\lambda)e^{i\lambda\xi_1 x}, y_2(0,\lambda)e^{i\lambda\xi_2 x}, A_3 e^{i\lambda\xi_3 x}, A_4 e^{i\lambda\xi_4 x}, B_5 e^{i\lambda\xi_5 x}, B_6 e^{i\lambda\xi_6 x}\right\},$$

$$h^{10}(x,\lambda) = \left\{y_1(0,\lambda)e^{i\lambda\xi_1 x}, y_2(0,\lambda)e^{i\lambda\xi_2 x}, y_3(0,\lambda)e^{i\lambda\xi_3 x}, A_4 e^{i\lambda\xi_4 x}, B_5 e^{i\lambda\xi_5 x}, B_6 e^{i\lambda\xi_6 x}\right\},$$

$$h^{11}(x,\lambda) = \left\{y_1(0,\lambda)e^{i\lambda\xi_1 x}, y_2(0,\lambda)e^{i\lambda\xi_2 x}, y_3(0,\lambda)e^{i\lambda\xi_3 x}, y_4(0,\lambda)e^{i\lambda\xi_4 x}, B_5 e^{i\lambda\xi_5 x}, B_6 e^{i\lambda\xi_6 x}\right\},$$

$$h^{12}(x,\lambda) = \left\{y_1(0,\lambda)e^{i\lambda\xi_1 x}, y_2(0,\lambda)e^{i\lambda\xi_2 x}, y_3(0,\lambda)e^{i\lambda\xi_3 x}, y_4(0,\lambda)e^{i\lambda\xi_4 x}, y_5(0,\lambda)e^{i\lambda\xi_5 x}, B_6 e^{i\lambda\xi_6 x}\right\}.$$

Kernels of these presentations are uniquely defined by the coefficients $c_{kj}\ (x)$ of system of equations (2.1).

Proof of this theorem is similar to the method of work $[4, 5]$.

4 Properties of scattering matrix

Here is constructed analytical and factorizational properties of elements of scattering matrix. By this object here is defined relation between the solutions at zero and asymptotics of solutions, as well as here is studied analytical and factorizational properties of scattering matrix.

Theorem 4.1 *Let the coefficients of system* (2.1) *are satisfied conditions* (2.2). *Then matrix*

$$P_1^k(\lambda) = \begin{pmatrix} S_{21}^k(\lambda) & S_{22}^k(\lambda) & S_{23}^k(\lambda) & S_{24}^k(\lambda) \\ S_{11}^2(\lambda) - S_{11}^1(\lambda) & S_{12}^2(\lambda) - S_{12}^1(\lambda) & S_{13}^2(\lambda) - S_{13}^1(\lambda) & S_{14}^2(\lambda) - S_{14}^1(\lambda) \\ S_{11}^3(\lambda) - S_{11}^1(\lambda) & S_{12}^3(\lambda) - S_{12}^1(\lambda) & S_{13}^3(\lambda) - S_{13}^1(\lambda) & S_{14}^3(\lambda) - S_{14}^1(\lambda) \\ S_{11}^4(\lambda) - S_{11}^1(\lambda) & S_{12}^4(\lambda) - S_{12}^1(\lambda) & S_{13}^4(\lambda) - S_{13}^1(\lambda) & S_{14}^4(\lambda) - S_{14}^1(\lambda) \end{pmatrix}, \tag{4.1}$$

$$P_2^k(\lambda) = \begin{pmatrix} S_{21}^k(\lambda) & S_{22}^k(\lambda) & S_{23}^k(\lambda) & S_{24}^k(\lambda) \\ S_{11}^1(\lambda) - S_{11}^3(\lambda) & S_{12}^1(\lambda) - S_{12}^3(\lambda) & S_{13}^1(\lambda) - S_{13}^3(\lambda) & S_{14}^1(\lambda) - S_{14}^3(\lambda) \\ S_{11}^2(\lambda) - S_{11}^3(\lambda) & S_{12}^2(\lambda) - S_{12}^3(\lambda) & S_{13}^2(\lambda) - S_{13}^3(\lambda) & S_{14}^2(\lambda) - S_{14}^3(\lambda) \\ S_{11}^4(\lambda) - S_{11}^3(\lambda) & S_{12}^4(\lambda) - S_{12}^3(\lambda) & S_{13}^4(\lambda) - S_{13}^3(\lambda) & S_{14}^4(\lambda) - S_{14}^3(\lambda) \end{pmatrix}, \tag{4.2}$$

$k = 1, 2, 3, 4,$

and their basic minors, without finite points, have inverse

$$\left[P_1^k(\lambda)\right]^{-1} = \left\|P_{ij}^{1k}(\lambda)\right\|_{i,j=1}^4, \quad \left[P_2^k(\lambda)\right]^{-1} = \left\|P_{ij}^{2k}(\lambda)\right\|_{i,j=1}^4,$$

moreover, we can give the following equality (without finite points):

$$S_{14}^k(\lambda) - S_{14}^1(\lambda) = I + G_{k+}(\lambda), k = 2, 3, 4, \tag{4.3}$$

$$S_{21}^k(\lambda) + (I + M(\lambda))S_{11}^k(\lambda) = (I + R_+(\lambda))^{-1}(I + R_-(\lambda)), k = \overline{1,4} \tag{4.4}$$

$$\sum_{i=1}^4 S_{1i}^1(\lambda)P_{i4}^{1k}(\lambda) = -(I + \delta_-^1(\lambda))^{-1}(I + \delta_+^1(\lambda)),\ k = 1, 2, 3, \tag{4.5}$$

$$\sum_{i=1}^4 S_{1i}^2(\lambda)P_{i3}^{1k}(\lambda) = -(I + \delta_-^2(\lambda))^{-1}(I + \delta_+^1(\lambda)),\ k = 1, 2, 4, \tag{4.6}$$

$$\sum_{i=1}^4 S_{1i}^3(\lambda)P_{i2}^{1k}(\lambda) = -(I + \delta_-^3(\lambda))^{-1}(I + \delta_+^1(\lambda)),\ k = 1, 3, 4, \tag{4.7}$$

$$\sum_{i=1}^4 S_{1i}^4(\lambda)P_{i2}^{2k}(\lambda) = -(I + \delta_-^4(\lambda))^{-1}(I + \delta_+^1(\lambda)),\ k = 2, 3, 4, \tag{4.8}$$

$$P_{11}^{1k} = P_{11}^{2k}(\lambda) = (I + K_+(\lambda))^{-1}(I + N_-(\lambda)),\ k = 1, 2, 3, 4 \tag{4.9}$$

where $G_{k+}(\lambda)(k = 1, 2, 3)$, $R_+(\lambda)$, $R_-(\lambda)$, $K_+(\lambda)$, $N_-(\lambda)$, $\delta_-^k(\lambda)$, $\delta_+(\lambda)\,(k = \overline{1,4})$ *are expressed by elements of second, third, sixth, seventh, twelfth presentations.*

Accordingly the set of zeros of functions we will call singular numbers of system of equations (2.1) on a semi-axis.

For simplicity, we confine ourselves to the case when there are no singular numbers.

Proof of this theorem follows from integral presentations of solutions and boundary conditions and similarly methods of work [4,5].

5 Inverse scattering problem

Inverse scattering problem of a system (2.1) is consist of restoration of coefficients of equations for given scattering matrix $S(\lambda)$ (there are no zeros of problem).

First of all here is installed relation between scattering matrix $S(\lambda)$ on semi-axis and transformation matrix $\prod(\lambda)$ which is related asymptotics $\{A_1, \dots, A_4, B_5, B_6\}$, by boundary values $\{y_1(0,\lambda), \dots, y_6(0,\lambda)\}$ on whole axis with additional conditions $c_{kj}\ (x) = 0,\ x < 0,\ k,j = \overline{1,6}$.

Theorem 5.1 *Let coefficients of the system* (2.1) *are satisfied conditions* (2.2) *and there are not singular specter. Then the transformation matrix*

$$\prod(\lambda) = \begin{pmatrix} I + R^7_{11-}(\lambda) & R^7_{12}(\lambda) & \dots & R^7_{15}(\lambda) & R^7_{16+}(\lambda) \\ R^7_{21-}(\lambda) & I + R^7_{22}(\lambda) & \dots & R^7_{25}(\lambda) & R^7_{26+}(\lambda) \\ \dots & \dots & \dots & \dots & \dots \\ R^7_{61-}(\lambda) & R^7_{62}(\lambda) & \dots & R^7_{65}(\lambda) & I + R^7_{66+}(\lambda) \end{pmatrix} \tag{5.1}$$

is expressed by elements of matrix $S(\lambda)$, $P^k_i(\lambda)$, $(i = 12;\ k = 1,2,3)$ *and analytic functions from factorizational equalities* (4.3)-(4.9).

Theorem 5.2 *Let coefficients* $c_{kj}\ (x) = 0$, $c_{kk}\ (x) = 0$ $(k,j = \overline{1,6})$ *are satisfied conditions* (2.2) *and there are not singular spectr. Then the coefficients of system* (2.1) *are uniquely defined by famous matrix* S *on semi-axis.*

Proof of this theorem follows from theorem 4. By factorizational equalities (4.3)-(4.9) by helping problem of Riemann we find all factorizational elements, and by helping of them here is restored $\prod(\lambda)$. By the work [1,5] it is obviously that by the matrix $\prod(\lambda)$ the coefficients of the system (2.1) are uniquely restored.

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