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Basis properties of the system of eigenfunctions of a fourth order eigenvalue problem with spectral and physical parameters in the boundary conditions

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Abstract. *In this paper we consider the eigenvalue problem for fourth order ordinary differential equation that describes the bending vibrations of a homogeneous rod, the both ends of which are fixed elastically and on these ends the tracking forces act. We investigate the spectral properties of this problem, including we study the basis properties of subsystems of root functions in the space* L_p, $1 < p < \infty$.

Keywords. fourth order eigenvalue problem, the bending vibrations of a homogeneous rod, location of eigenvalues, oscillatory properties of eigenfunctions, basis property of eigenfunctions

Mathematics Subject Classification (2010): 34B05, 34B09, 34B24, 34C10, 34K11

1 Introduction

We consider the following eigenvalue problem

$$y^{(4)}(x) = \lambda y(x),\ x \in (0,1), \tag{1.1}$$

$$y''(0) = y''(1) = 0, \tag{1.2}$$

$$y'''(0) - a\lambda y(0) = 0 \tag{1.3}$$

$$y'''(1) - c\lambda y(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, a and c are physical parameters such that $a > 0$ and $c < 0$.

The problem (1.1)-(1.4) arises when variables are separated in the dynamical boundary value problem describing bending vibrations of a homogeneous rod, the both ends of which are fixed elastically and on these ends the tracking forces act (see [6, 12]).

The Sturm-Liouville problems of second and fourth orders with spectral parameter in the boundary conditions has been considered in [1-4, 7-11]. In these papers, the spectral properties of considered problems are investigated, including the basis properties of the systems of root functions in the space L_p, $1 < p < \infty$. In a recent paper [3], the authors study the spectral properties of the eigenvalue problem of fourth order, which describes the bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, the left end of which is fixed and on the right end an inertial mass is concentrated.

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Hence in this problem the spectral parameter is contained in the boundary conditions at point $x = 1$.

The problem (1.1)-(1.4) was considered in [1] for $a = 0$. In this paper the oscillation properties of eigenfunctions and their derivatives were investigated. Moreover, the basis properties of the system of eigenfunctions in $L_p(0, 1)$, $1 < p < \infty$, also studied, necessary and sufficient conditions for the basicity of subsystems of eigenfunctions is obtained.

Note that the signs of the parameters a and c play an important role. In the case $a > 0$ and $c < 0$, then problem (1.1)-(1.4), can be treated as a spectral problem for a self-adjoint operator in the Hilbert space $H = L_2(0, 1) \oplus \mathbb{C}^2$ (see [3, 13]).

In this paper we investigate the location of the eigenvalues on the real axis, the structure of the root subspaces, the oscillation properties of the eigenfunctions and we study the basis properties in $L_p(0, 1)$, $1 < p < \infty$, of the subsystems of eigenfunctions of problem (1.1)-(1.4). It should be noted that the sufficient conditions for the basis property of the subsystems of the eigenfunctions of the problem (1.1)-(1.4) in $L_p(0, 1)$, $1 < p < \infty$, obtained by us differ sharply from the sufficient conditions obtained in [3].

2 Preliminaries

Below we require the following results.

Lemma 2.1 (see [5, Lemma 2.1]). *Let $y(x, \lambda)$ be a nontrivial solution of equation* (1.1) *for $\lambda > 0$. If y, y', y'', Ty are nonnegative and not all equal zero at $x = a$, then they are positive for $x > a$. If y, $-y'$, y'', $-Ty$ are nonnegative and not all equal zero at a, then they are positive for $x < a$.*

We introduce the boundary condition (see [1, 5])

$$y(1) \cos \delta - Ty(1) \sin \delta = 0, \tag{2.1}$$

where $\delta \in [0, \pi)$.

Alongside the problem (1.1)-(1.4) we shall consider the problems (1.1), (1.2), (1.3), (2.1). By making the change of variables $x' = 1 - x$ and applying the results of [1] and [9] we have the following statement.

Theorem 2.1 . *The eigenvalues of the problem* (1.1), (1.2), (1.3), (2.1) *are real, simple, except the case $\delta = \frac{\pi}{2}$, where $\lambda = 0$ is a double eigenvalue (with geometric multiplicity is two), and form an infinitely increasing sequence $\{\lambda_k(\delta)\}_{k=1}^{\infty}$ such that $\lambda_1(\delta) = 0$, $\lambda_k(\delta) > 0$ for $\delta \in \left[0, \frac{\pi}{2}\right)$ and $k \geq 2$, $\lambda_1(\delta) < 0$, $\lambda_2(\delta) = 0$, $\lambda_k(\delta) > 0$ for $\delta \in \left(\frac{\pi}{2}, \pi\right)$ and $k \geq 3$, and $\lambda_1\left(\frac{\pi}{2}\right) = \lambda_2\left(\frac{\pi}{2}\right) = 0$, $\lambda_k\left(\frac{\pi}{2}\right) > 0$, $k \geq 3$. Moreover, the eigenfunction $y_k^{(\delta)}(x)$, $k \geq 3$, corresponding to the eigenvalue $\lambda_k(\delta)$ has exactly $k - 1$ simple zeros in the interval* $(0, 1)$.

Remark 2.1 If $\delta \in [0, \frac{\pi}{2})$, then $y_2^{(\delta)}(x)$ has exactly one zero in $(0, 1)$.

For each fixed $\lambda \in \mathbb{C}\backslash\{0\}$ there exists a unique (up to a constant factor) nontrivial solution

$$\begin{gathered} y(x, \lambda) = (\sin \rho - \sinh\, \rho)\, (\cos \rho x + \cosh \rho x) - \\ (\cos \rho - \cosh \rho)\, (\sin \rho x + \sinh \rho x) - 2a\rho\, (\sinh \rho \sin \rho x - \sin \rho\, \sinh \rho x) \end{gathered} \tag{2.2}$$

of problem (1.1), (1.2), (1.3), where $\rho = \sqrt[4]{\lambda}$.

For $\lambda > 0$ the equation $\sin \rho - \sinh\, \rho = 0$ has no solutions, for $\lambda < 0$ the equation $\sin \rho - \sinh\, \rho = 0$ has solutions $\xi_k = -4\eta_k^4$, $k \in \mathbb{N}$, where η_k, $k \in \mathbb{N}$, are the positive roots of the equation $\tan\eta = \tanh\eta$, and $\eta_1 > \frac{\pi}{2}$. The function $y(x,\lambda)$ for $x \in [0,1]$, $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus\{0\}$, can be rewritten as follows:

$$y(x,\lambda) = (\sin\rho - \sinh\ \rho) \times$$
$$\Big\{(\cos\rho x + \cosh\rho x) - \tfrac{(\cos\rho - \cosh\rho)}{(\sin\rho - \sinh\ \rho)}\ (\sin\rho x + \sinh\rho x) - \tag{2.3}$$
$$\tfrac{2a\rho}{(\sin\rho - \sinh\ \rho)}\ (\sinh\rho\sin\rho x - \sin\rho\sinh\rho x)\Big\}.$$

It is easy to verify that the following relation holds

$$\lim_{\lambda\to 0}\Big\{(\cos\rho x + \cosh\rho x) - \tfrac{(\cos\rho - \cosh\rho)}{(\sin\rho - \sinh\ \rho)}\ (\cos\rho x + \cosh\rho x) - \tag{2.4}$$
$$\tfrac{2a\rho}{(\sin\rho - \sinh\ \rho)}\ (\sinh\rho\sin\rho x - \sin\rho\sinh\rho x)\Big\} = -\,6\left(x - \tfrac{1}{3}\right).$$

We consider the function $\frac{y'''(1)}{y(1,\lambda)}$ in a sufficiently small deleted neighborhood of $\lambda = 0$. By virtue of (2.2) we obtain

$$\frac{y'''(1,\lambda)}{y(1,\lambda)} = \rho^3\frac{1 - \cos\rho\ \cosh\rho + 2a\rho^4(\cos\rho\sinh\rho - \sin\rho\cosh\rho)}{\sin\rho\ \cosh\rho - \cos\rho\ \sinh\rho}, \tag{2.5}$$

which implies that

$$\frac{y'''(1,\lambda)}{y(1,\lambda)} = \frac{1}{4}\lambda + o(\lambda)\ \text{ as }\ \lambda \to 0. \tag{2.6}$$

We observe that the function

$$F(\lambda) = \begin{cases} \frac{y'''(1,\lambda)}{y(1,\lambda)} & if\ \ \lambda \neq 0, \\ 0 & if\ \ \lambda = 0 \end{cases}$$

is well defined for

$$\lambda \in B \equiv (\mathbb{C}\backslash\mathbb{R}) \cup (-\infty, \lambda_2(0)) \cup \left(\bigcup_{k=3}^{\infty}(\lambda_{k-1}(0), \lambda_k(0))\right),$$

and is a meromorphic function of finite order; the eigenvalues $\lambda_k(\pi/2)$, $k \in \mathbb{N}$, and $\lambda_k(0)$, $k \in \mathbb{N}, k \geq 2$, of problem (1.1), (1.2), (1.3), (2.1) for $\delta = \frac{\pi}{2}$ and $\delta = 0$ are the zeros and poles of this function, respectively.

Lemma 2.2 *The following relations hold:*

$$\frac{dF(\lambda)}{d\lambda} = \begin{cases} y^{-2}(1,\lambda)\left\{\int_0^1 y^2(x,\lambda)dx + ay^2(0,\lambda)\right\} & if\ \ \lambda \neq 0, \\ \frac{1}{4} & if\ \ \lambda = 0, \end{cases} \quad \lambda \in B, \tag{2.7}$$

$$\lim_{\lambda\to -\infty} F(\lambda) = -\,\infty\,. \tag{2.8}$$

Proof. The proof of the relation (2.7) is similar to that of [3, Lemma 3.3]. The relation (2.7) implies from (2.5).

By virtue of Theorem 2.1 and relations (2.7), (2.8) for each $\delta \in (0, \pi)$ we have

$$\lambda_2(0) < \lambda_3(\delta) < \lambda_3(0) < \lambda_4(\delta) < \ldots \ . \tag{2.9}$$

Moreover, we also have

$$\begin{gathered} 0 = \lambda_1(\delta) = \lambda_1(0) < \lambda_2(\delta) < \lambda_2(0) \ \text{ for } \ \delta \in (0, \tfrac{\pi}{2}), \\ \lambda_1\left(\tfrac{\pi}{2}\right) = \lambda_2\left(\tfrac{\pi}{2}\right) = 0 \ \text{ and } \ \lambda_1(\delta) < 0 = \lambda_2(\delta) \ \text{ for } \ \delta \in \left(\tfrac{\pi}{2}, \pi\right). \end{gathered} \tag{2.10}$$

As an immediate consequence of (2.5) we obtain the following statement.

Lemma 2.3 *The number of zeros in* $(0, 1)$ *of the solution* $y(x, \lambda)$ *of problem* (1.1), (1.2), (1.3) *tends to* ∞ *as* $\lambda \to \pm\infty$.

Consider the equation

$$y(x, \lambda) = 0$$

for $x \in (0, 1]$ and $\lambda \in \mathbb{R}$. The zeros of this equation are functions of λ.

Lemma 2.4 *Every zero* $x(\lambda) \in (0, 1]$ *of the function* $y(x, \lambda)$ *is simple and is a* C^1 *function of* $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. If $\lambda > 0$, then by Lemma 2.1 it follows that every zero $x(\lambda) \in (0, 1]$ of the function $y(x,\lambda)$ is simple. If $\lambda < 0$ and $y(x_0, \lambda) = y'(x_0, \lambda) = 0$ for $x_0 \in (0, 1]$, then the function $y(x, \lambda)$ solves the problem defined on $[0, x_0]$ and determined by equation (1.1) with the boundary conditions (1.2), $y''(0) = 0$, $y(x_0) = y'(x_0) = 0$, which contradicts the condition $\lambda < 0$ in view of Theorem 2.1. The smoothness of $x(\lambda)$ follows from the well-known implicit function theorem. Lemma is proved.

Lemma 2.5 *Let* $0 < \mu < \nu$. *If* $y(x, \mu)$ *has* l *zeros in the interval* $(0, 1)$, *then* $y(x, \mu)$ *has at least* l *zeros in this interval.*

The proof of this lemma is similar to that of [3, Lemma 3.6].

Lemma 2.6 *If* $\lambda \in (\lambda_{k-1}(0), \lambda_k(0)]$, $k = 2, 3, \ldots$, *then the function* $y(x, \lambda)$ *has exactly* $k - 1$ *simple zeros in* $(0, 1)$.

Proof. It is obvious that as $\lambda > 0$ varies, the zeros of the solution $y(x, \lambda)$ can enter or leave the interval $(0, 1)$ only through the endpoints $x = 0$ or $x = 1$. But these zeros cannot pass through the endpoint $x = 0$, since in this case by (1.2), (1.3) we have $y(0, \lambda) = y''(0, \lambda) = Ty(0, \lambda) = 0$, which contradicts the boundary condition $y''(1, \lambda) = 0$ in view of Lemma 2.1. Hence the zeros of $y(x, \lambda)$ cannot pass through the endpoint $x = 1$, since by Lemma 2.3 and 2.5 the number of zeros does not decrease. Then zeros of the function $y(x, \lambda)$ enter through the endpoint $x = 1$. By (2.4) in the right neighborhood of $\lambda = 0$ the function has $y(x, \lambda)$ has exactly one simple zero in the interval $(0, 1)$. Moreover, by Theorem 2.1, the function $y(x, \lambda_k(0))$, $k = 2, 3, \ldots$, has $k - 1$ simple zeros in $(0, 1)$.

Now the assertion of this theorem follows from the preceding considerations with the use of Lemma 2.4. The proof of this lemma is complete.

3 Operator interpretation and main properties of eigenvalues and eigenfunctions of problem (1.1)-(1.4)

The problem (1.1)-(1.4) can be reduced to the eigenvalue problem for the linear operator L in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with inner product

$$(\hat{y}, \hat{v}) = (\{y, m, n\}, \{v, s, t\}) = \int_0^1 y(x)\overline{v(x)}\, dx + \ |a|^{-1} m\bar{s} \ + \ |c|^{-1} n\bar{t}, \tag{3.1}$$

where

$$L\hat{y} = L\{y, m, n\} = \{y^{(4)}(x),\, y'''(0),\, y'''(1)\}$$

with the domain

$$D(L) = \{\{y\,(x),\, m,\, n\} \ : \ y \in W_2^4(0,1),\ y^{(4)} \in L_2(0,1),\\ y''(0) = y''(1) = 0,\ m = ay(0),\ n = cy(1)\}$$

dense everywhere in H [13]. It is obvious that the operator L is well defined in H. The problem (1.1)-(1.4) takes the form

$$L\hat{y} = \lambda\hat{y},\ \hat{y} \in D(L),$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of problem (1.1)-(1.4) and those the operator L coincide; moreover, between the eigenfunctions, there is a one-to-one correspondence

$$y_k(x) \leftrightarrow \ \{y_k(x),\, m_k,\, n_k\},\ m_k = ay_k(0),\ n_k = cy_k\,(1).$$

Theorem 3.1 *L is a self-adjoint operator in H. The system of eigenvectors $\{y_k(x),\, m_k,\, n_k\}$ of the operator L forms a Riesz basis (after normalization) in the space H.*

The proof of this theorem is similar to that of [3, Theorem 5.1].

It follows immediately from the first part of Theorem 3.1 that the following result is valid.

Lemma 3.1 *All eigenvalues of the boundary value problem* (1.1)-(1.4) *are real and form an at most countable set without finite limit point.*

It is easy to see that the eigenvalues of problem (1.1)-(1.4) are the roots of the equation

$$y'''(1,\lambda) - c\lambda y(1,\lambda) = 0. \tag{3.2}$$

Remark 3.1 If $\lambda \neq 0$ is an eigenvalue of problem (1.1)-(1.4) then by the second part of Lemma 2.1 it follows that that $y(1,\lambda) \neq 0$.

Each nonzero root (with regard of multiplicities) of equation (3.2) is a root of the equation

$$F(\lambda) = c\lambda, \tag{3.3}$$

as well.

Theorem 3.2 *The eigenvalues of the boundary value problem* (1.1)-(1.4) *form an infinitely nondecreasing sequence $0 = \lambda_1 = \lambda_2 < \lambda_3 < \ \ldots \ < \lambda_k < \ \ldots$. The eigenfunction $y_k(x)$, corresponding to the eigenvalue λ_k, for $k \geq 3$ has $k-1$ simple zeros in $(0,1)$.*

Proof. By virtue of the relations (2.7)-(2.10), we have

$$\lim_{\lambda\to-\infty} F(\lambda) = -\infty,$$

$$\lim_{\lambda\to\lambda_k(0)-0} F(\lambda) = +\infty, \quad \lim_{\lambda\to\lambda_k(0)+0} F(\lambda) = -\infty, \ k\in\mathbb{N}\backslash\{1\}.$$

Hence the function $F(\lambda)$ takes each value in $(-\infty,+\infty)$ at a unique point in each of intervals $(-\infty,\lambda_2(0))$ and $(\lambda_{k-1}(0),\lambda_k(0))$, $k\in\mathbb{N}\backslash\{1,2\}$. Since $c<0$, it follows that the function $c\lambda$ is strictly decreasing in the interval $(-\infty,+\infty)$.

It follows from the preceding considerations that in the interval $(\lambda_{k-1}(0),\lambda_k(0))$, $k\in\mathbb{N}$ and $k\geq 3$, there exists a unique $\lambda=\lambda_k^*$ such that (3.3) is satisfied, i.e., condition (1.4) is satisfied. Therefore, λ_k^* is an eigenvalue of the boundary value problem (1.1)-(1.4) and $y_k^*(x)=y(x,\lambda_k^*)$ is the corresponding eigenfunction. Hence, by Lemma 2.6, the function $y_k^*(x)$ has exactly $k-1$ simple zeros in $(0,1)$.

Observe that equation (3.3) has an unique solution $\lambda=0$ in the interval $(-\infty,\lambda_2(0))$ which is double eigenvalue of problem (1.1)-(1.4). The eigen-subspace corresponding to this eigenvalue is $\{b+dx : b,\, d\in\mathbb{R}\}$. Therefore, we will assume that $\lambda_1=\lambda_2=0$ and the corresponding eigenfunctions are $y_1(x)=1$ and $y_2(x)=\left(x-\frac{1}{3}\right)$, respectively.

Now we get that $\lambda_k^*=\lambda_k$ and $y_k^*(x)=y_k(x)$ for $k\geq 3$ which completes the proof of this theorem.

It follows from [9, formulas (3.3), (3.4)] that

$$\sqrt[4]{\lambda_k(0)} = (k-1)\pi + O\left(\frac{1}{k}\right), \tag{3.4}$$

$$u_k^{(0)}(x) = \sin(k-1)\pi x + O\left(\frac{1}{k}\right), \tag{3.5}$$

where relation (3.5) holds uniformly for $x\in[0,1]$.

Theorem 3.3 *The following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k-2)\pi + O\left(1/k\right), \tag{3.6}$$

$$y_k(x) = \sin(k-2)\pi x + O(1/k), \tag{3.7}$$

where relation (3.7) *holds uniformly for* $x\in[0,1]$.

Proof. The proof of this theorem is similar to that of [9, Theorem 3.1] with the use of Theorem 3.2.

4 Basis properties in $L_p(0,1)$, $p\in(1,\infty)$, of the subsystems of eigenfunctions of the boundary value problem (1.1)-(1.4)

Let

$$\delta_k = (\hat{y}_k, \hat{y}_k). \tag{4.1}$$

Then by virtue of conditions $a>0$, $c<0$ and (3.1) it follows from (4.1) that

$$\delta_k = \ ||y_k||_{L_2}^2 + a^{-1}m_k^2 - c^{-1}n_k^2 > 0. \tag{4.2}$$

It follows from Theorem 3.1 that the system of eigenvectors $\{\hat{v}_k\}_{k=1}^{\infty}$, $\hat{v}_k=\delta_k^{-\frac{1}{2}}\hat{y}_k$, of operator L forms an orthonormal basis (i.e. Riesz basis) in H.

Let r and s ($r \neq s$) be arbitrary fixed natural numbers and

$$\tilde{\Delta}_{r,\,s} = \begin{vmatrix} a\delta_r^{-1/2} y_r(0) & a\delta_s^{-1/2} y_s(0) \\ c\delta_r^{-1/2} y_r(1) & c\delta_s^{-1/2} y_s(1) \end{vmatrix} = ac\delta_r^{-1}\delta_s^{-1} \begin{vmatrix} y_r(0) & y_s(0) \\ y_r(1) & y_s(1) \end{vmatrix}, \tag{4.3}$$

$$\Delta_{r,\,s} = \begin{vmatrix} y_r(0) & y_s(0) \\ y_r(1) & y_s(1) \end{vmatrix}. \tag{4.4}$$

By (4.2) it follows from (4.3) and (4.4) that

$$\tilde{\Delta}_{r,\,s} \neq 0 \Leftrightarrow \Delta_{r,\,s} \neq 0. \tag{4.5}$$

Theorem 4.1 *If $\Delta_{r,\,s} \neq 0$, then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,\,s}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$; if $\Delta_{r,\,s} = 0$, then this system is incomplete and nonminimal in the space $L_2(0,1)$.*

The proof of Theorem 4.1 in the case $p = 2$ is similar to that of [4, Theorem 4.1] with the use of Theorem 3.1 and relation (4.5), in the case $p \in (1, +\infty)\backslash\{2\}$ is similar to that of [9, Theorem 5.1] with the use of asymptotic formulas (3.4)-(3.7).

Using Theorem 4.1, we can obtain sufficient conditions for the subsystem of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,\,s}^{\infty}$ of problem (1.1)-(1.4) to form a basis in $L_p(0,1), 1 < p < \infty$.

Theorem 4.2 *Let r and s ($r \neq s$) be arbitrary fixed natural numbers having different parity. Then the system of eigenfunctions $\{y_k(x)\}_{k=1,\,k\neq r,\,s}^{\infty}$ of problem* (1.1)-(1.4) *forms a basis in the space $L_p(0,1), 1 < p < \infty$, which is an unconditional basis for $p = 2$.*

Proof. By Remark 3.1 it follows from Theorem 3.2 that

$$\Delta_{r,\,s} = y_r(1)y_s(1) \begin{vmatrix} \frac{y_r(0)}{y_r(1)} & \frac{y_s(0)}{y_s(1)} \\ 1 & 1 \end{vmatrix} = y_r(1)y_s(1) \left\{ \frac{y_r(0)}{y_r(1)} - \frac{y_s(0)}{y_s(1)} \right\}. \tag{4.6}$$

By virtue of Theorem 4.1 we have

$$\operatorname{sgn} \frac{y_k(0)}{y_k(1)} = (-1)^{k+1},\ k \in \mathbb{N}. \tag{4.7}$$

By using (4.7), from (4.6) we obtain

$$\Delta_{r,\,s} = y_r(1)y_s(1) \left\{ (-1)^{r+s} \left| \frac{y_r(0)}{y_r(1)} \right| - \left| \frac{y_s(0)}{y_s(1)} \right| \right\}. \tag{4.8}$$

Hence, if r and s have different parity, then it follows from (4.8) that $\Delta_{r,\,s} \neq 0$. Therefore, the statement of this theorem follows from Theorem 4.1. The proof of this theorem is complete.

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