Transactions of NAS of Azerbaijan, Issue Mathematics, **37** (4), 120–131 (2017).
Series of Physical-Technical and Mathematical Sciences

Uniformly boundedness of the operator-valued functions arising in the solution of convolution differential-operator equations

Hummet K. Musaev

Received: 15.05.2017 / Revised: 12.10.2017 / Accepted: 19.11.2017

Abstract. *In this paper, the properties of the operator valued functions arising in the solution of convolution differential-operator equations are investigated in weighted Besov spaces. Especially, the uniformly boundedness of the derivatives for operator-valued functions are established. These results applied to obtain the boundary value problems for abstract integro-differential equations.*

Keywords. uniformly positive operators, vector valued weighted Besov spaces, operator-valued functions, convolution differential-operator equations.

Mathematics Subject Classification (2010): 34G10, 45J05

1 Introduction

In recent years, the maximal regularity properties of differential operator equations has been studied extensively, e.g. in [1-4, 5-8, 13]. Moreover, convolution-differential equations (CDEs) have been studied, e.g. in [13,14] and the refence therein. However, the convolution-differential operator equation (CDOE) is a relatively less investigated subject. In this direction we can reminded the works [9-11] . The main aim of the present paper is to establish the uniformly boundedness of the operator-valued functions arising in the solution of the degenerate CDOE

$$(L+\lambda)\,u=\sum_{k=0}^{l}a_k*\frac{d^{[k]}u}{dx^{[k]}}+A*u+\lambda u=f \tag{1.1}$$

in E-valued weighted Besov spaces, where E is a Banach space, $A=A(x)$ is a linear operator in E, $a_k=a_k(x)$ are complex-valued functions, λ is a complex parameter, $\gamma\,(x)$ is a measurable positive function in $(-\infty,\infty)$ and

$$u^{[k]}=\left(\gamma(x)\frac{d}{dx}\right)^k u.$$

We prove that the solution of equation (1.1) can be represented in the form

$$u(x)=F^{-1}\left[\hat{A}(\xi)\ \left(\lambda+L\left(\xi\right)\right)\right]^{-1}\hat{f}. \tag{1.2}$$

H.K. Musaev
Baku State University, Baku, Azerbaijan
E-mail: gkm55@mail.ru

The methods of proof based on operator-valued multiplier theorems, theory of elliptic operators, vector-valued convolution integrals, operator theory and etc.

Let $x = (x_1, x_2, ..., x_n) \in \Omega \subset R^n$. Let $\gamma(x)$ be a measurable positive function in domain $\Omega \subset R^n$. $L_{p,\gamma}(\Omega; E)$ denotes the space of all strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \gamma(x) dx \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x\in\Omega} [\|f(x)\|_E \gamma(x)].$$

For $\gamma(x) \equiv 1$ the spaces $L_{p,\gamma}(\Omega; E)$ will denoted by $L_p(\Omega; E)$.

Weight function γ satisfies A_p condition (i.e. $\gamma \in A_p$) if there is a constant C such that

$$\left(\frac{1}{|Q|} \int_Q \gamma(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x)\, dx \right)^{p-1} \le C,\ p \in (1, \infty)$$

for all cubes $Q \subset R^n$.

Remark 1.1 By virtue of [12] the following weighted functions

$$\gamma(x) = |x|^\alpha, x \in R,\ -1 < \alpha < p-1, \gamma(x) = \prod_{k=1}^N \left(1 + \sum_{j=1}^n |x_j|^{\alpha_{jk}} \right)^{\beta_k},$$

belong to A_p class, where $x = (x_1, x_2, ..., x_n)$, $\alpha_{jk} \ge 0$, $N \in \mathbf{N}$, x_k, $\beta_k \in R$, $\mathbf{N}$ =1, 2, ... is the set of natural numbers.

Let $S = S(R^n; E)$ denote a Schwartz class, i.e., a space of E-valued rapidly desreasing smooth functions on R^n and $S'(R^n; E)$ denotes the space of E-valued tempered distributions. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where α_i are integers. An E-valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S'(R^n; E)$, if the equality

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

holds for all $\varphi \in S$.

Let $f \in S'(R^n; E)$. Let F denote the Fourier transform. Through this section the Fourier transform of a function f will be denoted by $\widehat{f}$, $Ff = \hat{f}$ and $\frac{d}{d\xi} A(\xi)$ by $A'(\xi)$. It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n} \widehat{f},\ D_\xi^\alpha(F(f)) = F\left[(-ix_1)^{\alpha_1} ... (-ix_n)^{\alpha_n} f \right].$$

Let E_1 and E_2 be two Banach spaces. $\mathcal{L}(E_1, E_2)$ denotes the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ this space will denoted by $\mathcal{L}(E)$.

The function $\Psi \in L_\infty(R^n; \mathcal{L}(E_1, E_2))$ is called a multiplier from $B^s_{p,q,\gamma}(R^n; E_1)$ to $B^s_{p,q,\gamma}(R^n; E_2)$ for $p, q \in (1, \infty)$ if the map

$$u \to Ku = F^{-1} \Psi(\xi) Fu, u \in S(R^n; E_1)$$

is well defined and extends to a bounded linear operator

$$K : B^s_{p,q,\gamma}(R^n; E_1) \to B^s_{p,q,\gamma}(R^n; E_2).$$

Let $\mathbb{C}$ be the set of complex numbers and

$$S_{\varphi}=\{\lambda:\lambda\in\mathbb{C},\ |\arg\lambda|\leq\varphi\}\cup\{0\}\,,\ \ 0\leq\varphi<\pi.$$

A linear operator $A=A(x)$ is said to be uniformly $\varphi-$positive in a Banach space E, if $D(A(x))$ is dense in E and does not depend on x,

$$\left\|(A(x)+\lambda I)^{-1}\right\|_{\mathcal{L}(E)}\leq M(1+|\lambda|)^{-1}$$

with $M>0$, $\lambda\in S_{\varphi}$, $\varphi\in[0,\pi)$, where I is the indentity operator in E. Sometimes instead of $A+\lambda I$ we will write $A+\lambda$ and denote it by A_{λ}.

Let $E(A)$ denote the space $D(A)$ with graphical norm

$$\|u\|_{E(A)}=(\|u\|^{p}+\|Au\|^{p})^{\frac{1}{p}}\,,\qquad\qquad 1\leq p\leq\infty.$$

Let $A=A(t)\in S'(R;\mathcal{L}(D(A),E))$. Then the Fourier transformation of $A(t)$ in the sense of Schwartz distributions is defined as follows:

$$\left(\widehat{A}u,\varphi\right)=(Au,\widehat{\varphi})\,,\ \ u\in D(A)\text{ and }\varphi\in S(R).$$

Let $A=A(t)$ be a uniformly $\varphi-$positive operator in E. Then, it is differentiable if

$$\left(\frac{d}{dt}A\right)u=A^{'}(t)u=\lim_{h\to 0}\frac{A(t+h)u-A(t)u}{h}<\infty$$

for all $u\in E(A)$.

Let $A=A(t)$ be a uniformly $\varphi-$positive operator in E and $u\in B^{s}_{p,q}(R;E(A))$ and

$$(A*u)(t)=\int_{R}A(t-y)u(y)dy.$$

Let $y\in R, m\in N$ and $e_i, , i=1,2,...,n$, be standart unit vectors in R^{n}. Let

$$\triangle_{i}(y)f(x)=f(x+ye_{i})-f(x),...,\triangle_{i}^{m}(y)f(x)=\triangle_{i}(y)\left[\triangle_{i}^{m-1}(y)f(x)\right]=$$
$$=\sum_{k=0}^{m}(-1)^{m+k}C_{m}^{k}f(x+kye_{i}).$$

Let

$$\triangle_{i}^{m}(y)=\triangle_{i}^{m}(\Omega,y)=\begin{cases}\triangle_{i}^{m}(y)f(x)\text{ for }[x,\ x+mye_{i}]\subset\Omega\\ \quad 0\qquad\text{ for }[x,\ x+mye_{i}]\notin\Omega.\end{cases}$$

Let m be integers, s be positive numbers and

$$m>s,\quad 1\leq p\leq\infty,\ \ 1\leq q\leq\infty,\ y_{0}>0.$$

Let $\gamma\left(x\right)$ be a measurable positive function in domain $\Omega\subset R^{n}$.

Consider the E-valued weighted Besov space $B^{s}_{p,q,\gamma}(\Omega;E)$ defined as:

$$B^{s}_{p,q,\gamma}=B^{s}_{p,q,\gamma}(\Omega;E)=\{f:f\in L_{p,\gamma}(\Omega;E),\ \ \|f\|_{B^{s}_{p,q,\gamma}(\Omega;E)}$$
$$=\|f\|_{L_{p,\gamma}(\Omega:E)}+\sum_{i=1}^{n}\left(\int_{0}^{y_{0}}y^{-(sq+1)}\left\|\triangle_{i}^{m}(y)D_{i}f(x)\right\|^{q}_{L_{p,\gamma}(\Omega;E)}dy\right)^{\frac{1}{q}}<\infty,$$

$$\|f\|_{B^s_{p,\infty,\gamma}(\Omega;E)}=\sum_{i=1}^{n}\sup_{0<y\leq y_0}\frac{\left\|\triangle_i^m(y)D_if(x)\right\|_{L_{p,\gamma}(\Omega;E)}}{y^s},\,1\leq p\leq\infty,1\leq q\leq\infty\}\,.$$

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded to E. Let l is a positive integer and $D_k^l=\frac{\partial^l}{\partial x_k}$. Consider the E- valued function spaces defined by

$$B^{l,s}_{p,q,\gamma}(\Omega;E_0,E)=\Big\{u:u\in B^s_{p,q,\gamma}(\Omega;E_0),\ D^l_ku\in B^s_{p,q,\gamma}(\Omega;E),$$

$$\|u\|_{B^{l,s}_{p,q,\gamma}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q,\gamma}(\Omega;E_0)}+\sum_{k=1}^{n}\left\|D^l_ku\right\|_{B^s_{p,q,\gamma}(\Omega;E)}<\infty\Big\}.$$

For $\gamma\,(x)\equiv 1$ the spaces $B^s_{p,q,\gamma}(\Omega;E)$ and $B^{l,s}_{p,q,\gamma}(\Omega;E_0,E)$ will denoted by $B^s_{p,q}(\Omega;E)$ and $B^{l,s}_{p,q}(\Omega;E_0,E)$, respectively.

Consider the E-valued weighted Besov space $B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)$ defined as

$$B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)=\Big\{u:u\in B^s_{p,q,\gamma}(\Omega;E_0),\ D^{[l]}_ku\in B^s_{p,q,\gamma}(\Omega;E),$$

$$\|u\|_{B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q,\gamma}(\Omega;E_0)}+\sum_{k=1}^{n}\left\|D^{[l]}_ku\right\|_{B^s_{p,q,\gamma}(\Omega;E)}<\infty\Big\}.$$

The space $C(\Omega;E)$ and $C^{(m)}(\Omega;E)$ will denote the spaces of E-valued bounded, continuous and m-times continuously differentiable functions on Ω, respectively, and $D(\Omega;E)$ will denote the collection of infinitely differentiable E-valued functions with compact support on Ω.

2 Uniformly boundedness of the operator-valued functions

Consider the following degenerate convolution differential operator equation

$$(L+\lambda)\,u=\sum_{k=0}^{l}a_k*\frac{d^{[k]}u}{dx^{[k]}}+A*u+\lambda u=f(x), \tag{2.1}$$

in $E-$valued weighted Besov spaces, where $x\in R,\ A=A(x)$ is a linear operator in Banach space E, $a_k=a_k(x)$ are complex-valued functions, λ is a complex parameter.

We assume that

$$y=\int_0^x z^{-\gamma}dz<\infty. \tag{2.2}$$

It is clear to see that, under the substitution (2.2) the spaces $B^s_{p,q,\gamma}(R;E)$ and $B^{[l],s}_{p,q,\gamma}(R;E(A),E)$ are mapped isomorphically onto weighted spaces $B^s_{p,q,\tilde\gamma}(R;E)$, $B^{l,s}_{p,q,\tilde\gamma}(R;E(A),E)$ respectively, where

$$\tilde\gamma(y)=\gamma\left(x(y)\right).$$

Moreover, under the substitation (2.2) the degenerate problem (2.1) is redused to the following non-degenerate problem

$$(L+\lambda)\,u=\sum_{k=0}^{l}a_k*\frac{d^ku}{dx^k}+A*u+\lambda u=f(x), \tag{2.3}$$

considered in weighted space $B^s_{p,q,\tilde{\gamma}}(R;E)$, where A is a linear operator in Banach space E and a_k are complex numbers.

By applying the Fourier transform to equation (2.3) we obtain

$$\left[\sum_{k=0}^{l}\hat{a}_k(\xi)(i\xi)^k+\hat{A}(\xi)+\lambda\right]\hat{u}(\xi)=\hat{f}(\xi),$$

where $\hat{a}_k(\xi)$, $\hat{A}(\xi)$, $\hat{u}(\xi)$ and $\hat{f}(\xi)$ denote the Fourier transforms of $a_k(x)$, $A(x)$, $u(x)$ and $f(x)$ respectively.

Condition 1. Suppose

$$a_k\in L_1(R),\quad L(\xi)=\sum_{k=o}^{l}\hat{a}_k(\xi)(i\xi)^k\in S_{\varphi_1}$$

$$\varphi_1+\varphi<\pi,\ \ |L(\xi)|\geq C\max_k|\hat{a}_k(\xi)|\,|\xi|^l\,,$$

and $\hat{A}(\xi)$ is a uniformly $\varphi-$positive operator in a Banach space E with $\varphi\in[0,\pi)\,.$

Since $L(\xi)\in S_{\varphi_1}$ for all $\xi\in R$ and $\hat{A}(\xi)$ $\varphi-$positive, the operator $\hat{A}(\xi)+\lambda+L\,(\xi)$ is invertible in E, i.e. $\left[\hat{A}(\xi)+\lambda+\ L(\xi)\right]^{-1}\in\mathcal{L}(E)$. So we obtain that the solution of equation (2.3) can be represented in the form

$$u(x)=F^{-1}\left[\hat{A}(\xi)+(\lambda+\ L(\xi))\right]^{-1}\hat{f}\,. \tag{2.4}$$

For a coercive estimate of equation (2.3) it is sufficient to show the uniformly boundedness of the following operator-valued functions, respectively

$$G_1(\xi,\lambda)=\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1},$$

$$G_2(\xi,\lambda)=\hat{A}(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1},\ \ G_3(\xi,\lambda)=\lambda\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}.$$

Since,

$$\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\left\|a_k*\frac{d^ku}{dx^k}\right\|_{B^s_{p,q,\gamma(R;E)}}=\left\|F^{-1}G_1\left(\xi,\lambda\right)\hat{f}\right\|_{B^s_{p,q,\gamma(R;E)}},$$

$$\|A*u\|_{B^s_{p,q,\gamma(R;E)}}=\left\|F^{-1}G_2\left(\xi,\lambda\right)\hat{f}\right\|_{B^s_{p,q,\gamma(R;E)}},$$

$$\|\lambda u\|_{B^s_{p,q,\gamma(R;E)}}=\left\|F^{-1}G_3\left(\xi,\lambda\right)\hat{f}\right\|_{B^s_{p,q,\gamma(R;E)}}.$$

Lemma A. *Let Condition 1 be satisfied and* $\lambda\in S_\varphi$. *Then, operator-valued functions* $G_i(\xi,\lambda),\ i=1,2,3,$ *are uniformly bounded.*

Proof. First we prove the uniform boundedness of the operator function $G_3(\xi,\lambda)$.By using the well known inequalities [3, Lemma 2.3]

$$|\lambda + L(\xi)| \geq C\left(|\lambda| + |L(\xi)|\right),$$

for $L(\xi) \in S_{\varphi_1}$, $\lambda \in S_\varphi,\ \varphi_1 + \varphi < \pi$ and due to positivity of operator $\hat{A}(\xi)$ we have

$$\|G_3(\xi,\lambda)\|_{\mathcal{L}(E)} \leq M\,|\lambda| \left\|\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$
$$\leq M\,|\lambda|\left[1 + |\lambda| + |L(\xi)|\right]^{-1} \leq M.$$

Let us note that for the sake of simplicity we shall not change constants in every step. By virtue the resolvent properties of positive operators we obtain

$$\|G_2(\xi,\lambda)\|_{\mathcal{L}(E)} = \left\|\hat{A}(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$
$$= \left\|I - (\lambda + L(\xi))\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$
$$\leq 1 + |\lambda + L(\xi)|\left\|\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$
$$\leq 1 + M\,|\lambda + L(\xi)|\left(1 + |\lambda + L(\xi)|\right)^{-1} \leq 1 + M.$$

Next, let us consider $G_1(\xi,\lambda)$. It is clear to see that

$$\|G_1(\xi,\lambda)\|_{\mathcal{L}(E)} = \left\|\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$
$$\leq C\sum_{k=0}^{l} |\lambda|\,|\widehat{a}_k|\left[|\xi|\,|\lambda|^{-\frac{1}{l}}\right]^k\left\|\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}.$$

Taking into account Condition1, of the well-known Hausdorff-Youngs inequality and [3] we have

$$\|G_1(\xi,\lambda)\|_{\mathcal{L}(E)} \leq C\sum_{k=0}^{l} |\lambda|\,|\widehat{a}_k|\left[1 + |\xi|^l\,|\lambda|^{-1}\right]\left[1 + |\lambda + L(\xi)|\right]^{-1}$$
$$\leq C\sum_{k=0}^{l}\left[|\lambda|\,|\widehat{a}_k| + |\widehat{a}_k|\,|\xi|^l\right]\left[1 + |\lambda| + |L(\xi)|\right]^{-1}$$
$$\leq C\left[|\lambda| + |\xi|^l\sum_{k=0}^{l}|\widehat{a}_k|\right]\left[1 + |\lambda| + \left|\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\right|\right]^{-1} \leq C.$$

Through this section $\frac{d^m}{d\xi^m}\hat{a}_k\left(\xi\right)$, $\frac{d^m}{d\xi^m}\hat{A}\left(\xi\right)$ and $\frac{d^m}{d\xi^m}G_i(\xi,\lambda)$ will be denoted as $\hat{a}_k^{(m)}\left(\xi\right)$, $\hat{A}^{(m)}\left(\xi\right)$ and $G_i^{(m)}(\xi,\lambda)$ respectively.

Condition 2. Suppose

$$\hat{a}_k \in C^{(m)}(R),\ \ \hat{A}^{(m)}(\xi)\,\hat{A}^{-1}(\xi_0) \in C^{(m)}(R;\mathcal{L}(E)),\ \xi_0 \in R,\ \text{ and}$$

$$\left|\hat{a}_k^{(m)}(\xi)\right| \le M_1,\ \ \left|\xi^m \hat{a}_k^{(m)}(\xi)\right| \le M_2, \tag{2.5}$$

$$\left\|\hat{A}^{(m)}(\xi)\,\hat{A}^{-1}(\xi)\right\|_{\mathcal{L}(E)} \le M_3,\ \ \left\|\xi^m \hat{A}^{(m)}(\xi)\,A^{-1}(\xi)\right\|_{\mathcal{L}(E)} \le M_4, \tag{2.6}$$

where $m = 1, 2$ and M_i, $i = 1, 2, 3, 4$ are positive constants.

Lemma B. *Let Condition 1 and Condition 2 be satisfied, $\lambda \in S_\varphi$ for $|\lambda| \ge |\lambda_0|$ and $m = 1$.*

Then, operator functions $G_i'(\xi,\lambda)$, $i = 1, 2, 3$ are uniformly bounded.

Proof. First we shall prove that $G_2'(\xi,\lambda)$ is uniformly bounded.

$$\begin{aligned}
\|G_2'(\xi,\lambda)\|_{\mathcal{L}E)} &\le \left\|\hat{A}'(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}E)} \\
&+ \left\|\hat{A}(\xi)\hat{A}'(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)} \\
&+ \left\|\hat{A}(\xi)L'(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)} \\
&\le \left\|\hat{A}'(\xi)\hat{A}^{-1}(\xi)\hat{A}(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}
\end{aligned}$$

$$+ \left\|\hat{A}'(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)} \left\|\hat{A}(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$

$$+ \left\|L'(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)} \left\|\hat{A}(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}.$$

By using (2.5) and [3] we have

$$\begin{aligned}
\|G_2'(\xi,\lambda)\|_{\mathcal{L}(E)} &\le C\Bigg(\left\|\hat{A}'(\xi)\hat{A}^{-1}(\xi)\right\|_{\mathcal{L}(E)}\left\|\hat{A}(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)} \\
&+ \left\|\hat{A}'(\xi)\left[\hat{A}(\xi) + (\lambda + L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)} + |L'(\xi)|\left[1 + |\lambda + L(\xi)|\right]^{-1}\Bigg) \\
&\le C\,|L'(\xi)|\left[1 + |\lambda| + \left|\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\right|\right]^{-1}
\end{aligned}$$

$$\le C\left[\left|\sum_{k=0}^{l}\widehat{a}_k'(\xi)(i\xi)^k\right| + \left|\sum_{k=1}^{l}\widehat{a}_k(\xi)(i\xi)^{k-1}\right|\right]\left[1 + |\lambda| + \left|\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\right|\right]^{-1}. \tag{2.7}$$

By using (2.5) and (2.7), we obtain

$$\left|\sum_{k=0}^{l}\widehat{a}_k'(\xi)(i\xi)^k\right| \le C\left[1+|\lambda|+\left|\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\right|\right],$$

$$\left|\sum_{k=1}^{l}\widehat{a}_k(\xi)(i\xi)^{k-1}\right| \le C\left[1+|\lambda|+\left|\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\right|\right].$$

Finally

$$\left\|G_2'(\xi,\lambda)\right\|_{\mathcal{L}(E)} \le C\sum_{k=0}^{l}\left|\widehat{a}_k'(\xi)(i\xi)^k\right|\left[1+|\lambda|+\left|\sum_{k=0}^{l}\widehat{a}_k(\xi)(i\xi)^k\right|\right]^{-1} \le C. \tag{2.8}$$

In a smilar way, the uniformly boundedness of $G_3'(\xi,\lambda)$ is proved.

Next we shall prove the uniformly boundedness of the operator function $G_1'(\xi,\lambda)$. Similarly

$$\left\|G_1'(\xi,\lambda)\right\|_{\mathcal{L}(E)} \le \left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k'(\xi)(i\xi)^k\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$

$$+\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(ik)(i\xi)^{k-1}\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$

$$+\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k L'(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)}$$

$$+\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k\hat{A}'(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)}.$$

The uniformly boundedness of the first and second terms in the last expression can be easily proved. Really, by virtue of (2.5), (2.8) and conditions of the Lemma B we obtain, respectively

$$\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k'(\xi)(i\xi)^k\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$

$$\le\sum_{k=0}^{l}\left|\widehat{a}_k'(\xi)\right|\left\||\lambda|^{1-\frac{k}{l}}\,(i\xi)^k\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)} \le C,$$

$$\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(ik)(i\xi)^{k-1}\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$

$$\leq \sum_{k=1}^{l} |\lambda_0|^{-\frac{1}{l}} |\widehat{a}_k(\xi)| \, |\lambda|^{1-\frac{k-1}{l}} (i\xi)^{k-1} \left\| \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|_{\mathcal{L}(E)}$$

$$\leq C \sum_{k=1}^{l} |\lambda| \, |\widehat{a}_k(\xi)| \left[|\lambda|^{-\frac{1}{l}} |(i\xi)| \right]^{k-1} \left\| \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|_{\mathcal{L}(E)} \leq C$$

Now we show the uniformly boundedness of the operator-valued functions

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \, \widehat{a}_k(\xi)(i\xi)^k L^{'}(\xi) \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-2} \quad \text{and}$$

$$\sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \, \widehat{a}_k(\xi)(i\xi)^k \hat{A}'(\xi) \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-2}.$$

Similarly, with help of (2.7) , we have

$$\left\| \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \, \widehat{a}_k(\xi)(i\xi)^k L^{'}(\xi) \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-2} \right\|_{\mathcal{L}(E)}$$

$$\leq C \left| L'(\xi) \right| \left\| \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|$$

$$\times \sum_{k=0}^{l} |\lambda| \, |\widehat{a}_k(\xi)| \left[|\lambda|^{-\frac{1}{l}} |(i\xi)| \right]^{k} \left\| \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|_{\mathcal{L}(E)} \leq C.$$

Respectively

$$\left\| \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{l}} \, \widehat{a}_k(\xi)(i\xi)^k \hat{A}'(\xi) \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-2} \right\|_{\mathcal{L}(E)}$$

$$\leq C \left\| \hat{A}'(\xi) \hat{A}^{-1}(\xi) \hat{A}(\xi) \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|_{\mathcal{L}(E)}$$

$$\times \sum_{k=0}^{l} |\lambda| \, |\xi|^k \, |\lambda|^{\frac{-k}{l}} \left\| \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|_{\mathcal{L}(E)}$$

$$\leq C \left\| \hat{A}'(\xi) \hat{A}^{-1}(\xi) \right\|_{\mathcal{L}(E)} \left\| \hat{A}(\xi) \left[\hat{A}(\xi) + (\lambda + L(\xi)) \right]^{-1} \right\|_{\mathcal{L}(E)} \leq C.$$

In a similar way the uniformly boundedness of $G_3'(\xi, \lambda)$ is proved. Hence, operator functions $G_i'(\xi, \lambda)$, $(i = 1, 2, 3)$ are uniformly bounded.

Theorem 2.1 *Suppose the Conditions 1 and 2 is satisfied. Then, operator functions* $G_i^{(m)}(\xi, \lambda)$, $(i = 1, 2, 3; m = 0, 1, 2.)$ *are uniformly bounded and the following estimates hold:*

$$|\xi|^m \left\| G_i^{(m)}(\xi, \lambda) \right\|_{\mathcal{L}(E)} \leq C.$$

Proof. We prove the uniformly boundedness of the operator-function $G''_i(\xi,\lambda)$. For this, it is sufficient to prove the uniformly boundedness of $G''_2(\xi,\lambda)$.

It's known that

$$G''_2(\xi,\lambda)=\left\{\hat{A}'(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}-\hat{A}(\xi)\hat{A}'(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right.$$
$$\left.-\hat{A}(\xi)L'(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\}'.$$

After some transformations we have

$$\left\|G''_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le\left\|\hat{A}''(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}$$
$$+2\left\|\left[\hat{A}'(\xi)\right]^2\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)}+\left\|\hat{A}(\xi)\hat{A}''(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)}$$
$$+2\left\|\hat{A}'(\xi)L'(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)} \tag{2.9}$$
$$2\left\|\hat{A}(\xi)\left(\hat{A}'(\xi)+L'(\xi)\right)^2\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-3}\right\|_{\mathcal{L}(E)}$$
$$+\left\|\hat{A}(\xi)L''(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)}.$$

Due to the resolvent properties of the $\varphi-$positive operator $\hat{A}(\xi)$ and the conditions of the Lemma B, we have

$$\left\|\hat{A}''(\xi)\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}\right\|_{\mathcal{L}(E)}\le\left\|\hat{A}''(\xi)\hat{A}^{-1}(\xi)\right\|_{\mathcal{L}(E)}\left\|G_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le C,$$

$$\left\|\left[\hat{A}'(\xi)\right]^2\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-2}\right\|_{\mathcal{L}(E)}\le\left\|\hat{A}'(\xi)\hat{A}^{-1}(\xi)\right\|^2_{\mathcal{L}(E)}\left\|G_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le C.$$

Similarly, we obtain that the other terms of the expression (2.9) are also uniformly bounded. So, we obtain

$$\left\|G''_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le C. \tag{2.10}$$

The uniformly boundedness of the $G''_1(\xi,\lambda)$ and $G''_3(\xi,\lambda)$ are shown similarly. Since, operator functions $G''_1(\xi,\lambda)$ and $G''_3(\xi,\lambda)$ contain the similar terms as are in $G''_2(\xi,\lambda)$.So, we proved the $\|G''_i(\xi,\lambda)\|_{\mathcal{L}(E)}\le C$ for $i=1,2,3$.Hence, operator functions $G_i^{(m)}(\xi,\lambda)$ are uniformly bounded.

Now, we prove that $|\xi|^m\left\|G_i^{(m)}(\xi,\lambda)\right\|_{\mathcal{L}(E)}$ is uniformly bounded. First we prove that $|\xi|^m\left\|G_2^{(m)}(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le C$, for $m=1$. For $m=0$ it is proved.

For the sake simplicity, we redenote the $\left[\hat{A}(\xi)+(\lambda+L(\xi))\right]^{-1}$ by $H(\xi,\lambda)$. Then we have

$$G_1(\xi,\lambda)=\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^kH(\xi,\lambda),$$
$$G_2(\xi,\lambda)=\hat{A}(\xi,\lambda)H(\xi,\lambda),\ G_3(\xi,\lambda)=\lambda H(\xi,\lambda).$$

It is clear to see that

$$|\xi|\left\|G_2'(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le|\xi|\left(\left\|\hat{A}'(\xi)H(\xi,\lambda)\right\|_{\mathcal{L}(E)}+\left\|\hat{A}(\xi)\hat{A}'(\xi)H^2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\right.$$
$$\left.+\left\|\hat{A}(\xi)L'(\xi)H^2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\right)\le C\left(\left\|\xi\hat{A}'(\xi)\hat{A}^{-1}(\xi)\right\|_{\mathcal{L}(E)}\left\|G_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\right.$$
$$\left.+\left\|\xi\hat{A}'(\xi)\hat{A}^{-1}(\xi)\right\|_{\mathcal{L}(E)}\left\|G_2(\xi,\lambda)\right\|^2_{\mathcal{L}(E)}+|\xi|\left|L'(\xi)\right|\left\|G_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\left\|H(\xi,\lambda)\right\|_{\mathcal{L}(E)}\right).$$

Using (2.5), (2.6) and (2.7) we get

$$|\xi|\left\|G_2'(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le C\left|L'(\xi)\right|\left[1+|\lambda+L(\xi)|\right]^{-1}\le C. \tag{2.11}$$

Similarly

$$\left\|\xi G_1'(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le\xi\left(\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k'(\xi)(i\xi)^kH(\xi,\lambda)\right\|_{\mathcal{L}(E)}\right.$$
$$+\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)\,(ik)\,(i\xi)^{k-1}H(\xi,\lambda)\right\|_{\mathcal{L}(E)}$$
$$+\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^kL'(\xi)H^2(\xi,\lambda)\right\|_{\mathcal{L}(E)}$$
$$\left.+\left\|\sum_{k=0}^{l}|\lambda|^{1-\frac{k}{l}}\,\widehat{a}_k(\xi)(i\xi)^k\hat{A}'(\xi)H^2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\right)$$
$$\le\sum_{k=0}^{l}\left|\xi\widehat{a}_k'(\xi)\right|\left\|(i\xi)^k|\lambda|^{1-\frac{k}{l}}\,H(\xi,\lambda)\right\|_{\mathcal{L}(E)}$$
$$+\sum_{k=0}^{l}\left|\widehat{a}_k(\xi)\right|\left\|(i\xi)^k|\lambda|^{1-\frac{k}{l}}\,H(\xi,\lambda)\right\|_{\mathcal{L}(E)}+\sum_{k=0}^{l}|\lambda|\,|\xi|^k\,|\lambda|^{-\frac{k}{l}}\left|\xi L'(\xi)\right|\left\|H(\xi,\lambda)\right\|^2_{\mathcal{L}(E)}$$
$$+\sum_{k=0}^{l}|\lambda|\,|\xi|^k\,|\lambda|^{-\frac{k}{l}}\left\|G_2(\xi,\lambda)\right\|_{\mathcal{L}(E)}\left\|\hat{A}'(\xi)\hat{A}^{-1}(\xi)\right\|_{\mathcal{L}(E)}.$$

Then by (2.5)-(2.8) and the conditions of the Lemma B we obtain from the above $\left\|\xi G_1'(\xi,\lambda)\right\|_{\mathcal{L}(E)}\le C$.

In a similar way we have operator functions $\xi G_3'(\xi,\lambda)$ are uniformly bounded.

Finally we prove that

$$|\xi|^m \left\| G_i^{(m)}(\xi,\lambda) \right\|_{\mathcal{L}(E)} \leq C \;\text{ for }\; m=2. \qquad (2.12)$$

From the representations of $G_i(\xi,\lambda)$, $i = 1,2,3$, it is easy to see that the operator-valued functions $\xi^m G_i^{(m)}(\xi,\lambda)$, for $m = 2$, contain similar terms to $\xi G_i'(\xi,\lambda)$. Taking into account representations of $\xi G_i'(\xi,\lambda)$ in a smiliar way, we have $|\xi|^m \left\| G_i^{(m)}(\xi,\lambda) \right\|_{\mathcal{L}(E)} \leq C$.

Remark 2.1 The results obtained have good applications. With the application of the inequalites obtained, we have coercive estimates for convolution differential-operator equations. It is also proved that the operator-valued functions $G_i(\xi,\lambda)$ are uniformly bounded multipliers in $B^s_{p,q,\gamma}(R;E)$.

References

1. Agarwal, R., O' Regan, D., Shakhmurov, V.B.: *Separable anisotropic differential operators in weighted abstract spaces and applications*, J. Math. Anal. Appl. **338**, 970–983 (2008).
2. Amann, H.: *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applictions*, Math.Nachr. **186**, 5–56 (1997).
3. Dore, G. and Yakubov, S.: *Semigroup estimates and non coercive boundary value problems*, Semigroup Form, **60**, 93–121 (2000).
4. Girardi, M., Weis, L.: *Operator-valued multiplier theorems on Besov spaces*, Math.Nachr., **251**, 34–51 (2003).
5. Guliev, V.S.: *To the theory of multipliers of Fourier integrals for functions with values in Banach spaces*, Trudy Math. Inst. Steklov, **214** (17), 164–181 (1996).
6. Krein, S.G.: Linear Differential Equations in Banach Space, A.M.S., *Providence, RI* (1982).
7. Musaev, H.K., Shakhmurov, V.B.: *Regularity properties of degenerate convolution-elliptic equations*, Boundary Value Problems, **50**, 1–19 (2016).
8. Shakhmurov, V.B.: *Embedding and maximal regular differential operators in Banach-valued weighted spaces*, Acta mathematica Sinica **22** (5), 1493–1508 (2006).
9. Shakhmurov, V.B., Shahmurov R.: *Maximal B-Reqular Integro-differential equaions*. Chinese annals of mathematics. **30** (1), 39–50 (2009).
10. Shakhmurov, V.B., Musaev, H.K.: *Separability properties of convolution-differential operator equations in weighted L_p spaces.* Appl. and Comput. Math., **14** (2), 221–233 (2015).
11. Shakhmurov, V.B., Musaev, H.K.: *Maximal regular convolution differential equations in weighted Besov spaces*, Appl. and Comput. Math., **16** (2), 190–200 (2017).
12. Triebel, H.: *Spaces of distributions with weights. Multiplier on L_p spaces with weights.* Math. Nachr. **78**, 339–356 (1977).
13. Yakubov, S., Yakubov, Ya.: Differential-operator equations. Ordinary and Partial Differential equations, *Chapmen and Hall /CRC, Boca Raton*, 536 p. (2000).
14. Veronica, P.: *Solutions of second-order integro-differential equations on periodic Besov spaces*, Proc. Edin. Math. Soc. **50**, 477–492 (2007).