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# Probability method and Wiman-Valiron type estimations for differential equations

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**Abstract.** *In the paper we consider evolution equations of the form*

$$u'(t) \pm A(t)\, u(t) = 0$$

*in Hilbert space, where $A(t)$ is a self-adjoint operator with a discrete spectrum. The Wiman-Valiron type estimations are established for solutions of equations (1.1) such that $\|u(t)\| \to \infty$ as $t \to \infty$ (or $t \to 0$), that characterize the behavior of the solution depending on behavior of Fourier coefficients of the solutions in the system of eigen functions of the given operator $A(t)$.*

**Keywords.** probability, entire functions, maximum modulus, maximum term, estimates of Wiman-Valiron type, logarithmic measure, distributions, spectrum, operator, differential equations, Hilbert space

**Mathematics Subject Classification (2010):** 34K08, 34K10

## 1 Introduction

In the paper we consider evolution equation of the form

$$u'(t) + A(t)\, u(t) = 0 \tag{1.1}$$

in Hilbert space, where $A(t)$ is a self-adjoint operator with a discrete spectrum. The Wiman-Valiron type estimations are established for solutions of equations (1.1) such that $\|u(t)\| \to \infty$ as $t \to \infty$ (or $t \to 0$), that characterize the behavior of the solution depending on behavior of Fourier coefficients of the solutions in the system of eigen functions of the given operator $A(t)$.

Let $H$ be a Hilbert space, $A(t)$ a self adjoint operator with discrete spectrum. Let $\{\varphi_k(t)\}$ be a complete orthonormed system of eigen functions of the operator $A(t)$, $\{\lambda_k(t)\}$ be the sequence of its eigen values. Let the domain of definition $D(A(t))$ of the operator

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$A(t)$ be independent on $t$, and on $D(A)$ there exist a strong derivative $A'(t)$, one of the following conditions on the class of operators $A(t)$ be fulfilled:

$$\left(A'u(t), u(t)\right) \leq k(t)\left(A(t)u(t), u(t)\right), \tag{1.2}$$

$$\left(A'u(t), u(t)\right) \geq -k(t)\left(A(t)u(t), u(t)\right), \tag{1.3}$$

$$(u \in D(A), \quad t \in (0,T), \; 0 \leq T \leq \infty)$$

where $k(t) \geq 0$, $k(t) \in L_1(0,\infty)$ (In the case when $A(t)$ is a constant operator, in these conditions we can accept $k \equiv 0$).

It is appropriate to give brief review on Wiman-Valiron theory and its relation with differential equations. This theory has appeared 100 years before in the field of entire functions in the papers of Wiman [11] and Valiron [10] and is intensively developed in our days. Extensive papers are published in very advanced periodic journals (see: e.i. [1]). The essence of this theory that plays a key role in our studies in the field of differential equations is as follows.

Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function,

$$M(r) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \max_n |a_n| r^n$$

be the maximum of the modulus and maximum term of the function $f(z)$ in the ring of radius $r$. These functions are very significan in theory of entire functions. It is known that $M(r) \to \infty$, $\mu(r) \to \infty$ as $r \to \infty$ and always $\mu(r) \leq M(r)$ . It is difficult to calculate $M(r)$ even for simple entire functions. But very often it suffices to be able to estimate it from above by the function $\mu(r)$, that is simpler to calculate. There arises a natural question: is it possible to estimate $M(r)$ from above by $\mu(r)$? In other words, is it possible to find a function $\psi(y) > 0$, $y > 0$ (or a class of such functions) that the inequality of type

$$M(r) \leq \psi(\mu(r))$$

may be fulfilled in some sense?

Historically such a result was first established by Wiman and Valiron and has the following form of estimation

$$M(r) \leq \mu(r)(\log \mu(r))^{\frac{1}{2}+\varepsilon}, \quad \varepsilon > 0. \tag{1.4}$$

And this inequality is fulfilled outside of possibly of some set $E \subset (0,\infty)$ of finite logarithmic measure (the set $E$ is called exceptional set).

In 1963, the American mathematician Rosenbloom [2] strengthened and generalized the Wiman-Valiron result for a wider class of functions $\Psi(y)$.

The Rosenbloom's result is the following. Let $f(z)$ be an entire function, $\Psi(y)$ be a positive increasing function such that the integral

$$\int^\infty \left(\int^y \Psi(t)\, dt\right)^{-\frac{1}{2}} dy < \infty \tag{1.5}$$

is finite.

Then outside of, possibly, finite weighted measure, the inequality of the form $E$

$$M(r) \leq \mu(r)\sqrt{\Psi(\log M(r))} \tag{1.6}$$

is valid.

In N. Suleymanov's paper [3–9] the estimations of type (1.6) were established for solutions of the evolution equations of the form

$$u' \pm A(t)\,u = O, \tag{1.7}$$

wherein the functions

$$M(t) = \|u(t)\|\,,\ \mu(t) = \max_k |(u(t)\,, \varphi_k(t))|\,,$$

play the role of functions $M(r)$ and $\mu(r)$, where $\{\varphi_k(t)\}$ are eigen functions of the operator $A(t)$. In the paper [3–9] theory of Wiman-Valiron type estimations was constructed in conformity to theory of differential equations of type (1.7). For obtaining such results a method based on probability was offered. Owing to this method, stronger and more exact results compared with Wiman-Valirons Rosenbloom results were obtained simultaneously for the same entire functions. Earlier such Wiman-Valiron type reults were not known even for constant operators $A(t)$.

## 2 Problem statement

us consider the function

$$g(t) = \frac{1}{2}\log(u(t)\,, u(t))\,,$$

where $u(t)$ is the solution of equation (1.1). Calculate the derivatives, $g'$ and $g''$ find:

$$g' = -\frac{(Au, u)}{(u, u)}, \tag{2.1}$$

$$g'' = -\frac{(A'u, u)}{(u, u)} + 2\frac{(Au, Au)(u, u) - (Au, u)^2}{(u, u)^2}. \tag{2.2}$$

By the Schwarts inequality it holds the inequality

$$(Au, u)^2 \le (Au, Au)(u, u)\,.$$

Taking into account (1.2) and (1.4), from (1.5) it follows the inequality

$$g'' \ge -k(t)\,\frac{(Au, u)}{(u, u)} + d = k(t)\,g' + d, \tag{2.3}$$

where

$$d = 2\frac{(Au, Au)(u, u) - (Au, u)^2}{(u, u)^2}.$$

As by the Schwarts inequality $d > 0$, then from (1.6) we have:

$$g'' - k(t)\,g' > 0, \quad d \le g'' - k(t)\,g'. \tag{2.4}$$

Assign to the solution $u(t)$ of equation (1.1) some random variable $X$ with discrete distribution (dependent on $t$ parameter) that is determined by the formula

$$p_k \equiv P(X = \lambda_k(t)) = \frac{|(u(t)\,, \varphi_k)|^2}{\|u\|^2} \equiv \frac{c_k(t)^2}{\|u\|^2},$$

where $c_k(t)$ are Fourier coefficients of the functions $u(t)$ in the system $\{\varphi_k(t)\}$.

Having calculated the mathematical expectation $MX$ and variance $DX$, we find:

$$MX = \sum_{1}^{\infty} \lambda_k p_k = \frac{1}{\|u\|^2} \sum c_k(t)^2 \lambda_k.$$

As

$$u\left(t\right) = \sum_{1}^{\infty} c_k\left(t\right) \varphi_k\left(t\right), \;\; Au = \sum^{\infty} c_k(t)\lambda_k \varphi_k,$$

$$\left(Au, u\right) = \sum_{1}^{\infty} \lambda_k c_k^2, \;\; \left(Au, Au\right) = \sum_{1}^{\infty} \lambda_k^2 c_k^2,$$

then taking account (1.4) we have:

$$MX = \frac{(Au, u)}{(u, u)} = -g'\left(t\right). \tag{2.5}$$

The variance is calculated by the formula:

$$DX = MX^2 - \left(MX\right)^2 = MX^2 - g'^2. \tag{2.6}$$

Here

$$MX^2 = \sum_{1}^{\infty} \lambda_k^2 p_k = \frac{1}{(u, u)} \sum_{1}^{\infty} \lambda_k^2 c_k^2 = \frac{(Au, Au)}{(u, u)}.$$

Consequently,

$$DX = \frac{(Au, Au)}{(u, u)} - g'^2 = \frac{(Au, Au)\left(u, u\right) - \left(Au, u\right)^2}{\left(u, u\right)^2} = \frac{1}{2} d.$$

Hence it follows that

$$0 < DX \leq \frac{1}{2}\left(g'' - k\left(t\right) g'\right). \tag{2.7}$$

**Lemma 2.1** *Let $N\left(t; \lambda\right)$ be the amount of eigen values $\lambda_k\left(t\right)$ of the operator $A\left(t\right)$ not exceeding the number $\lambda$ (with regard to multiplicity). Denote*

$$\mu\left(t\right) = \max_k \left|\left(u\left(t\right), \varphi_k\left(t\right)\right)\right| = \max_k \left|c_k\left(t\right)\right|,$$

*where $c_k\left(t\right)$ are Fourier coefficients of the function $u\left(t\right)$.*

It is valid the inequality

$$e^{2g(t)} \leq \mu\left(t\right)^2 \varDelta N\left(t; g', g''\right), \tag{2.8}$$

where

$$\varDelta N\left(t; g', g''\right)$$

$$\equiv N\left(t; \left|g'\right| + C\sqrt{g'' - k\left(t\right) g'}\right) - N\left(t; \left|g'\right| - C\sqrt{g'' - k\left(t\right) g'}\right), \;\; c = const > 0 \tag{2.9}$$

(Remark. Here and in what follows $C$ denotes a positive constant possibly dependent on $t$ parameter but not always the same).

The proof is similar to the similar lemma from the paper [6] (p. 84) that is based on theory of probability.

The essence of this method is as follows (in brief). By the Chebyshev inequality, the probability is estimated from above by the variance $DX$ by the inequality of the form

$$P\left(|X-MX|>\varepsilon\right)\leq\frac{DX}{\varepsilon^2},\quad \varepsilon>0.$$

Accept $\varepsilon=C\sqrt{DX},\ \ C=const>1$ and get

$$1-C^{-2}\leq P\left(\left|X+g'\right|\leq C\sqrt{DX}\right)=\sum{}'p_k=\sum{}'\frac{c_k^2}{\|u\left(t\right)\|^2}$$

(here summation $\sum'$ is taken over those $k$, for which $|\lambda_k+g'|\leq C\sqrt{g''-kg'}$).

Hence it immediately follows the inequality of the form

$$\|u\left(t\right)\|^2\leq C\mu\left(t\right)^2\cdot\sum{}'1.$$

It is clear that the sum $\sum_k' 1\left\{k:|\lambda_k+g'|\leq C\sqrt{DX}\right\}$ just gives $\Delta N\left(t;g',g''\right)$.
The lemma is proved.

Some Wiman-Valiron type estimations are immediately obtained from the lemma's statement. Let, for example, for some function $Q\left(t,y\right)$ the inequality of the following form be fulfilled

$$\Delta N\left(t;g',g''\right)\leq Q\left(t;g\left(t\right)\right). \tag{2.10}$$

Then by the Lemma, hence it follows the estimation of the form

$$\|u\left(t\right)\|\leq\mu\left(t\right)\sqrt{Q\left(t;\log\|u\left(t\right)\|\right)}.$$

In particular, if $Q\left(t,y\right)=y^k$, then we get (1.4) type estimation. It should be noted that just the above mentioned papers [3–9] were devoted to fulfilment of inequalities of type (2.6). The following theorem is the main result of the present paper.

**Theorem 2.1** *Let $u\left(t\right)$ be the solution of equation (1.1) such that $\|u\left(t\right)\|\to\infty$ as $t\to\infty$ (or $t\to 0$). Let as $\lambda>\delta>0,\ \ \lambda\to\infty$ for the function $N\left(t,\lambda\right)$ the following condition be fulfilled*

$$N\left(t,\lambda+\delta\right)-N\left(t,\lambda-\delta\right)\leq C\left(\delta+\sqrt{\lambda}\right)\cdot\lambda^s. \tag{2.11}$$

*Let $\psi\left(y\right)$ be a continuous positive non-decreasing function on $\left(0,\infty\right)$ such that for some $\alpha>0$ the following integral is finite*

$$\int^{\infty}\left(\int^{y}\Psi\left(t\right)dt\right)^{-\alpha}dy<\infty. \tag{2.12}$$

*Then outside of (possibly) some set $E\subset\left(0,\infty\right)$ of finite weighted measure the following Wiman-Valiron Rosenbloom type estimation is valid:*

$$\|u\left(t\right)\|\leq\mu\left(t\right)\sqrt{\Psi\left(\log\mu\left(t\right)\right)}. \tag{2.13}$$

*We carry out proof (without loss of generality) for the particular case of the operator $A\left(t\right)$ having substituted $N\left(\lambda\right)=\lambda$ or $N\left(\lambda\right)=\lambda^3$ and $k\left(t\right)\equiv k=const>0$.*

Introduce change of variables:

$$\xi = \xi(t) = \frac{1}{k}e^{kt}, \;\; t = t(\xi) = \frac{1}{k}\ln k\xi, \tag{2.14}$$

where $t(\xi)$ is the inverse function to the function $\xi(t)$.

Assume $\tilde{g}(\xi) = g(t) = g(t(\xi))$. We have:

$$g'(t) = \tilde{g}e^{kt}, \;\; g''(t) = \tilde{g}''e^{2kt} + k\tilde{g}'e^{kt}$$

$$g'' - kg' = \tilde{g}''e^{2kt}, \;\; \sqrt{g'' - kg'} = \sqrt{\tilde{g}''}\cdot e^{kt}.$$

It is clear that $\tilde{g}'' > 0$. From (2.9) we get:

$$\Delta N\left(t; g', g''\right) = 2C\sqrt{g'' - kg'} = 2C\sqrt{\tilde{g}''}e^{kt}.$$

Choose the function $\psi(y)$ so that the inequality of type

$$\Delta N\left(t, g', g''\right) \le \sqrt{\psi(g)},$$

be fulfilled or that is equivalent

$$\Delta N\left(t, \tilde{g}', \tilde{g}''\right) \le \sqrt{\psi(\tilde{g})}.$$

Then we have

$$\sqrt{\tilde{g}''} \le C\sqrt{\psi(\tilde{g})}.$$

Having again denoted $\tilde{g}(\xi) \equiv h(\xi)$, we get:

$$\sqrt{h''(\xi)} \le \sqrt{\psi(h)}.$$

Then:

$$h'' \le \psi(h), \;\; h''h' \le \psi(h)h', \;\; \left(h'^2\right)' \le 2\psi(h)h',$$

$$h'^2 \le 2\int^h \psi(t)\,dt, \;\; h' \le \left(2\int^h \psi(t)\,dt\right)^{1/2} \equiv \psi_1(h).$$

Denote $E = \{\xi : h'(\xi) > \psi_1(h(\xi))\}$. We have:

$$mesE = \int_E d\xi \le \int_E \frac{h'(\xi)\,d\xi}{\Psi_1(h)} \le \int^\infty \frac{dh}{\Psi_1(h)} = \int^\infty \left(\int^y \Psi(t)\,dt\right)^{-\frac{1}{2}} dy < \infty.$$

Consequently, outside of $E, mesE < \infty$ it holds the inequality

$$\Delta N\left(t, g', g''\right) \le \sqrt{\psi(g)}. \tag{2.15}$$

Then by the Lemma, it holds estimation (2.13).

Note that in the expression of the function $\Delta N(t, \lambda, \delta)$ instead of parameters $\lambda$ and $\delta$ we can accept $\lambda = |g'|$ and $\delta = \sqrt{g'' - kg'}$.

Then we get $\lambda > \delta > 0$ outside of the some set $E, mesE < \infty$. Indeed, after change of variables (2.12) we have:

$$\lambda = C\left|\tilde{g}'\right|, \;\; \delta = C\sqrt{\tilde{g}''}.$$

Denote $(\tilde{g}' \equiv h(\xi))$

$$E = \left\{\left|\tilde{g}'\right| < C\sqrt{\tilde{g}''}\right\} = \left\{h < C\sqrt{h'}\right\} = \left\{h^2 < Ch'\right\} = \left\{1 < C\frac{h'}{h^2}\right\}.$$

Consequently,

$$mesE=\int_E d\xi\le C\int_E \frac{h'(\xi)\,d\xi}{h(\xi)^2}\le C\int^{\infty}\frac{dh}{h^2}<+\infty.$$

Therefore, outside of $E$, $mesE<\infty$, we have $\lambda>\delta>0$.
Now let $N(\lambda)=\lambda^3$. Then from (2.9) we get:

$$\varDelta N\left(t,g',g''\right)=6cg'^2\sqrt{g''-kg'}+2c^3\sqrt{\left(g''-kg'\right)^3}. \tag{2.16}$$

After change of variables $\xi=\frac{1}{k}e^{kt}$, $t=\frac{1}{k}\ln k\xi$, $h=\tilde{g}$ we have:

$$\varDelta N\left(t,h',h''\right)=c_1h'^2\sqrt{h''}+c_2\left(h''\right)^{3/2}. \tag{2.17}$$

We choose the function $\psi(y)$ from condition (1.5) so that the condition of the form

$$\varDelta N\left(t,h',h''\right)\le\sqrt{\psi(h)} \tag{2.18}$$

be fulfilled.

Hence we get the system of inequalities

$$\left.\begin{array}{l} h'^2\sqrt{h''}\le\sqrt{\psi(h)}, \\ \left(h''\right)^{3/2}\le\sqrt{\psi(h)}. \end{array}\right\} \tag{2.19}$$

Let s consider the first inequality. We have:

$$h'^2\sqrt{h''}=\sqrt{\psi(h)},\ \ h'^4h''\le\psi(h)\,,\ \ h'^5h''\le\psi(h)\,h',$$

$$\left(h'^6\right)'\le 6\psi(h)\,h', h'^6\le 6\int^h\psi(t)\,dt,$$

$$h'(\xi)\le\left(6\int^h\psi(t)\,dt\right)^{1/6}\equiv\psi_2(h)\,.$$

Denote

$$E_1=\left\{h'(\xi)>\psi_2(h(\xi))\right\},$$

we have:

$$mesE_1=\int_{E_1}d\xi\le\int^{\infty}\frac{dh}{\psi_2(h)}=\int^{\infty}\left(6\int^h\psi(t)\,dt\right)^{-\frac{1}{6}}dh<+\infty.$$

Consequently outside of $E_1$, $mesE_1<\infty$ it holds the inequality$h'(\xi)\le\psi_2(h(\xi))$. This is equivalent to the fact that the first inequality of the system (2.19) is fulfilled outside of $E_1$. The same holds for the second inequality. Thus, inequality (2.18) is fulfilled outside of some set $E$ of finite measure. Then by the Lemma it follows the estimation of type (2.13).

**Remark 2.1** In general situation we use condition (2.7) of the theorem and introduce change of variables having assumed:

$$\xi(t)=\int_0^t\left[\exp\int_0^p k(s)\,ds\right]d\rho.$$

Similar results are obtained when fulfilling the conditions of the form (1.2). But this case is more suitable when inverse-parabolic type equation (1.1) is considered.

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