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On properties of functions from Sobolev-Morrey type spaces with dominant mixed derivatives

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Abstract. *In this paper it is constructed a new Sobolev-Morrey type spaces with dominant mixed derivatives. Utilizing integral representation of generalized derivatives of functions defined on n-dimensional domains satisfying flexible φ-horn condition an embedding theorem is proved. Also, it is proved that the generalized derivatives of functions from this spaces satisfies the generalized Hölder condition.*

Keywords. Sobolev-Morrey type spaces with dominant mixed derivatives, integral representation, embedding theorem,generalized Hölder condition.

Mathematics Subject Classification (2010): 2010 Mathematics Subject Classification: 26A33, 46E30, 42B35, 46E35

1 Introduction

The fact that some mixed derivatives $D^\nu f$ cannot be estimated in terms of the derivatives of f entering the definition of the norm of W_p^l, H_p^l and $B_{p,\theta}^l$, leads to the necessity of consideration of the function spaces of another type in which the key role is played by mixed derivatives. The function spaces $S_p{}^l W$ $(l \in N^n)$ and $S_p{}^l H$ with dominant mixed derivative were introduced and studied by S. M. Nikolskii [14] and later by A. D. Dzhabrailov [3]; the spaces $S_p{}^l W$ were extended to the case when $l = (l_1, l_2, \dots, l_n)$, where $l_j \geq 0$ cannot be an integer. The spaces $S^l_{\mathbf{p},\mathbf{a},æ,\tau} W(G)$ $(l \in N^n;\ p \in [1;\infty)^n; a \in [0,1]^n; æ \in (0,\infty)^n; \tau \in [1,\infty])$ type Sobolev-Morrey with dominant mixed derivatives were introduced and studied in [11].

Example Let us consider an equation of the form

$$u_{xy}^{(2)} + u_x^{(1)} + u_y^{(1)} + u = f(x),$$

in our case the solution of this equation is sought in the space $S^{(1,1)}W$. One can look for the solution of equation in the space$W^{(2,2)}$, but then this solution will require additional derivatives, in other words, in our case the solution belongs to a wider class.

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In this paper we intorduce the Sobolev-Morrey type spaces $S^l_{p,\varphi,\beta}W(G)$ with dominant mixed derivatives and studied differential and differential-difference properties of fucntions from this spaces. Here $G \subset R^n$; $l \in N^n$, $1 \le p < \infty$; $\beta \in [0,1]^n$; $\varphi(t) = (\varphi_1(t_1), \ldots, \varphi_n(t_n))$, $\varphi_j(t_j) > 0 (t_j > 0)$ is Lebesgue measurable functions; $\lim_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim_{t_j \to +\infty} \varphi_j(t_j) = \infty$. we denote the set of vector-functions $\varphi(t)$ by A. Let $e_n = \{1, 2, \ldots, n\}$,$e \subseteq e_n$; $l = (l_1, \ldots, l_n), l_j > 0$ are integers $(j \in e_n)$; and $l^e = (l^e_1, \ldots, l^e_n)$, where $l^e_j = l_j > 0$ for $j \in e$; $l^e_j = 0$ for $j \in e_n \setminus e = e'$; $|\varphi([t]_1)|^{-\beta} = \prod_{j \in e_n} (\varphi_j([t_j]_1))^{-\beta_j}$, $[t]_1 = \min\{1, t\}$. For any $x \in R^n$ put

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j \in e_n \right\}$$

and

$$\int_{a^e}^{b^e} f(x) dx^e = \left(\prod_{j \in e} \int_{a_j}^{b_j} dx_j \right) f(x),$$

i.e., integration is carried out only with respect to the variables x_j whose indices belong to e.

Definition 1.1 *Denote by $S^l_{p,\varphi,\beta}W(G)$ the space of locally summable functions f on G having the generalized derivatives $D^{l^e} f (e \subseteq e_n)$ on G with the finite norm*

$$\|f\|_{S^l_{p,\varphi,\beta}W(G)} = \sum_{e \subseteq e_n} \left\| D^{l^e} f \right\|_{p,\varphi,\beta;G}, \tag{1.1}$$

where

$$\|f\|_{L_{p,\varphi,\beta}(G)} = \|f\|_{p,\varphi,\beta;G} = \sup_{\substack{x \in G,\\ t>0}} \left(|\varphi([t]_1)|_1^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right). \tag{1.2}$$

The space $S^l_{p,\varphi,\beta}W(G)$ in the case when, $\varphi_j(t_j) = t_j$, $\beta_j = \frac{\varkappa_j a_j}{p}$ $(j \in e_n)$ coincides with the space Sobolev-Morrey $S^l_{p,a,\varkappa}W(G)$ with dominant mixed derivatives studied in [10], in the case $\beta_j = 0$ $(j \in e_n)$ coincides with space $S^l_p W(G)$. The spaces $W^l_{p,\varphi,\beta}(G)$ introduced and studied in [13] .

Note that spaces type Morrey with different norms were introduced and studied in the papers [1],[4]-[9] and [12] .

Let for any $t_j > 0$ $(j \in e_n)$, there exists a positive constant $c > 0$ such that $|\varphi([t]_1)| \le C$, then the embeddings $L_{p,\varphi,\beta}(G) \hookrightarrow L_p(G)$, $S^l_{p,\varphi,\beta}W(G) \hookrightarrow S^l_p W(G)$ hold i.e.

$$\|f\|_{p;G} \le C\|f\|_{p,\varphi,\beta;G}, \quad \|f\|_{S^l_p W(G)} \le C\|f\|_{S^l_{p,\varphi,\beta}W(G)} \tag{1.3}$$

The spaces $L_{p,\varphi,\beta}(G)$ and $S^l_{p,\varphi,\beta}W(G)$ are complete.

Definition 1.2 *The open set $G \subset R^n$ is said to be an open set with condition of type flexible φ-horn if for some $\theta \in (0,1]^n$, $T \in (0,\infty)$ for any $x \in G$ there exists the vector-function*

$$\rho(\varphi(t), x) = (\rho_1(\varphi_1(t_1), x), \ldots, \rho_n(\varphi_n(t_n), x)), \quad 0 \le t_j \le T_j$$

with the following properties:

1) for all $j \in e_n$ $\rho(\varphi_j(t_j), x)$ is absolutely continuous on $[0, T_j]$, $\left|\rho'_j(\varphi_j(t_j), x)\right| \le 1$ for almost all $t_j \in [0, T_j]$,

2) $\rho_j(0, x) = 0$, $x + \bigcup_{0 \le t \le T} [\rho(\varphi(t), x) + \varphi(t)\theta I] \subset G$.

In particular, $\varphi(t) = t^{\lambda}\ \left(t^{\lambda} = \left(t^{\lambda_1}, t^{\lambda_2}, ..., t^{\lambda_n}\right)\right)$ is the set V and $x + V$ will be said to be a set of flexible λ-horn introduced in [2].

Lemma 1.1 *Let $G \subset R^n$, $1 \le p < \infty$, and $f \in S_p^l W(G)$. Then we can construct the sequence $h_s = h_s(x)$ $(s = 1, 2, \ldots)$ of infinitely differentiable finite in R^n functions for which*

$$\lim_{s\to\infty} \|f - h_s\|_{S_p^l W(G)} = 0. \tag{1.4}$$

Proof. Let $G = \bigcup\limits_{\lambda=1}^{M} G^{\lambda}$. For obtaining equality (1.4) we estimate the norm $\|f - h_s\|_{S_p^l W(G)}$

$$\|f - h_s\|_{S_p^l W(G)} = \sum_{e\subseteq e_n} \|D^{l^e}(f - h_s)\|_{p,G}. \tag{1.5}$$

The sequence $h_s(x)$ $(s = 1, 2, \ldots)$ is determined by the equalities

$$h_s(x) = F(x, \varphi(t))|_{t=\frac{1}{s}} = \sum_{\lambda=1}^{M} \eta_{\lambda}(x) f_{\varphi^{\lambda}(t)}(x),$$

here the averaging functions are determined as follows:

$$f_{\varphi^{\lambda}(t)}(x) = \int_{R^n} f(x + \varphi^{\lambda}(t)y) K_{\lambda}(y)dy,$$

where $K_{\lambda}(y) \in C_0^{\infty}(R^n)$ $(\lambda = 1, 2, \ldots, M)$, $suppK_{\lambda}(\cdot) \subset [-1; 1]$ and

$$\int_{R^n} K_{\lambda}(y)dy = 1,$$

the functions $\eta_{\lambda} = \eta_{\lambda}(x)$ $(\lambda = 1, 2, \ldots, M)$ determine the expansion of a unit in the domain G, i.e.

1)$0 \le \eta_{\lambda}(x) \le 1$ in R^n;

2) $\eta_{\lambda}(x) = 0$ in $G \setminus G_{\lambda}$ for all $\lambda = 1, 2, \ldots, M$;

3) $\sum\limits_{\lambda=1}^{M} \eta_{\lambda}(x) = 1$ in G;

4) $|D^{\alpha}\eta_{\lambda}(x)| \le C_{\lambda}$, $C_{\lambda} = const$ for all $\lambda = 1, 2, \ldots, M$ and $\alpha \ge 0$.

Obviously,

$$f(x) - h_s(x) = \sum_{\lambda=1}^{M} \eta_{\lambda}(x)(f(x) - f_{\varphi^{\lambda}(t)}(x)).$$

Consequently,

$$\|f(\cdot) - h_s(\cdot)\|_{S_p^l W(G)} \le \sum_{\lambda=1}^{M} \|\eta_{\lambda}(\cdot)(f(\cdot) - f_{\varphi^{\lambda}(t)}(\cdot))\|_{S_{p,}^l W(G^{\lambda})}$$

$$\le C \sum_{\lambda=1}^{M} \|(f(\cdot) - f_{\varphi^{\lambda}(t)}(\cdot))\|_{S_p^l W(G^{\lambda})}, \tag{1.6}$$

$$\|(f(\cdot) - f_{\varphi^{\lambda}(t)}(\cdot))\|_{S_p^l W(G^{\lambda})} = \sum_{e\subseteq e_n} \|D^{l^e}(f(\cdot) - f_{\varphi^{\lambda}(t)}(\cdot))\|_{p,G^{\lambda}}. \tag{1.7}$$

As much as small for rather small t, as a consequence of continuity of L_p- average functions, belonging to the space $L_p(G^\lambda)$, from (1.6) and (1.7) it follows

$$\|f(\cdot) - h_s(\cdot)\|_{S_p^l W(G)} < \varepsilon,$$

In other words,

$$\lim_{s\to\infty} \|f - h_s\|_{S_p^l W(G)} = 0.$$

Assuming that $\varphi_j(t)$ $(j = 1, 2, ..., n)$ are also differentiable on $[0, T_j] j \in e_n$, we can show that for $f \in S_p^l W(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x \in U \subset G)$

$$D^\nu f(x) = \sum_{e\subseteq e_n} (-1)^{|\nu|+|l^e|} \prod_{j=e'} (\varphi_j(T_j))^{-1-\nu_j}$$

$$\times \int_{0^e}^{T^e} \int_{R^n} M_e^{(\nu)} \left(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}, x\right)\right)}{\varphi(t^e + T^{e'})}, \rho'\left(\varphi\left(t^e + T^{e'}\right), x\right) \right)$$

$$\times D^{l_e} f(x + y) \prod_{j\in e} (\varphi_j(t_j))^{l-\nu_j-2} \prod_{j\in e} \varphi_j'(t_j)\, dt^e, \tag{1.8}$$

where $t^e + T^{e'} = t_j (j \in e)$; $t^e + T^{e'} = T_j (j \in e')$; $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j \geq 0$ $(j \in e_n)$ are integers. Recall that the set $x + \bigcup_{0\leq t_j\leq T_j} [\rho(\varphi(t), x) + \varphi(t)\Theta I]$ (1.4). It can be shown that for $f \in S_p^l W(G)$, $1 \leq p \leq \infty$ and $l \in N^n$, if $l_j - \nu_j > 0 (j \in e_n)$ then on G exists a generalized derivative $D^\nu f \in L_p(G)$ and for it is valid with the same kernels, for all $x \in U \subset G$ (1.8).

Let $L_e(\cdot, y, z) \in C_0^\infty(R^n)$ $(j \in e_n)$ be such that

$$S(L_e) \subset I_{\varphi(T)} = \left\{ y : |y_j| < \frac{1}{2}\varphi_j(T_j), \;\; j = 1, 2, ..., n \right\}.$$

Assume that for any $0 < T_j \leq 1$ $(j \in e_n)$

$$V = \bigcup_{0<t_j\leq T_j} \left\{ y : \frac{y}{\varphi(t^e + T^{e'})} \in S(L_e) \right\}.$$

It is clear that $V \subset I_{\varphi(T)}$ and suppose that $U + V \subset G$.

Lemma 1.2 *Let* $1 \leq p \leq q \leq r \leq \infty$; $0 < \eta_j, t_j < T_j \leq 1$ $(j \in e_n)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \geq 0$ *be entire,* $(j \in e_n)$; $\Phi \in L_{p,\varphi,\beta}(G)$ *and*

$$\mu_j = l_j - \nu_j - (1 - \beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right),$$

$$E_\eta^e(x) = \prod_{j=e'} (\varphi_j(T_j))^{-1-\nu_j} \int_{0^e}^{\eta^e} L_e(x; t, T)\Phi(x + y)$$

$$\times \prod_{j\in e} (\varphi_j(t_j))^{l-\nu_j-2} \prod_{j\in e} \varphi_j'(t_j)\, dt^e, \tag{1.9}$$

$$E^e_{\eta,T}(x)=\prod_{j=e'}(\varphi_j(T_j))^{-1-\nu_j}\int_{\eta^e}^{T^e}L_e(x;t,T)\Phi(x+y)$$
$$\times\prod_{j\in e}(\varphi_j(t_j))^{l-\nu_j-2}\prod_{j\in e}\varphi'_j(t_j)\,dt^e, \tag{1.10}$$

where

$$L_e(x;t,T)$$
$$=\int_{R^n}M_e\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'},x\right)\right)}{\varphi\left(t^e+T^{e'}\right)},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)dy. \tag{1.11}$$

Then for any $\overline{x}\in U$ the following inequalities are true

$$\sup_{\overline{x}\in U}\left\|E^e_\eta\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le C_1\left\|\Phi\right\|_{p,\varphi,\beta;G}\times$$
$$\prod_{j\in e'}(\varphi_j(t_j))^{-\nu_j-(1-\beta_jp)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j\in e_n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}\prod_{j\in e}(\varphi_j(\eta_j))^{-\mu_j}\ (\mu_j>0), \tag{1.12}$$

$$\sup_{\overline{x}\in U}\left\|E^i_{\eta,T}\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le C_2\left\|\Phi\right\|_{p,\varphi,\beta;G}\times$$
$$\prod_{j\in e'}(\varphi_j(t_j))^{-\nu_j-(1-\beta_jp)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j\in e_n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}\times$$
$$\begin{cases}\prod_{j\in e}(\varphi_j(T_j))^{-\mu_j} & \text{for } \mu_j>0\\ \prod_{j\in e}\ln\frac{\varphi_j(T_j)}{\varphi_j(\eta_j)} & \text{for } \mu_j=0\\ \prod_{j\in e}(\varphi_j(\eta_j))^{-\mu_j} & \text{for } \mu_j<0,\end{cases} \tag{1.13}$$

where $U_{\psi(\xi)}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j(\xi_j),j=1,2,...,n\right\}$ and $\psi\in N$, C_1, C_2 are the constants independent of φ, ξ, η and T.

Proof. Applying sequentially the Minkowsky generalized inequality for any $\overline{x}\in U$

$$\left\|E^i_\eta\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\int_{0^e}^{\eta^e}\left\|H\left(\cdot,t^e+T^{e'}\right)\right\|_{q,U_{\psi(\xi)}(\overline{x})}$$
$$\times\prod_{j=e}(\varphi_j(t_j))^{l_j-\nu_j-2}\varphi'_j(t_j)\,dt^e, \tag{1.14}$$

where

$$H(x,t^e+T^e)=\int_{R^n}L_e(x,t^e+T^e)\,\Phi(x+y)\,dy.$$

Estimate the norm $\left\|H\left(\cdot,t^e+T^{e'}\right)\right\|_{q,U_{\psi(\xi)}(\overline{x})}$. From the Holder inequality $(q\le r)$ we have

$$\left\|H\left(\cdot,t^e+T^{e'}\right)\right\|_{q,U_{\psi(\xi)}(\overline{x})}\le\left\|H\left(\cdot,t^e+T^{e'}\right)\right\|_{r,U_{\psi(\xi)}(\overline{x})}\prod_{j\in e_n}(\psi_j(\xi_j))^{\frac{1}{q}-\frac{1}{r}}. \tag{1.15}$$

Further, we will assume that there exists a function $\widetilde{L}_e(x)$ such that

$$|L_e(x,y,z)| \le C|\widetilde{L}_e(x)| \ \text{ for all } \ (y,z) \in R^{2n}.$$

Let χ be a characteristic function of the set $S(L_e)$. Noting that $1 \le p \le r \le \infty$, $s \le r$(as $\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}$), and

$$|L_e \Phi| = \left(|\Phi|^p |L_e|^s\right)^{\frac{1}{r}} \left(|\Phi|^p \chi\right)^{\frac{1}{q}-\frac{1}{r}} \left(|L_e|^s\right)^{\frac{1}{s}-\frac{1}{r}}$$

and again apply for $|H|$ the Holder inequality $\left(\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we obtain

$$\left\| H\left(\cdot, t^e + T^{e'}\right)\right\|_{r, U_{\psi(\xi)}(\overline{x})}$$

$$\le \sup_{x \in U_{\psi(\xi)}(\overline{x})} \left(\int_{R^n} |\Phi(x+y)|^p \chi\left(\frac{y}{\varphi(t^e + T^{e'})}\right) dy \right)^{\frac{1}{p}-\frac{1}{r}}$$

$$\times \sup_{y \in V} \left(\int_{U_{\psi(\xi)}(\overline{x})} |\Phi(x+y)|^p \, dx \right)^{\frac{1}{r}} \left(\int_{R^n} \left| \widetilde{L}_e\left(\frac{y}{\varphi(t^e + T^{e'})}\right)\right|^s dy \right)^{\frac{1}{s}}. \tag{1.16}$$

For any $x \in U$ we have

$$\int_{R^n} |\Phi(x+y)|^p \chi\left(\frac{y}{\varphi(t^e + T^{e'})}\right) dy$$

$$\le \int_{G_{\varphi\left(t^e + T^{e'}\right)}(\overline{x})} |\Phi(y)|^p \, dy \le \prod_{j \in e'} \left(\varphi_j(T_j)\right)^{\beta_j p} \prod_{j \in e} \left(\varphi_j(t_j)\right)^{\beta_j p} \|\Phi\|^p_{p,\varphi,\beta;G}. \tag{1.17}$$

For $y \in V$ $(U_\psi + V \subset G_\varphi)$

$$\int_{U_{\psi(\xi)}(\overline{x})} |\Phi(x+y)|^p \, dx$$

$$\le \int_{G_{\varphi(\xi)}(\overline{x}+y)} |\Phi(x)|^p \, dx \le \|\Phi\|^p_{p,\varphi,\beta;G} \prod_{j \in e_n} \left(\psi_j([\xi]_1)\right)^{\beta_j p} \tag{1.18}$$

and

$$\int_{R^n} \left| \widetilde{L}_e\left(\frac{y}{\varphi(t)}\right)\right|^s dy = \prod_{j \in e'} \left(\varphi_j(T_j)\right) \prod_{j \in e} \left(\varphi_j(t_j)\right) \left\|\widetilde{L}_e\right\|^s_s. \tag{1.19}$$

From inequalities (1.15)- (1.18) it follows that

$$\|H(\cdot,t)\|_{r,U_{\psi(\xi)}(\overline{x})} \le C_1 \|\Phi\|_{p,\varphi,\beta;G}$$

$$\times \prod_{j \in e'} \left(\varphi_j(T_j)\right)^{1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{j \in e} \left(\varphi_j(t_j)\right)^{1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}$$

$$\times \prod_{j \in e_n} \left(\psi_j(\xi_j)\right)^{\left(\frac{1}{q}-\frac{1}{r}\right)} \prod_{j \in e_n} \left(\psi_j([\xi_j]_1)\right)^{\frac{\beta_j p}{q}}. \tag{1.20}$$

Substituting inequalities in (1.14) for $(r = q)$, we obtain (1.12). Inequality (1.13) is proved in the same way.

From inequality (1.12) it follows that

$$\left\|E^e_\eta\right\|_{q,\psi,\beta_1;U} \le C \|\Phi\|_{p,\varphi,\beta;G}. \tag{1.21}$$

where $\beta_1 = \frac{\beta p}{q} C$ are the constants independent of Φ.

2 Main results

Prove two theorems on the properties of the functions from the space $S^l_{p,\varphi,\beta}W(G)$.

Theorem 2.1 *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \le p \le q \le \infty$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ be entire $j \in e_n$, $\mu_j > 0 (j \in e_n)$ and let $f \in S^l_{p,\varphi,\beta}W(G)$. Then the following embeddings hold*

$$D^\nu : S^l_{p,\varphi,\beta}W(G) \to L_{q,\psi,\beta^1}(G) \text{ and } D^\nu : S^l_{p,\varphi,\beta}W(G) \to S^{l^1}_{q,\psi,\beta^1}W(G), (l^1 \in N^n)$$

i.e. for $f \in S^l_{p,\varphi,\beta}W(G)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are true

$$\|D^\nu f\|_{q,G} \le C_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{s_{e,j}} \left\| D^{l^e} f \right\|_{p,\varphi,\beta;G} \tag{2.1}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \le C_2 \|f\|_{S^l_{p,\varphi,\beta}W(G)}, \ (p \le q < \infty), \tag{2.2}$$

where

$$s_{e,j} = \begin{cases} \mu_j, & \text{for } j \in e, \\ -\nu_j - (1 - \beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right), & j \in e' \end{cases}$$

and if

$$\mu_j - l^1_j > 0, j \in e_n$$

then

$$\|D^\nu f\|_{W^{l^1}_q(G)} \le C_3 \sum_{e \subseteq e_n} \prod_{j \in e'} (\varphi_j(T_j))^{s_{e,j}} \prod_{j \in e} (\varphi_j(T_j))^{s_{e,j} - l^1_j} \left\| D^{l^e} f \right\|_{p,\varphi,\beta;G} \tag{2.3}$$

$$\|D^\nu f\|_{S^{l^1}_{q,\psi,\beta_1}W(G)} \le \|f\|_{S^l_{p,\varphi,\beta}W(G)}, \ p \le q < \infty. \tag{2.4}$$

In particular, if

$$\mu^0_j = l_j - \nu_j - (1 - \beta_j p)\frac{1}{p} > 0, j \in e_n \tag{2.5}$$

then $D^\nu f(x)$ is continuous on G, i.e.

$$\sup_{x \in G} |D^\nu f(x)| \le C_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} (\varphi_j(T_j))^{s^0_{e,j}} \left\| D^{l^e} f \right\|_{p,\varphi,\beta;G} \tag{2.6}$$

where

$$s^0_{e,j} = \begin{cases} \mu^0_j, & j \in e, \\ -\nu_j - (1 - \beta_j p)\frac{1}{p}, & j \in e', \end{cases}$$

$0 < T_j \le \min\{1, T_{0j}\}$, T_0 is a fixed number; C_1, C_2, C_3, C_4 are the constants independent of f, C_1 and C_3 are independent also on T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu}f$ on G. Indeed, from the condition $B_T^e < \infty$ $(e \subseteq e_n)$ it follows that for $f \in S^l_{p,\varphi,\beta}W(G) \to S^l_p W(G)$, there exists $D^{\nu}f \in L_p(G)$ and for it integral representation (1.8) with the same kernels is valid.

Based around the Minkowsky inequality, from identities (1.8) we get

$$\|D^{\nu}f\|_{q,G} \le \sum_{e\subseteq e_n} \|E_T^e\|_{q,G}\,. \tag{2.7}$$

By means of inequality (1.12) for $U = G$,$D^{l^e}f = \Phi$ $\eta = T$, $\xi \to \infty$ we get inequality (2.1). By means of inequality (1.13) for $\eta = T$, we get inequality (2.2). For proof (2.3) in identity (1.8) instead ν_j we take $\nu_j + l_j^1 (j \in e)$. Again with inequality (1.12) for $U = G$,$D^{l^e}f = \Phi$ $\eta = T$, $\xi \to \infty$ we get inequality (2.3).

Now let conditions (2.5) be satisfied, then based around identities (1.8) from inequality (2.7) we get

$$\left\|D^{\nu}f - D^{\nu}f_{\varphi(T)}\right\|_{\infty,G} \le C \sum_{\varnothing\ne e\subseteq e_n} \prod_{j\in e} (\varphi_j(T_j))^{\mu_j^0} \left\|D^{l^e}f\right\|_{p,\varphi,\beta;G}.$$

where $\mu_j^0 = \mu_j(q=\infty)$.

As $T_j \to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^{\nu}f$ is continuous on G. Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2.2 *Let all the conditions of theorem 1 be fulfilled. Then for $\mu_j > 0$ $(j \in e_n)$ the generalized derivatives $D^{\nu}f$ satisfies on G the generalized Holder condition, i.e. the following inequality is valid:*

$$\|\varDelta(\gamma,G)\,D^{\nu}f\|_{q,G} \le C\,\|f\|_{S^l_{p,\varphi,\beta}W(G)} \prod_{j\in e_n} (\varphi_j(|\gamma_j|))^{\sigma_j}, \tag{2.8}$$

where σ_j $(j \in e_n)$ any number, satisfying the inequality:

$$\begin{cases} 0 \le \sigma_j \le 1, \ \text{if} \ \mu_j > 1 \ \text{for} \ j \in e, \\ 0 \le \sigma_j < 1, \ \text{if} \ \mu_j = 1 \ \text{for} \ j \in e, \ \text{and} \ 0 \le \sigma_j \le 1 \ \text{for} \ j \in e' \\ 0 \le \sigma_j \le \mu_j, \ \text{if} \ \mu_j < 1 \ \text{for} \ j \in e. \end{cases} \tag{2.9}$$

If $\mu_j^0 > 0$ $(j \in e_n)$, then

$$\sup_{x\in G} |\varDelta(\gamma,G)D^{\nu}f(x)| \le C\|f\|_{S^l_{p,\varphi,\beta}W(G)} \prod_{j\in e_n} (\varphi_j(|\gamma_j|))^{\sigma_j^0}. \tag{2.10}$$

where σ_j^0 satisfy the same conditions as σ_j, but replaced μ_j in μ_j^0.

Proof. According to lemma 8.6 from [2] there exists a domain

$$G_\omega \subset G\,(\omega = \zeta r(x)\,,\ \zeta > 0\ r(x) = \rho(x, \partial G)\,,\ x \in G)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identies (1.8) with the same kernels are valid. After same transformations, from (1.8) we get

$$|\varDelta(\gamma,G)\,D^{\nu}f(x)| \le C_1 \sum_{e\subseteq e_n} \prod_{j\in e'} (\varphi_j(T_j))^{-1-\nu_j}$$

$$\times \int_0^{|\gamma_1^e|} \cdots \int_0^{|\gamma_n^e|} \prod_{j\in e} (\varphi_j(t_j))^{l_j-\nu_j-2} \prod_{j\in e} \varphi_j'(t_j) dt^e$$

$$\times \int_{R^n} \left| L_e^{(\nu)} \left(\frac{y}{\varphi(t^e+T^{e'})}, \frac{\rho\left(\varphi\left(t^e+T^{e'},x\right)\right)}{\varphi(t^e+T^{e'})}, \rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\right|$$

$$\times \left| \Delta(\gamma,G) D^{l^e} f(x+y)\right| dy + C_2 \sum_{e\subseteq e_n} \prod_{j\in e'} (\varphi_j(T_j))^{-2-\nu_j}$$

$$\times \prod_{j\in e_n} |\gamma_j| \int_{|\gamma_1^e|}^{T_1^e} \cdots \int_{|\gamma_n^e|}^{T_n^e} \prod_{j\in e} (\varphi_j(t_j))^{l_j-\nu_j-3} \prod_{j\in e} \varphi_j'(t_j) dt^e$$

$$\times \int_{R^n} \left| L_e^{(\nu+1)} \left(\frac{y}{\varphi(t^e+T^{e'})}, \frac{\rho\left(\varphi\left(t^e+T^{e'},x\right)\right)}{\varphi(t^e+T^{e'})}, \rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\right|$$

$$\times \int_0^1 \cdots \int_0^1 \left| D^{l^e} f(x+y+\gamma_1 u_1+\ldots+\gamma_n u_n)\right| dydu$$

$$= C_1 \sum_{e\subseteq e_n} E_\gamma^e(x,t) + C_2 \sum_{e\subseteq e_n} E_{\gamma,T}^e(x,t), \tag{2.11}$$

where $0 < T_j \le \{1, T_{0,j}\}$, $j \in e_n$. We also assume that $|\gamma_j| < T_j (j \in e_n)$. Consequently, $|\gamma_j| < \min(\omega_j, T_j)(j \in e_n)$. If $x \in G \setminus G_\omega$ then by definition

$$\Delta(\gamma,G) D^\nu f(x) = 0.$$

Based around (2.9) we have

$$\|\Delta(\gamma,G) D^\nu f\|_{q,G} \le C_1 \sum_{e\subseteq e_n} \left\|E_\gamma^e(\cdot,\gamma)\right\|_{q,G_\omega} +$$

$$+C_2 \sum_{e\subseteq e_n} \left\|E_{\gamma,T}^e(\cdot,\gamma)\right\|_{q,G_\omega}. \tag{2.12}$$

By means of inequality (1.13), for $D^{l_e} f = \Phi$, $\eta = |\gamma_j|$ $(j \in e_n)$ we get

$$\left\|E_{|\gamma|}^e(\cdot,\gamma)\right\|_{q,G_\omega} \le C_1 \left\|D^{l^e} f\right\|_{p,\varphi,\beta;G} \prod_{j\in e} (\varphi_j(|\gamma_j|))^{\mu_j}. \tag{2.13}$$

ad by means of inequality (1.14) for $D^{l^e} f = \Phi$, $\eta = |\gamma_j|$ we get

$$\left\|E_{|\gamma|,T}(\cdot,\gamma)\right\|_{q,G_\omega} \le C_2 \left\|D^{l^e} f\right\|_{p,\varphi,\beta;G} \prod_{j\in e'} \varphi_j(|\gamma_j|) \prod_{j\in e} (\varphi_j(|\gamma_j|))^{\mu_j-1}. \tag{2.14}$$

From inequalities (2.12) - (2.14) we get the required inequality.

Now suppose that $|\gamma_j| \ge \min(\omega_j, T_j)$, $(j \in e_n)$. Then

$$\|\Delta(\gamma,G) D^\nu f\|_{q,G} \le 2\|D^\nu f\|_{q,G} \le C(\omega T)\|D^\nu f\|_{q,G} \prod_{j\in e_n} (\varphi_j(|\gamma_j|))^{\sigma_j}.$$

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (2.1)), in this case we get estimation (2.8).

Theorem 2.2 is proved.

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