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Inverse spectral and inverse nodal problems for Sturm-Liouville equations with point $\delta'-$interaction

Manaf Dzh. Manafov

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Abstract. *Inverse spectral and inverse nodal problems are studied for Sturm-Liouville equations with $\delta'-$interaction. We obtain uniqueness and reconstruction using the nodal set of eigenfunctions for the given problem.*

Keywords. Sturm-Liouville equations · Inverse spectral and inverse nodal problems · Point $\delta'-$interaction

1 Introduction

We consider the Sturm-Liouville boundary value problem (BVP) L :

$$ly := -y'' + q(x)y = \lambda y, \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{1.1}$$

$$U(y) := y'(0) = 0, \;\; V(y) := y(\pi) = 0, \tag{1.2}$$

$$I(y) := \begin{cases} y\left(\frac{\pi}{2}+0\right) - y\left(\frac{\pi}{2}-0\right) = \alpha y'\left(\frac{\pi}{2}\right), \\ y'\left(\frac{\pi}{2}+0\right) = y'\left(\frac{\pi}{2}-0\right) \equiv y'\left(\frac{\pi}{2}\right), \end{cases} \tag{1.3}$$

where $q(x)$, $\alpha \neq 0$ are real, and $q(x) \in W_2^1(0,\pi)$, λ is a spectral parameter.

Without loss of generality we assume that

$$\int_0^\pi q(x)dx = 0. \tag{1.4}$$

Notice that, we can understand problem (1.1) and (1.3) as studying the equation

$$-y'' + \left(\alpha\delta'\left(x - \frac{\pi}{2}\right) + q(x)\right) y = \lambda y, \;\; x \in (0,\pi), \tag{1.5}$$

where $\delta'(x)$ is the Dirac derivative function (see [1]).

On the separable Hilbert space $L^2([0,\pi])$ consider the linear differential operator

$$L : y(x) \to -y''(x) + q(x)\, y(x)$$

M.Dzh. Manafov
Adıyaman University, Faculty of Science and Arts, Department of Mathematics, Adıyaman, 02040, Turkey
Institute of Mathematics and Mechanics, Azerbaijan Natinonal Academy of Sciences, Baku, AZ 1141, Azerbaijan
E-mail: mmanafov@adiyaman.edu.tr

with a dense domain

$$D(L)=\Big\{y(x)\in W_2^2\Big[(0,\pi)\setminus\Big\{\frac{\pi}{2}\Big\}\Big]\cap W_2^0[(0,\pi)]:$$
$$y\Big(\frac{\pi}{2}+0\Big)-y\Big(\frac{\pi}{2}-0\Big)=\alpha y'\Big(\frac{\pi}{2}\Big),$$
$$y'\Big(\frac{\pi}{2}+0\Big)=y'\Big(\frac{\pi}{2}-0\Big),y'(0)=0,y(\pi)=0\Big\}.$$

We will consider inverse problems of recovering $q(x)$ the given spectral and nodal characteristics. We denote the BVP (1.1)-(1.3) by $L=L(q)$.

Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, monographs [4,8,12–14,17]). Inverse nodal problems consist in constructing operators from the given nodes (zeros) of eigenfunctions (see [5,7, 11,15,16]).

In this present paper we obtain some results on inverse spectral and inverse nodal problems and establish connections between them.

2 Inverse spectral problems

In this section we study so-called incomplete inverse problem of recovering the potential $q(x)$ from a part of the spectrum $BVP\ L$.The technique employed is similar to those used in [17].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on the intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},\pi\right)$. Denote $\langle y,z\rangle:=yz'-y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (1.3), then

$$\langle y,z\rangle_{x=\frac{\pi}{2}-0}=\langle y,z\rangle_{x=\frac{\pi}{2}+0}, \tag{2.1}$$

i.e. the function $\langle y,z\rangle$ is continuous on $(0,\pi)$.

Let $\varphi(x,\lambda)$ be solution of equation (1.1) satisfying the initial conditions $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=0$ and the matching conditions (1.3). Then $U(\varphi)=0$. Denote

$$\Delta(\lambda):=-V(\varphi)=-\varphi(\pi,\lambda). \tag{2.2}$$

By virtue of (2.1) and the Liouville's formula (see [2, p.83]), $\Delta(\lambda)$ does not depend on x. The function $\Delta(\lambda)$ is called characteristic function on L.

Lemma 2.1 *The $BVP\ L$ has a countable set of eigenvalues $\{\lambda_n\}_{n\ge1}$. All eigenvalues are real, simple and $n\to\infty$*

$$\sqrt{\lambda_n}=n+\frac{1}{\pi n}\left(w_0+(-1)^{n-1}w_1\right)+\frac{1}{\pi n^2}w_2+o\left(\frac{1}{n^2}\right), \tag{2.3}$$

$$w_0=\tfrac{2}{\alpha}+\tfrac12\int_0^\pi q(t)dt,\ w_1=\tfrac12\left(\int_{\frac{\pi}{2}}^\pi q(t)dt-\int_0^{\frac{\pi}{2}}q(t)dt\right),$$

$$w_2=-\tfrac{1}{4\alpha}\Big\{4\int_0^\pi q(t)dt+\left[q(0)+(\alpha-1)\,q\left(\tfrac{\pi}{2}\right)-\alpha q(\pi)\right] \tag{2.4}$$
$$+\left[\alpha\left(\int_0^\pi q(t)dt\right)^2+\left(\tfrac12-\alpha\right)\left(\int_0^{\frac{\pi}{2}}q(t)dt\right)^2\right]\Big\}.$$

Proof. Let $\lambda = k^2, \tau := Imk$. From (2.2), in order to find eigenvalues, we have to construct a solution $\varphi(x,\lambda)$ for all interval $(0,\pi)$. For this reason we construct the following equation

$$-y'' + \left(\alpha\delta'\left(x - \frac{\pi}{2}\right) + q(x)\right) y = \lambda y, \quad x \in (0,\pi),$$

which has a solution $\varphi(x,\lambda)$. For $|\lambda| \to \infty$ uniformly in x one has:

$$\begin{aligned}\varphi(x,\lambda) &= \cos kx + \frac{1}{2k}\int_0^x q(t)dt.\sin kx \\ &+ \frac{1}{4k^2}\left\{(q(x) - q(0)) - \frac{1}{2}\left[\int_0^x q(t)dt\right]^2\right\}\cos kx \\ &+ o\left(\frac{1}{k^2}\exp(|\tau| x)\right), \ x < \frac{\pi}{2}. \end{aligned} \tag{2.5}$$

$$\begin{aligned}\varphi'(x,\lambda) &= -k\sin kx + \frac{1}{2}\int_0^x q(t)dt.\cos kx \\ &+ \frac{1}{4k}\left\{-(q(x) - q(0)) + \frac{1}{2}\left[\int_0^x q(t)dt\right]^2\right\}\sin kx \\ &+ o\left(\frac{1}{k}\exp(|\tau| x)\right), \ x < \frac{\pi}{2}. \end{aligned} \tag{2.6}$$

Using the matching conditions (1.3) we write $\varphi(x,\lambda)$ in $\left(\frac{\pi}{2},\pi\right)$:

$$\begin{aligned}\varphi(x,\lambda) = &-k\frac{\alpha}{2}\left[\sin kx + \sin k(\pi - x)\right] + \left[\left[1 + \frac{\alpha}{4}\int_0^x q(t)dt\right]\cos kx\right. \\ &\left.+\frac{\alpha}{4}\left[\int_0^{\frac{\pi}{2}} q(t)dt - \int_{\frac{\pi}{2}}^x q(t)dt\right]\right]\cos k(\pi - x) \\ &+\frac{1}{4k}\left\{2\int_0^x q(t)dt + \frac{1}{2}\left[q(0) + (\alpha - 1)\,q\left(\frac{\pi}{2}\right) - \alpha q(x)\right]\right. \\ &\left.+\frac{1}{2}\left[\alpha\left(\int_0^x q(t)dt\right)^2 - \left(\alpha - \frac{1}{2}\right)\left(\int_0^{\frac{\pi}{2}} q(t)dt\right)^2\right]\right\}\sin kx \\ &+\frac{1}{4k}\left\{\frac{1}{2}\left[q(0) + (\alpha - 1)\,q\left(\frac{\pi}{2}\right) - \alpha q(x)\right] + \frac{1}{2}\left[\alpha\left(\int_{\frac{\pi}{2}}^x q(t)dt\right)^2\right.\right. \\ &\left.\left.-\alpha\int_0^{\frac{\pi}{2}} q(t)dt\int_{\frac{\pi}{2}}^x q(t)dt + \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} q(t)dt\right)^2\right]\right\}\sin k(\pi - x) \\ &+o\left(\frac{1}{k}\exp(|\tau| x)\right). \end{aligned} \tag{2.7}$$

It follows from (2.7) that for $|\lambda| \to \infty$

$$\Delta(\lambda) = \frac{\alpha}{2}\left(k\sin k\pi - w_0\cos k\pi + w_1 - \frac{w_2}{k}\sin k\pi\right) + o\left(\frac{1}{k}\exp(|\tau|\pi)\right). \tag{2.8}$$

Denote $G_\delta = \{k : |k-n| \geq \delta, n = 0, \pm 1, \pm 2, ...\}$, $\delta > 0$. Let us show that

$$|\sin k\pi| \geq C_\delta \exp(|\tau|\pi), \quad k \in G_\delta, \tag{2.9}$$

$$|\Delta(\lambda)| \geq C_\delta |k| \exp(|\tau|\pi), \quad k \in G_\delta, |k| \geq k^*, \tag{2.10}$$

for sufficiently large $k^* = k^*(\delta)$.

Let $k = \sigma + i\tau$. It is sufficient to prove (2.9) for the domain

$$D_\delta = \left\{ k : \sigma \in \left[-\frac{1}{2}, \frac{1}{2} \right], \tau \geq 0, |k| \geq \delta \right\}.$$

Denote $\theta(k) = |\sin k\pi| \exp(-|\tau|\pi)$. Let $k \in D_\delta$. For $\tau \geq 1$, $\theta(k) \geq C_\delta$. Since $\sin k\pi = (\exp(ik\pi) - \exp(-ik\pi))/2i$, we have for $\tau \geq 1$, $\theta(k) \geq \frac{1}{4}$. Thus, (2.9) is proved. Further, using (2.8) we get for $k \in G_\delta$,

$$\Delta(\lambda) = \frac{\alpha}{2} k \sin k\pi \left(1 + O\left(\frac{1}{k}\right)\right),$$

and consequently (2.10) is valid.

Denote

$$\Gamma_n = \left\{ \lambda : |\lambda| = \left(n + \frac{1}{2}\right)^2 \right\}.$$

By virtue of (2.8)

$$\Delta(\lambda) = f(\lambda) + g(\lambda), \ f(\lambda) = k \sin k\pi, \ |g(\lambda)| \leq C \exp(|\tau|\pi).$$

According to (2.9), $|f(\lambda)| > |g(\lambda)|$, $\lambda \in \Gamma_n$, for sufficiently large n $(n \geq n^*)$. Applying Rouche's theorem (see [3]) to the circle $\sigma_n(\delta) = \{k : |k-n| \leq \delta\}$, we conclude that for sufficiently large n, in $\sigma_n(\delta)$ there is exactly one zero of $\Delta(k^2)$. Since $\delta > 0$ is arbitrary, we must have

$$k_n = n + \varepsilon_n, \ \ \varepsilon_n = o(1), \ n \to \infty. \tag{2.11}$$

Substituting (2.11) into (2.8) we get

$$n \sin \varepsilon_n \pi - \left(w_0 \cos \varepsilon_n \pi + (-1)^{n-1} w_1 \right) + \frac{1}{n} w_2 = 0. \tag{2.12}$$

Then $\sin \varepsilon_n \pi = O\left(\frac{1}{n}\right)$, i.e. $\varepsilon_n = O\left(\frac{1}{n}\right)$. Using (2.12) once more we obtain more preciesly

$$\varepsilon_n = \frac{1}{\pi n}\left(w_0 + (-1)^{n-1} w_1 \right) + \frac{1}{\pi n^2} w_2. \tag{2.13}$$

Substituting (2.13) into (2.11), we get (2.3). Since the $BVP\ L$ is selfadjoint (see [10]), all eigenvalue $\{\lambda_n\}_{n\geq 1}$ are real and simple.

Together with L we consider a $BVP\ \widetilde{L} = L(\widetilde{q})$ of the same form, but with different coefficient $\widetilde{q}$. The following theorem has been proved in [6] for the classical Sturm-Liouville equation. We show it also holds for (1.1)-(1.3).

Theorem 2.1 *If for any* $n \in \mathbb{N}$

$$\lambda_n = \widetilde{\lambda}_n, \langle y_n, \widetilde{y}_n \rangle_{x=\frac{\pi}{2}-0} = 0,$$

then $q(x) = \widetilde{q}(x)$ *almost everywhere* (*a.e.*) *on* $(0, \pi)$.

Proof. Since

$$-y''(x,\lambda)+q(x)y(x,\lambda)=\lambda y(x,\lambda),\ -\widetilde{y}''(x,\lambda)+\widetilde{q}(x)\widetilde{y}(x,\lambda)=\lambda\widetilde{y}(x,\lambda),$$

$$y(0,\lambda)=1,\ y'(0,\lambda)=0 \qquad \widetilde{y}(0,\lambda)=1,\ \widetilde{y}'(0,\lambda)=0$$

it follows from (2.1) that

$$\int_0^{\frac{\pi}{2}} r(x)y(x,\lambda)\,\widetilde{y}(x,\lambda)\,dx=\langle y,\widetilde{y}\rangle_{x=\frac{\pi}{2}-0}, \tag{2.14}$$

where $r(x)=q(x)-\widetilde{q}(x)$. Since $\langle y_n,\widetilde{y}_n\rangle_{x=\frac{\pi}{2}-0}=0$ for $n\in\mathbb{N}\cup\{0\}$, it follows from (2.14) that

$$\int_0^{\frac{\pi}{2}} r(x)y(x,\lambda_n)\,\widetilde{y}(x,\lambda_n)\,dx=0,\ n\in\mathbb{N}\cup\{0\}. \tag{2.15}$$

For $x\le\frac{\pi}{2}$ the following representation holds (see [8,12]):

$$y(x,\lambda)=\cos kx+\int_0^x K(x,t)\cos kt dt,$$

where $K(x,t)$ is a continuous function which does not depend on λ. Hence

$$2y(x,\lambda)\,\widetilde{y}(x,\lambda)=1+\cos 2k\pi x+\int_0^x V(x,t)\cos 2kt dt, \tag{2.16}$$

where $V(x,t)$ is a continuous function which does not depend on λ. Substituting (2.16) into (2.15) and taking the relation $\int_0^\pi r(x)dx=0$ into account, we calculate

$$\int_0^{\frac{\pi}{2}}\left(r(x)+\int_x^{\frac{\pi}{2}} V(t,x)r(t)dt\right)\cos 2k_n x dx=0,\ n\in\mathbb{N}\cup\{0\},$$

which implies from the completeness of the function cosine, that

$$r(x)+\int_x^{\frac{\pi}{2}} V(t,x)r(t)dt=0 \text{ a.e. on } \left(0,\frac{\pi}{2}\right).$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution, it follows that $r(x)=0$ a.e. on $\left(0,\frac{\pi}{2}\right)$. To prove that $q(x)=\widetilde{q}(x)$ a.e. on $\left(\frac{\pi}{2},\pi\right)$ we will consider the supplementary problem $\widehat{L}$:

$$-y''(x,\lambda)+q_1(x)y(x,\lambda)=\lambda y(x,\lambda),\ \ q_1(x)=q(\pi-x),\ 0<x<\frac{\pi}{2}$$

$$U(y):=y'(0,\lambda)=0$$

$$y\left(\frac{\pi}{2}-0,\lambda\right)+\alpha y'\left(\frac{\pi}{2}-0,\lambda\right)=y\left(\frac{\pi}{2}+0,\lambda\right),\ \ y'\left(\frac{\pi}{2}-0,\lambda\right)=y'\left(\frac{\pi}{2}+0,\lambda\right).$$

It follows from (2.1) that

$$\langle y_n,\widetilde{y}_n\rangle_{x=\frac{\pi}{2}+0}=0.$$

A direct calculation implies that $\widehat{y}_n(x):=y_n(\pi-x)$ is the solution to the supplementary problem $\widehat{L}$ and $\widehat{y}_n\left(\frac{\pi}{2}-0\right)=y_n\left(\frac{\pi}{2}+0\right)$. Thus for the supplementary problem $\widehat{L}$ the assumption conditions in Theorem 2.1 are still satisfied. If we repeat the above arguments then yields $r(\pi-x)=0$ on $0<x<\frac{\pi}{2}$, that is $q(x)=\widetilde{q}(x)$ a.e. on $\left(\frac{\pi}{2},\pi\right)$.

3 Inverse nodal problems

In this section, we consider the inverse nodal problems with $\delta'-$ interaction. We obtain uniqueness theorem and a procedure of recovering the potential $q(x)$ on the whole interval $(0,\pi)$ from a dense subset of nodal points. We recall that these results were given for regular Sturm-Liouville problems defined on the interval $(0,1)$ in [7].

The eigenfunctions of the $BVP\ L$ have the form $y_n(x)=\varphi(x,\lambda_n)$. We note that $y_n(x)$ are real-valued functions. Substituting (2.2) into (2.4) and (2.6) we obtain the following asymptotic formulae for $n\to\infty$ uniformly in x:

$$y_n(x)=\cos nx+\frac{1}{2n}\left(\int_0^x q(t)dt-\frac{1}{\pi}\left(w_0+(-1)^{n-1}w_1\right)x\right)\cos nx$$
$$+o\left(\frac{1}{n}\right),\quad x<\frac{\pi}{2}, \tag{3.1}$$

$$y_n\ (x)=\frac{\alpha}{2}n\left[(-1)^n-1\right]\sin nx+\Bigg\{1-\frac{\alpha}{4}\left((-1)^n-1\right)\int_0^x q(t)dt$$
$$+(-1)^n\frac{\alpha}{2}\int_0^{\frac{\pi}{2}}q(t)dt-\frac{\alpha}{2}\left[(-1)^n\left(1-\frac{x}{\pi}\right)+\frac{x}{\pi}\right]\left(w_0+(-1)^{n-1}w_1\right)\Bigg\}\cos nx$$
$$+\frac{1}{4n}\Bigg\{\frac{2}{\pi}\left(w_0+(-1)^{n-1}w_1\right)[\alpha\left((-1)^n-1\right)$$
$$+\alpha\left(w_0+(-1)^{n-1}\right)\left[\left(\frac{x}{\pi}\right)^2-(-1)^n\left(1-\frac{x}{\pi}\right)^2\right]-\left(4+\alpha\int_0^x q(t)dt\right)\Bigg]$$
$$+2\int_0^x q(t)dt+\frac{1}{2}\left[1-(-1)^n\right]\left[q(0)+(\alpha-1)\,q\left(\frac{\pi}{2}\right)-\alpha q\left(x\right)\right]$$
$$+\frac{1}{2}\left[\alpha\left(\int_0^x q(t)dt\right)^2-\left(\alpha-\frac{1}{2}\right)\left(\int_0^{\frac{\pi}{2}}q(t)dt\right)^2\right]-(-1)^n\frac{1}{2}\left[\alpha\left(\int_{\frac{\pi}{2}}^x q(t)dt\right)^2\right.$$
$$\left.\left.-\alpha\int_0^{\frac{\pi}{2}}q(t)dt\int_{\frac{\pi}{2}}^x q(t)dt+\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}q(t)dt\right)^2\right]\right\}\sin nx$$
$$+\frac{\alpha}{2n}\left[(-1)^n\left(\frac{x}{\pi}-1\right)-\frac{x}{\pi}\right]w_2\cos nx+o\left(\frac{1}{n}\right),\quad x>\frac{\pi}{2}. \tag{3.2}$$

We used the same method employed in [[9], ch.1, pp. 14-18] to prove the eigenfunction $y_n(x)$ corresponding eigenvalue λ_n exactly having $\left[\frac{n}{2}\right]+n-1$ simple zeros in $(0,\pi)$. More precisely, $\left[\frac{n}{2}\right]$ of $\left[\frac{n}{2}\right]+n-1$ simple zeros lies in $\left(0,\frac{\pi}{2}\right)$ and the rest is in $\left(\frac{\pi}{2},\pi\right)$:

$$0<x_n^1<x_n^2<\cdots<x_n^{\left[\frac{n}{2}\right]}<\frac{\pi}{2}<x_n^{\left[\frac{n}{2}\right]+1}<\cdots<x_n^{\left[\frac{n}{2}\right]+n-1}<\pi.$$

A set $X_L:=\left\{x_n^j\right\}_{n\geq 2,\,j=\overline{1,\left[\frac{n}{2}\right]+n-1}}$ is called the set of all nodal points of the $BVP\ L$. Let $X_L^k:=\left\{x_{2m+k}^j\right\}_{m\geq 1,\,j=3m-k-1}$, $k=0,1$. Clearly, $X_L^0\cup X_L^1=X_L$.

Set

$$\mu_n^0:=0,\ \mu_n^{\left[\frac{n}{2}\right]+n}:=\pi,\ \mu_n^j:=\frac{j}{n}\pi \text{ for } n=2m+1$$

and

$$\gamma_n^j := \mu_n^j - \frac{\pi}{2n} \text{ for } n = 2m.$$

Inverse nodal problems consist in recovering the problem $q(x)$ from the given set X_L of nodal points or from a certain its part.

Taking (3.1)-(3.2) into account, we obtain the following asymptotic formulae for nodal points as $n \to \infty$ uniformly in j:

for $x_n^j \in \left(0, \frac{\pi}{2}\right)$:

$$x_n^j = \gamma_n^j + \frac{1}{2n^2}\left(\int_0^{\gamma_n^j} q(t)dt - \frac{1}{\pi}(w_0 - w_1)\gamma_n^j\right) + o\left(\frac{1}{n^2}\right), \quad n = 2m, \tag{3.3}$$

$$x_n^j = \gamma_n^j + \frac{1}{2n^2}\left(\int_0^{\gamma_n^j} q(t)dt - \frac{1}{\pi}(w_0 + w_1)\gamma_n^j\right) + o\left(\frac{1}{n^2}\right), \; n = 2m+1, \tag{3.4}$$

for $x_n^j \in \left(\frac{\pi}{2}, \pi\right)$:

$$x_n^j = \gamma_n^j + \frac{\beta}{2n^2}$$

$$\times\left(\int_0^{\gamma_n^j} q(t)dt - \frac{1}{2\pi}(w_0 - w_1)\left(2\gamma_n^j - \pi\right) + d_1\right) + o\left(\frac{1}{n^2}\right), n = 2m, \tag{3.5}$$

$$\frac{x_n^j}{n} = \mu_n^j + \frac{1}{2n^2}$$

$$\times\left(\int_0^{\mu_n^j} q(t)dt - \frac{1}{2\pi}(w_0 + w_1)\left(2\mu_n^j - \pi\right) + d_2\right) + o\left(\frac{1}{n^2}\right), \; n = 2m+1, \tag{3.6}$$

where

$$\beta = \frac{8}{\alpha\pi} - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} q(t)dt - 1,$$

$$d_1 = \frac{4}{\pi}\left(\int_0^{\frac{\pi}{2}} q(t)dt\right)^2 - \frac{4}{\alpha}, \; d_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} q(t)dt - \frac{1}{\alpha}. \tag{3.7}$$

Using these formula we arrive at the following assertion.

Theorem 3.1 *Fix $k \in \{0,1\}$ and $x \in [0,\pi]$. Let $\left\{x_n^j\right\} \subset X_L^k$ be chosen such that $\lim_{n\to\infty} x_n^j = x$. Then there exists a finite limit*

$$g_k(x) := \lim_{n\to\infty} 2n \begin{bmatrix} nx_n^j - j\pi + \frac{\pi}{2}, & if\ x_n^j \in \left(0, \frac{\pi}{2}\right), \\ \frac{1}{\beta}\left(nx_n^j - j\pi + \frac{\pi}{2}\right), & if\ x_n^j \in \left(\frac{\pi}{2}, \pi\right), n = 2m, \\ x_n^j - j\pi, & if\ x_n^j \in \left(\frac{\pi}{2}, \pi\right), n = 2m+1 \end{bmatrix}, \tag{3.8}$$

and

$$g_k(x) = \int_0^x q(t)dt - \frac{1}{\pi}\left(w_0 + (-1)^{k-1} w_1\right)x, \;\; x \le \frac{\pi}{2},$$

$$g_k(x) = \int_0^x q(t)dt - \frac{1}{2\pi}\left(w_0 + (-1)^{k-1} w_1\right)(2x - \pi) + d_k, \; x \ge \frac{\pi}{2} \tag{3.9}$$

where d_1 and d_2 are defined by (3.7).

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Theorem 3.2 *Fix* $k \in \{0,1\}$ *. Let* $X \subset X_L^k$ *be a subset of nodal points which is dense on* $(0,\pi)$ *. Let* $X = \widetilde{X}$*. Then* $q(x) = \widetilde{q}(x)$ *a.e. on* $(0,\pi)$ *. Thus the specification of* X *uniquely determines the potential* $q(x)$ *on* $(0,\pi)$ *. The function* $q(x)$ *can be constructed via the formula*

$$q(x) = g_k'(x) + \frac{1}{\pi}\left(g_k(\pi) - g_k(0)\right), \tag{3.10}$$

where $g_k(x)$ *is calculated by (3.9).*

Proof. Formula (3.10) follows from (3.9), (1.4) and (2.4). Note that by (3.9), we have

$$g_k'(x) = q(x) - \frac{w_0 + (-1)^{k-1}w_1}{\pi},\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{3.11}$$

hence

$$g_k(\pi) - g_k(0) = \int_0^\pi q(x)dx - \left(w_0 + (-1)^{n-1}w_1\right). \tag{3.12}$$

Then (3.10) can be derived directly from (3.11) and (3.12). Note that if $X = \widetilde{X}$, then (3.8) yields $g_k(x) \equiv \widetilde{g}_k(x)$, $x \in [0,\pi]$. By (3.10), we obtain $q(x) = \widetilde{q}(x)$ a.e. on $(0,\pi)$.

By a similar way, in general the following theorem can be proved.

Theorem 3.3 *Let* $X \subset X_L$ *be a dense on* $(0,\pi)$ *subset of nodal points. Let* $X = \widetilde{X}$*. Then* $q(x) = \widetilde{q}(x)$ *a.e. on* $(0,\pi)$*.*

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