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Estimations of the norm of functions from Sobolev-Morrey type space reduced by polynomials

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Abstract. *In this paper, the integral inequalities as estimation of the norms of functions reduced by polynomials, are proved.*

Keywords. integral inequalities, Sobolev-Morrey type space, φ -horn, generalized derivative, embedding theorem

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1 Introduction

In this paper, by means of integral representation method, we estimate the norms of functions from Sobolev-Morrey type spaces $W^l_{p,\varphi,\beta}(G)$ $l \in N^n$, $p \in [1;\infty)$ (introduced and studied from point of view of imbedding theory in the paper [1]), reduced by polynomials, determined in n-dimensional domains satisfying the flexible φ -horn condition. In other words, we prove the inequalities

$$\|D^\nu (f - P_{l-1})\|_{q,G} \le C|\widetilde{Q}(1)| \|f\|_{\omega^l_{p,\varphi,\beta}(G)} \tag{1.1}$$

where

$$\widetilde{Q}(T) = \max_{1\le i\le n} \widetilde{Q}^i(T) = \int\limits_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j-1-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty.$$

Note that the inequalities of the norm form

$$\|D^\nu (f - P_{l-1})\|_{q,G} \le C \sum_{i=1}^n |\widetilde{Q}_i(1)| \|D_i^{l_i} f\|_{\omega^l_p(G)}$$

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well proved in the paper [2]. But unlike the papers [2]- [6], here not only the space $\omega^l_{p,\underset{\sim}{\varphi},\beta}(G)$, is taken, furthermore, in our case the exponent of power at $\varphi_i(t)$ in the expression $\widetilde{Q}(1)$, is greater when in the right side of inequalities (1.1) instead of the space $\omega^l_{p,\varphi,\beta}(G)$ the space $\omega^l_p(G)$ in written. First such results belong to Sobolev. He proved that the function from the space L^r_p arrives as $|x| \to \infty$ at polynomials.

The Sobolev-Morrey type space $W^l_{p,\varphi,\beta}(G)$ is understood as a space of locally summable on $G \subset R^n$ functions having on f, generalized derivatives $D^{l_i}_i$ $(i = 1, 2, \ldots, n)$ with the finite norm

$$\|f\|_{W^l_{p,\varphi,\beta}(G)} = \|f\|_{L_{p,\varphi,\beta}(G)} + \|f\|_{\omega_{p,\varphi,\beta}(G)}\,, \tag{1.2}$$

here

$$\|f\|_{\omega_{p,\varphi,\beta}(G)} = \sum_{i=1}^{n} \left\| D^{l_i}_i f \right\|_{L_{p,\varphi,\beta}(G)}\,,$$

$$\|f\|_{L_{p,\varphi,\beta}(G)} = \|f\|_{p,\varphi,\beta;G} =$$

$$= \sup_{\substack{x\in G,\\ t>0}} |\varphi\left([t]_1\right)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)}\,, \tag{1.3}$$

$$G_{\varphi(t)}\left(x\right) = G \cap I_{\varphi(t)}\left(x\right) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j\left(t\right), \quad j = 1, 2, ..., n \right\},$$

where $l \in N^n$, $p \in [1;\infty)$, $\varphi\left(t\right) = \left(\varphi_1\left(t\right), ..., \varphi_n\left(t\right)\right)$ is a vector-function, $\varphi_j\left(t\right) > 0$ $(t > 0)$ by Lebesgue measurable, and $\lim\limits_{t\to+0} \varphi_j\left(t\right) = 0$, $\lim\limits_{t\to+\infty} \varphi_j\left(t\right) = \infty$,$|\varphi\left([t]_1\right)|^{-\beta} = \prod\limits_{j=1}^{n} \left(\varphi_j\left([t]_1\right)\right)^{-\beta_j}$, $[t]_1 = \min\left\{1, t\right\}$, $\beta_j \in [0, 1]$, $j = 1, 2, ..., n$. Let for any $t > 0$ $|\varphi\left([t]_1\right)| \le C$, where $C > 0$ is a positive constant. Then the embeddings

$$L_{p,\varphi,\beta}\left(G\right) \to L_p\left(G\right), W^l_{p,\varphi,\beta}\left(G\right) \to W^l_p\left(G\right) (W^l_{p,\varphi,0} \equiv W^l_p)$$

$$\|f\|_{p,G} \le c \|f\|_{p,\varphi,\beta;G}\,, \|f\|_{W^l_p(G)} \le c \|f\|_{W^l_{p,\varphi,\beta}(G)}\,, \tag{1.4}$$

hold.

Let $L_i\left(\cdot, y, z\right) \in C^\infty_0\left(R^n\right)$ be such that

$$S\left(L_i\right) = supp\, L_i \subset I_{\varphi(T)} = \left\{ y : |y_j| < \frac{1}{2}\varphi_j\left(T\right), j = 1, 2, \ldots, n \right\}.$$

Assume that for any $0 < T \le T_0 \le 1$ (T_0 is a fixed number)

$$V = \bigcup_{0<t\le T} \left\{ y : \frac{y}{\varphi\left(t\right)} \in S\left(L_i\right) \right\}.$$

It is clear that $V \subset I_{\varphi(T)}$ and suppose that $U + V \subset G$, where U–is an open set, contained in the domain G. Let

$$G_{\varphi(t)}(U) = \big(U + I_{\varphi(t)}(x)\big) \cap G = Z,$$

and let

$$U + V \subset G_{\varphi(t)}(U).$$

Throughout this paper a domain $G \subset R^n$ will satisfy φ-horn condition (see [6], [2]).

2 Main results

Theorem 2.1 *Let $G \subset R^n$ satisfy the condition of flexible φ-horn,* $1 \leq p \leq q \leq \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ *be entire* $j = 1, 2, ..., n$,

$$\widetilde{Q}^i(T) = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - 1 - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\varphi_i'(t)}{\varphi_i(t)} dt < \infty. \tag{2.1}$$

and let $f \in W^l_{p,\varphi,\beta}(G)$. *Then*

$$\|D^\nu (f - P_{l-1})\|_{q,G} \leq C |\widetilde{Q}(1)| \|f\|_{\omega^l_{p,\varphi,\beta}(G)} \tag{2.2}$$

where $\widetilde{Q}(1) = \max_{1 \leq i \leq n} \widetilde{Q}^i(1)$ *and C the constant independent of f.*

Proof. Under conditions of Theorem 2.1, there exist generalized derivatives $D^\nu f$ in G. Indeed, if $\widetilde{Q}^i(1) < \infty$ for every $i = 1, 2, \ldots, n; p \leq q$ it follows that

$$Q^i(T) = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - 1} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty,$$

and $f \in W^l_{p,\varphi,\beta}(G) \to W^l_p(G)$, then there exist generalized derivatives $D^\nu f$ and for it the identity obtained in [1] $(\rho(\varphi(t), x) = -x\varphi(t), 0 < t \leq T = 1)$ is valid:

$$D^\nu f(x) = P^{(\nu)}_{l-1} + \sum_{i=1}^n \int_0^1 \int_{R^n} L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x) \right)$$

$$\times D_i^{l_i} f(x+y) \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - 1} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt dy. \tag{2.3}$$

The support of this identity is contained in the flexible φ horn $x + V(\lambda, x, \theta) \subset G$. Hence, by the Minkowski inequality, we have:

$$\|D^\nu (f - P_{l-1})\|_{q,U} \leq \sum_{i=1}^n \|A_i(x,t)\|_{q,U} \tag{2.4}$$

here

$$A_i(x,t) = \int_0^1 \int_{R^n} L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \rho'(\varphi(t), x) \right) \times$$

$$D_i^{l_i} f(x+y) \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - 1} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt dy \tag{2.5}$$

Applying generalized Minkowski inequality (2.5) for $A_i(x,t)$ defined by equality (2.5), we get

$$\|A_i(\cdot, t)\|_{q,U} \leq C \int_0^1 \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - 1} \|B_i(\cdot, t)\|_{q,U} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt, \tag{2.6}$$

here

$$B_i(x,t) = \int\limits_{R^n} D_i^{l_i} f(x+y) L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right) dy$$

From the Hölder inequality $(q \le r \le \infty)$ we have

$$\|B_i(\cdot,t)\|_{q,U} \le \|B_i(\cdot,t)\|_{r,U} (\text{mes} U)^{\frac{1}{q}-\frac{1}{r}}. \tag{2.7}$$

Let $1 \le p \le r\infty$, $s \le r$ $\left(\frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{r}\right)$

$$\left|L_i^{(\nu)} D_i^{l_i} f\right| = \left(\left|L_i^{(\nu)}\right|^p \left|D_i^{l_i} f\right|^s\right)^{\frac{1}{r}} \left(\left|D_i^{l_i} f\right|^p \chi\right)^{\frac{1}{p}-\frac{1}{r}} \left(\left|L_i^{(\nu)}\right|^s\right)^{\frac{1}{s}-\frac{1}{r}}$$

and apply Hölder inequality for $\left(\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{s} - \frac{1}{r}\right) = 1\right)$, we have

$$|B_i(x,t)| \le$$

$$\le \left(\int\limits_{R^n} \left|D_i^{l_i} f(x+y)\right|^p \left|L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right)\right|^s dy \right)^{\frac{1}{r}} \times$$

$$\left(\int\limits_{R^n} \left|D_i^{l_i} f(x+y)\right|^p \chi \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right) dy \right)^{\frac{1}{p}-\frac{1}{r}} \times$$

$$\left(\int\limits_{R^n} \left|L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right)\right|^s dy \right)^{\frac{1}{s}},$$

Let χ be a characteristic function of the set $S(L_i)$. We have

$$\|B_i(x,t)\|_{r,U}$$

$$\le \sup_{x \in U} \left(\int\limits_{R^n} \left|D_i^{l_i} f(x+y)\right|^p \chi \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right) dy \right)^{\frac{1}{p}-\frac{1}{r}}$$

$$\times \sup_{y \in V} \left(\int\limits_{U} \left|D_i^{l_i} f(x+y)\right|^p dx \right)^{\frac{1}{r}}$$

$$\times \left(\int\limits_{R^n} \left|L_i^{(\nu)} \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right)\right|^s dy \right)^{\frac{1}{s}}, \tag{2.8}$$

Let a functions $L_i^{(\nu)}$ satisfy the condition: $\exists C > 0$, for any $(x,y) \in R^n \times R^n$

$$\left|L_i^{(\nu)}(x,y,z)\right| \le C \left|L_i^{\nu}(x)\right|.$$

For any $x \in U$ we have

$$\int\limits_{R^n} \left| D_i^{l_i} f(x+y) \right|^p \chi\left(\frac{y}{\varphi(t)}\right) dy \leq \int\limits_{(U+V)_{\varphi(t)}(\overline{x})} \left| D_i^{l_i} f(y) \right|^p dy \leq$$

$$\leq \int\limits_{Z} \left| D_i^{l_i} f(y) \right|^p dy \leq \left\| D_i^{l_i} f \right\|_{p,\varphi,\beta;Z}^p \cdot \prod_{j=1}^{n} (\varphi_j(t))^{\beta_j p}. \tag{2.9}$$

For $y \in V$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| D_i^{l_i} f(x+y) \right|^p dx \leq \int\limits_{Z} \left| D_i^{l_i} f(x) \right|^p dx \leq \left\| D_i^{l_i} f \right\|_{p,\psi,\beta;Z}^p, \tag{2.10}$$

$$\int\limits_{R^n} \left| L_i^{\nu}\left(\frac{y}{\varphi(t)}\right) \right|^s dy \leq \| L_i^{\nu} \|_s^s \prod_{j=1}^{n} \varphi_j(t). \tag{2.11}$$

From inequalities (2.8)-(2.11) it follows that

$$\| B_i(\cdot,t) \|_{r,U} \leq C_1 \| L_i^{\nu} \|_s^s \prod_{j=1}^{n} (\varphi_j(t))^{\frac{1}{s}+\beta_j p\left(\frac{1}{p}-\frac{1}{r}\right)} \left\| D_i^{l_i} f \right\|_{p,\varphi,\beta;Z}. \tag{2.12}$$

Inequalities (2.6), (2.7) and (2.12) for $(r=q)$ we have

$$\| A_i(\cdot,t) \|_{q,U} \leq C |\widetilde{Q}^i(1)| \left\| D_i^{l_i} f \right\|_{p,\varphi,\beta;Z} \tag{2.13}$$

Substituting the inequality (2.13) in (2.4) for $U = G$, we obtain the inequality (2.2).

This completes the proof of Theorem 2.1.

The following theorem is proved analogously to Theorem 2.1..

Theorem 2.2 *Let all the conditions of theorem 1 be fulfilled. Furthermore, let $l^1 \in N^n$ and if*

$$\widetilde{Q}_1^i(T) = \int\limits_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - l_j^1 - (1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{\varphi_i'(t)}{\varphi_i(t)^{1-l_i}} dt < \infty,$$

then

$$\| D^{\nu}(f - P_{l-1}) \|_{\omega_q^{l^1}(G)} \leq C |\widetilde{Q}_1(1)| \| f \|_{\omega_{p,\varphi,\beta}^{l}(G)} \tag{2.14}$$

where $\widetilde{Q}_1(1) = \max\limits_{1 \leq i \leq n} \widetilde{Q}_1^i(1)$ and C the constant independent of f.

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