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On some Hardy-Sobolev's type variable exponent inequality and its application

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Abstract. *In this paper, it has been proved a Sobolev's type variable exponent inequality*

$$\left\| \frac{u(x)}{x(l-x)} \right\|_{p(x);(0,l)} \leq \frac{C}{l} \left\| u'(x) \right\|_{p(x);(0,l)}, \qquad \forall u \in \dot{W}^1_{p(.)}(0,l)$$

where the exponent function $p : (0,l) \to (1,\infty)$, *is a monotone increasing near little neighborhood of origin and monotone decreasing near* l *satisfying the conditions:*

$$\int_a^l t^{-\frac{1}{p'(t)}} \frac{dt}{t} \leq C_2 a^{-\frac{1}{p'(a)}}, \quad \textit{and} \quad \int_a^l t^{-\frac{1}{p'(l-t)}} \frac{dt}{t} \leq C_1 a^{-\frac{1}{p'(l-a)}},$$

for $0 < a < l$. *Applying this inequality and Browder-Minty theory methods, it has been proved an existence result of solution for some variable exponent equation.*

Keywords. variable exponent spaces, inequality, solvability, Dirichlet problem

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1 Introduction

One of the results of this paper is the following assertion on variable exponent boundedness of the conjugate Hardy operator $\int_x^l f(t)\,dt$ on finite interval $(0,l)$.

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Theorem 1.1 *Let $p:(0,l)\to(1,\infty)$ be a measurable function such that $1<p(x)\le p^+<\infty$. Assume that, p monotone decreases on a little neighborhood $(l-\varepsilon,l)$ of the right end of interval $(0,l)$.*

Then it holds an inequality

$$\Big\|\frac{1}{l-x}\int_x^l f(t)dt\Big\|_{p(.);(0,l)}\le C\big\|f(.)\big\|_{p(.);(0,l)} \tag{1.1}$$

for all measurable positive functions $f:(0,l)\to(0,\infty)$ if and only if

$$\int_a^l t^{-\frac{1}{p'(l-t)}}\frac{dt}{t}\le C_1 a^{-\frac{1}{p'(l-a)}},\quad 0<a<l, \tag{1.2}$$

where a positive constant C depends on p^+, C_1, ε, l and $p^-_{(0,l-\varepsilon)}>1$.

This assertion looks as following for an absolutely continuous functions.

Theorem 1.2 *Let $p:(0,l)\to(1,\infty)$ be a measurable function such that $1<p(x)\le p^+<\infty$. Assume that, p monotone decreases on a little neighborhood of right hand side of the interval $(0,l)$, i.e. on $(l-\varepsilon,l)$ by some $\varepsilon>0$.*

Then it holds an inequality

$$\Big\|\frac{u(x)}{l-x}\Big\|_{p(x);(0,l)}\le C\big\|u'(x)\big\|_{p(x);(0,l)} \tag{1.3}$$

for all absolutely continuous functions $u:(0,l)\to\mathbb{R}$ with $u(l)=0$ if and only if

$$\int_a^l t^{-\frac{1}{p'(l-t)}}\frac{dt}{t}\le C_1 a^{-\frac{1}{p'(l-a)}},\quad 0<a<l, \tag{1.4}$$

where a positive constant C depends on $p^+, C_1, C_2, \varepsilon, l$ and $p^-_{(0,l-\varepsilon)}>1$.

The proof of this assertions may be composed using the following boundedness result for the Hardy operator $Hf(x)=\frac{1}{x}\int_0^x f(t)dt$ in $L^p(.)(0,l)$ from [3].

Theorem 1.3 *Let $p:(0,l)\to(1,\infty)$ be a measurable function such that $1<p(x)\le p^+<\infty$. Assume that, p monotone increases on a little neighborhood $(0,\varepsilon)$ of the origin.*

Then it holds an inequality

$$\Big\|\frac{1}{x}\int_0^x f(t)dt\Big\|_{p(.);(0,l)}\le C\big\|f(.)\big\|_{p(.);(0,l)} \tag{1.5}$$

for all measurable positive functions $f:(0,l)\to(0,\infty)$ if and only if

$$\int_a^l t^{-\frac{1}{p'(t)}}\frac{dt}{t}\le C_2 a^{-\frac{1}{p'(a)}},\quad 0<a<l, \tag{1.6}$$

where a positive constant C depends on p^+, C_2, ε, l and $p^-_{(\varepsilon,l)}>1$.

Combining Theorems 1.1 and 1.2 one gets the following result.

Theorem 1.4 *Let $p : (0,l) \to (1,\infty)$ be a measurable function such that $1 < p(x) \leq p^+ < \infty$. Assume that, p monotone increases near origin, decreases near l on a little neighborhood, i.e. $p \uparrow$ on $(0,\varepsilon)$ and $p \downarrow$ on $(l-\varepsilon, l)$ for some $\varepsilon > 0$.*

Then it holds the inequality

$$\Big\|\frac{u(x)}{x(l-x)}\Big\|_{p(x);(0,l)} \leq \frac{C}{l}\big\|u'(x)\big\|_{p(x);(0,l)} \tag{1.7}$$

for all absolutely continuous functions $u : (0,l) \to \mathbb{R}$ with $u(0) = u(l) = 0$ if and only if (1.6) *and* (1.4) *is satisfied; moreover, a positive constant C in* (1.7) *depends on $p^+, C_1, C_2, \varepsilon$ and $p^-_{(\varepsilon,l-\varepsilon)} > 1$*

For a proof of Theorem 1.4 let us note the inequality

$$\frac{u(x)}{x(l-x)} \leq \frac{2}{l}\Big(\frac{u(x)}{x} + \frac{u(x)}{l-x}\Big).$$

□

To prove Theorem 1.1 we shall use the inequality

$$\Big\|\frac{1}{x}\int_0^x f(t)\,dt\Big\|_{L^{p(\cdot)}(0,l)} \leq \|f\|_{L^{p(\cdot)}(0,l)} \tag{1.8}$$

for a measurable function $p : (0,l) \to (1,\infty)$ which is monotone increasing near the origin $(0,\varepsilon)$ and be such that $1 < p(x) \leq p^+ < \infty$. According to the resent works (see, e.g. [3], [4], [5]) for the inequality (1.5) it is necessary and sufficiently that the condition

$$\int_a^l x^{-\frac{1}{p'(x)}}\frac{dx}{x} \leq C a^{-\frac{1}{p'(a)}}, \quad 0 < a < l \tag{1.9}$$

be fulfilled. In our setting concerning Theorem 1.1 we have the condition (1.6) and the monotone decreasing of p near the right end of interval $(0,l)$, in $(l-\varepsilon, l)$ by some $\varepsilon > 0$. Insert a new exponent function $\tilde{p}(x) = p(l-x)$. Then we are in region of application of inequality (1.5) for the exponent $\tilde{p}$ interval $(0,l)$, space $L^{\tilde{p}(\cdot)}(0,l)$, and the operator H. Indeed, check the condition (1.9):

$$\int_a^l x^{-\frac{1}{\tilde{p}'(x)}}\frac{dx}{x},$$

inserting the expression of the function $\tilde{p}(x)$

$$= \int_a^l x^{-\frac{1}{p'(l-x)}}\frac{dx}{x},$$

using condition (1.6)

$$\leq C a^{-\frac{1}{p'(l-a)}} = C a^{-\frac{1}{\tilde{p}'(a)}},$$

that means the condition (1.9) is fulfilled for a function $\tilde{p}(x)$.

In order to show (1.3) it suffices to prove the inequality

$$\Big\|\frac{1}{(l-x)}\int_x^l g(t)\,dt\Big\|_{L^{p(\cdot)}(0,l)} \leq C\big\|g(.)\big\|_{p(.);(0,l)} \tag{1.10}$$

for any measurable function $g \in L^{p(\cdot)}(0,l)$. The monotone increasing near origin of the function $\tilde{p}(x)$ follows from its expression $\tilde{p}(x) = p(l-x)$ and the monotone decreasing of p near l. Let $f : (0,l) \to (0,\infty)$ be a measurable function and the condition (1.4) be satisfied. Using the cited above result, we get the inequality (1.8)

$$\Big\|\frac{1}{x}\int_0^x f(t)\,dt\Big\|_{L^{\tilde{p}(\cdot)}(0,l)} \le C\|f\|_{L^{\tilde{p}(\cdot)}(0,l)}.$$

Using the definition of variable exponent norm, calculate both hand sides of this inequality separately. In this way,

$$\Big\|\frac{1}{x}\int_0^x f(t)\,dt\Big\|_{L^{\tilde{p}(\cdot)}(0,l)} = \inf\Big\{\lambda > 0 : \int_0^l \Big(\frac{1}{x\lambda}\int_0^x f(t)\,dt\Big)^{\tilde{p}(x)} dx \le 1\Big\},$$

inserting the expression of the function $\tilde{p}(x)$

$$= \inf\Big\{\lambda > 0 : \int_0^l \Big(\frac{1}{x\lambda}\int_0^x f(t)\,dt\Big)^{p(l-x)} dx \le 1\Big\},$$

making the change of variable $z = l - x$ in the exterior integral

$$= \inf\Big\{\lambda > 0 : -\int_l^0 \Big(\frac{1}{(l-z)\lambda}\int_0^{l-z} f(t)\,dt\Big)^{p(z)} dz \le 1\Big\},$$

changing the limits of integration

$$= \inf\Big\{\lambda > 0 : \int_0^l \Big(\frac{1}{(l-z)\lambda}\int_0^{l-z} f(t)\,dt\Big)^{p(z)} dz \le 1\Big\},$$

changing the variable $t = l - y$ in the interior integral

$$= \inf\Big\{\lambda > 0 : \int_0^l \Big(-\frac{1}{(l-z)\lambda}\int_l^{z} f(l-y)\,dy\Big)^{p(z)} dz \le 1\Big\},$$

inserting $g(y) = f(1-y)$ and changing the limits of integration in the interior integral,

$$= \inf\Big\{\lambda > 0 : \int_0^l \Big(\frac{1}{(l-z)\lambda}\int_z^{l} g(y)\,dy\Big)^{p(z)} dz \le 1\Big\},$$

using the definition of variable exponent norm

$$= \Big\|\frac{1}{(l-z)}\int_z^l g(t)\,dt\Big\|_{L^{p(\cdot)}(0,l)}.$$

Applying the inequality (1.8), we get that is exceeded:

$$\le C\|f\|_{L^{\tilde{p}(\cdot)}(0,l)},$$

using the definition of variable exponent norm

$$= C \inf \Big\{ \lambda > 0 : \int_0^l \Big(\frac{f(x)}{\lambda} \Big)^{\tilde{p}(x)} dx \le 1 \Big\},$$

inserting the expression of the function $\tilde{p}(x)$,

$$= C \inf \Big\{ \lambda > 0 : \int_0^l \Big(\frac{f(x)}{\lambda} \Big)^{p(l-x)} dx \le 1 \Big\},$$

making the change of variable $z = l - x$ in the exterior integral

$$= C \inf \Big\{ \lambda > 0 : -\int_l^0 \Big(\frac{f(l-z)}{\lambda} \Big)^{p(z)} dz \le 1 \Big\},$$

changing the limits of integration

$$= C \inf \Big\{ \lambda > 0 : \int_0^l \Big(\frac{f(l-z)}{\lambda} \Big)^{p(z)} dz \le 1 \Big\},$$

by notation $g(z) = f(l-z)$

$$= C \inf \Big\{ \lambda > 0 : \int_0^l \Big(\frac{g(z)}{\lambda} \Big)^{p(z)} dz \le 1 \Big\},$$

using the definition of variable exponent norm

$$= C \big\| g(.) \big\|_{p(.);(0,l)}.$$

Therefore, the inequality (1.10) has been established, which completes the proof of inequality (1.5) since g is arbitrary.

□

Define $Lip_0(0,l)$ a class of Lipshitsz continuous functions $f : (0,l) \to \mathbb{R}$ such that $f(0) = f(l) = 0$. Define a norm

$$\|f\| = \|f\|_{L^1(0,l)} + \|f'\|_{L^{p(\cdot)}(0,l)}$$

in this class and close it in this norm. The obtained variable exponent space denote as $\dot{W}^1_{p(.)}(0,l)$. Space $\dot{W}^1_{p(.)}(0,l)$. is reflexive Banach space if $1 < p^-, p^+ < \infty$ (see, e.g. [1]).

Consider the Dirichlet problem

$$-\frac{d}{dx}\Big(\big| \frac{dy}{dx} \big|^{p(x)-2} \frac{dy}{dx} \Big) + \lambda \Big| \frac{y}{x(l-x)} \Big|^{p(x)-1} \frac{y}{x(l-x)} = F(x) \qquad (1.11)$$
$$y(0) = y(l) = 0,$$

where $\lambda > 0$ is a parameter, $F = \frac{f_0(x)}{x(l-x)} + \frac{df_1}{dx}$ with $f_0, f_1 \in L^{p'}(0,l)$.

Let $p(x)$ be measurable function satisfying Hardy's type inequality (1.11) to hold, e.g. it is satisfied conditions (1.6), (1.9) and $p(x)$ increases near origin decreases near l.

To prove the existence of solution of problem (1.11) we shall use a monotone operator method. To carry out its insert an operator $A : X \to X^*$, where $X = \dot{W}^1_{p(.)}(0,l)$, X^* its dual space. Insert an operator $A : u \in X \to A(u) \in X^*$. We define it as following

$$< A(u), \varphi >= \int_0^l |\frac{dy}{dx}|^{p(x)-2}\frac{dy}{dx}\frac{d\varphi}{dx}\,dx + \lambda\int_0^l |\frac{y}{x(l-x)}|^{p(x)}\varphi\,dx, \quad \forall\varphi \in X. \tag{1.12}$$

We say, $y = y(x)$ is a solution of problem (1.11) if

$$< A(y), \varphi >= (F, \varphi), \quad \forall\varphi \in X,$$

i.e.

$$\int_0^l |\frac{dy}{dx}|^{p(x)-2}\frac{dy}{dx}\frac{d\varphi}{dx}\,dx + \lambda\int_0^l |\frac{y}{x(l-x)}|^{p(x)}\,dx$$
$$= \int_0^l \frac{f_0(x)}{x(l-x)}\varphi(x)\,dx - \int_0^l f_1(x)\varphi'(x)dx.$$

In order to carry out the monotone operator method approach we have to show that the operator A is

1) monotone property, that is $\langle A(y_1) - A(y_2), y_1 - y_2\rangle \ \geq 0$,
2) coercivity, that is $\frac{<A(y),y>}{\|y\|} \to \infty$ as $\|y\| \to \infty$,
3) semi continuity, that is $< A(y_1+\lambda_k y_2), \varphi > \to < A(y_1+\lambda y_2), \varphi >$ as $\lambda_k \to \lambda$.

Verify 1): for any $y_1,\ y_2 \in X$ on base of inequality

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq 0$$

for all $a, b \in \mathbb{R}$ it follows

$$< A(y_1) - A(y_2), y_1 - y_2 >= \int_0^l \big(|y_1'|^{p(x)-2}y_1' - |y_2'|^{p(x)-2}y_2'\big)\big(y_1' - y_2'\big)\,dx$$
$$+\lambda\int_0^l \big(x(l-x)\big)^{-p(x)}\big(|y_1|^{p(x)-2}y_1 - |y_2|^{p(x)-2}y_2\big)\big(y_1 - y_2\big)\,dx \geq 0.$$

Verify 2). Let $\|y_k\|_{\dot{W}^1_{p(.)}(0,l)} \to \infty$. Then it is easily seen that $\|y_k'\|_{L^{p(\cdot)}(0,l)} \to \infty$ as $k \to \infty$. Indeed, if $\|y_k'\|_{L^{p(\cdot)}(0,l)} \leq C$ by some $C > 0$ not depending on $k \in \mathbb{N}$ on base of Holder's inequality for $p(.)$-norms (see, [1,2]) we have

$$\|y_k\|_{L^{p(\cdot)}(0,l)} \leq C_0\Big\|\frac{y_k}{x(l-x)}\Big\|_{L^{p(\cdot)}(0,l)} \cdot \max_{0<x<l} x(l-x)$$
$$\leq C_0 l^2 \cdot \Big\|\frac{y_k}{x(l-x)}\Big\|_{L^{p(\cdot)}(0,l)},$$

on base of Hardy's type inequality,

$$\leq C_0 l\,\|y_k'\|_{L^{p(\cdot)}(0,l)},$$

which contradicts the assumption $\|y_k\|_{L^{p(\cdot)}(0,l)} \to \infty$. Therefore, from $\|y_k\|_{\dot{W}^1_{p(\cdot)}(0,l)} \to \infty$ it follows $\|y'_k\|_{L^{p(\cdot)}(0,l)} \to \infty$.

On other hand, by the Holder's and Hardy's inequalities it follows that

$$\|y_k\|_{\dot{W}^1_{p(\cdot)}(0,l)} = \|y_k\|_{L^{p(\cdot)}(0,l)} + \|y'_k\|_{L^{p(\cdot)}(0,l)}$$
$$\leq \Big[C_0 l + 1\Big] \|y'_k\|_{L^{p(\cdot)}(0,l)}.$$

Therefore, to verify the convergence

$$\frac{< A(y_k), y_k >}{\|y_k\|_{\dot{W}^1_{p(\cdot)}(0,l)}} \to \infty \quad \text{as} \quad k \to \infty$$

it suffices to show that

$$\frac{< A(y_k), y_k >}{\|y'_k\|_{L^{p(\cdot)}(0,l)}} \to \infty \quad \text{as} \quad \|y'_k\|_{L^{p(\cdot)}(0,l)} \to \infty.$$

We have

$$\frac{< A(y_k), y_k >}{\|y'_k\|_{L^{p(\cdot)}(0,l)}} = \int_0^l \Big[\Big(\frac{y'_k}{\|y'_k\|_{L^{p(\cdot)}(0,l)}}\Big)^{p(x)} + \Big(\frac{y_k}{x(l-x)\|y'_k\|_{L^{p(\cdot)}(0,l)}}\Big)^{p(x)}\Big]\, dx$$
$$\geq \|y'_k\|^{p^- -1}_{L^{p(\cdot)}(0,l)} \int_0^l \Big(\frac{y'_k}{\|y'_k\|_{L^{p(\cdot)}(0,l)}}\Big)^{p(x)} dx$$
$$\geq \|y'_k\|^{p^- -1}_{L^{p(\cdot)}(0,l)} \to \infty$$

as $\|y'_k\|_{L^{p(\cdot)}(0,l)} \to \infty$.

We have used that $\int_0^l \Big(\frac{y'_k}{\|y'_k\|_{L^{p(\cdot)}(0,l)}}\Big)^{p(x)} dx = 1$, by definition. Therefore,

$$\frac{< A(y_k), y_k >}{\|y_k\|} \geq \|y'_k\|^{p^- -1}_{L^{p(\cdot)}(0,l)} \to \infty$$

as $k \to \infty$ provided $\lambda \geq 0$.

The coercivity has been shown.

3) Verify the semi continuity. Let the number sequence $\lambda_k \to \lambda$ and $y_1, y_2, \varphi \in X$. Then by using the Lebesgue majorant theorem and the convergence $\lambda_k \to \lambda$ it follows that

$$< A(y_1 + \lambda_k y_2), \varphi >= \int_0^l \Big|\frac{dy_1}{dx} + \lambda_k \frac{dy_2}{dx}\Big|^{p(x)-2}\Big(\frac{dy_1}{dx} + \lambda_k \frac{dy_2}{dx}\Big)\frac{d\varphi}{dx}\, dx$$
$$+\lambda \int_0^l \Big(\frac{|y_1 + \lambda_k y_2|}{x(l-x)}\Big)^{p(x)-2}\Big(\frac{y_1 + \lambda_k y_2}{x(l-x)}\Big)\varphi\, dx$$
$$\to \int_0^l \Big|\frac{dy_1}{dx} + \lambda \frac{dy_2}{dx}\Big|^{p(x)-2}\Big(\frac{dy_1}{dx} + \lambda \frac{dy_2}{dx}\Big)\frac{d\varphi}{dx}\, dx$$

$$-\lambda \int_0^l \Big(\frac{|y_1+\lambda y_2|}{x(l-x)}\Big)^{p(x)-2} \big(\frac{y_1+\lambda y_2}{x(l-x)}\big)\varphi\, dx$$

as $k \to \infty$.

Therefore, all conditions 1)-3) have been verified. By assertion of the Browder-Minty theory it follows: *there exists a unique solution* $y \in \dot{W}^1_{p(.)}(0,l)$ *of the problem* (1.11).

Verify $f \in X^*$ by using Holder's and Hardy's inequalities.

$$|(F,\varphi)| = \Big|\int_0^l \frac{f_0}{x(l-x)}\varphi\, dx - \int_0^l f_1(x)\varphi'(x)\, dx\Big| \le \Big|\int_0^l f_0 \frac{\varphi}{x(l-x)}\, dx\Big|$$

$$+\Big|\int_0^l f_1\varphi'\, dx\Big| \le \|f_0\|_{L^{p'(\cdot)}(0,l)} \Big\|\frac{\varphi}{x(l-x)}\Big\|_{L^{p(\cdot)}(0,l)} + \|f_1\|_{L^{p'(\cdot)}(0,l)} \Big\|\varphi'\Big\|_{L^{p(\cdot)}(0,l)}$$

$$\le C\Big(\|f_0\|_{L^{p'}(0,l)} + \|f_1\|_{L^{p'}(0,l)}\Big)\|\varphi'\|_{L^{p(\cdot)}(0,l)}$$

$$\le C\Big(\|f_0\|_{L^{p'}(0,l)} + \|f_1\|_{L^{p'}(0,l)}\Big)\|\varphi\|_{\dot{W}^1_{p(.)}(0,l)}.$$

Hence $F \in X^*$ if $f_0, f_1 \in L^{p'(\cdot)}(0,l)$.

To show the uniqueness apply the monotone condition on operator A. Let y_1, y_2 are two solutions of the problem (1.11) from space $\dot{W}^1_{p(.)}(0,l)$. Then

$$\int_0^l \Big(\big|\frac{dy_1}{dx}\big|^{p(x)-2}\frac{dy_1}{dx} - \big|\frac{dy_2}{dx}\big|^{p(x)-2}\frac{dy_2}{dx}\Big)\big(\frac{dy_1}{dx} - \frac{dy_2}{dx}\big)\, dx$$

$$+\lambda \int_0^l \Big(|y_1|^{p(x)-2}y_1 - |y_2|^{p(x)-2}y_2\Big)\big(y_1 - y_2\big)\, dx = 0.$$

That implies $y_1 = y_2$.

Hence, it has been proved the following result on the solvability of the Dirichlet problem (1.11).

Theorem 1.5 *Let* $p : (0,l) \to (1,\infty)$ *be a measurable function that is monotone increasing near the origin and decreasing near* l. *Moreover, are satisfied the conditions* (1.6), (1.4) *for the function* p. *Then there exists a unique solution of the Dirichlet problem* (1.11) *from space* $\dot{W}^1_{p(.)}(0,l)$ *for any measurable functions* f_0, $f_1 \in L^{p'}(0,l)$ *and* $\lambda \ge 0$.

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