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Parabolic maximal operator and its commutators in parabolic generalized Orlicz-Morrey spaces

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Abstract. *In this paper, we give sufficient conditions for the boundedness of parabolic maximal operator and its commutators in parabolic generalized Orlicz-Morrey spaces.*

Keywords. parabolic generalized Orlicz-Morrey space · parabolic maximal operator · commutator · BMO

Mathematics Subject Classification (2010): 42B25, 42B35

1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the parabolic maximal operator and the parabolic singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \ \ (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$(a) \ \ \rho(A_t x) = t\rho(x), \ \ t > 0, \ \ \text{for every} \ \ x \in \mathbb{R}^n;$$
$$(b) \ \ \rho(0) = 0, \ \ \rho(x - y) = \rho(y - x) \geq 0$$
$$\text{and} \ \ \rho(x - y) \leq k(\rho(x - z) + \rho(y - z));$$
$$(c) \ \ dx = \rho^{\gamma-1} d\sigma(w) d\rho, \ \ \text{where} \ \ \rho = \rho(x), w = A_{\rho^{-1}} x$$
$$\text{and } d\sigma(w) \text{ is a measure on the unit ellipsoid } S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}.$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([3,6]). The parabolic balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x - y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)| = v_\rho r^\gamma$, where v_ρ is the

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volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x,r) = B(x,r)$. Note that in the standard parabolic case $P = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

The parabolic maximal function $M^P f$ of a function $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$M^P f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy.$$

If $P = I$, then $M \equiv M^I$ is the Hardy-Littlewood maximal operator. It is well known that the parabolic maximal operators play an important role in harmonic analysis (see [7], [15]).

In this work we present the characterization for parabolic maximal operator M^P (Theorem 4.1) and its commutators M_b^P (Theorem 4.2) in parabolic generalized Orlicz-Morrey spaces $M_{\Phi,\varphi,P}(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 On Young Functions and Orlicz Spaces

Orlicz space was first introduced by Orlicz in [12, 13] as a generalizations of Lebesgue spaces L_p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L_1 space when L_1 space does not work.

First, we recall the definition of Young functions.

Definition 2.1 *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.

Note that by the convexity of Φ and concavity of Φ^{-1} we have the following properties

$$\begin{cases} \Phi(\alpha t) \le \alpha\Phi(t), \text{ if } 0 \le \alpha \le 1 \\ \Phi(\alpha t) \ge \alpha\Phi(t), \text{ if } \quad \alpha > 1 \end{cases} \text{and} \begin{cases} \Phi^{-1}(\alpha t) \ge \alpha\Phi^{-1}(t), \text{ if } 0 \le \alpha \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha\Phi^{-1}(t), \text{ if } \quad \alpha > 1. \end{cases} \tag{2.2}$$

Definition 2.2 *(Orlicz Space). For a Young function Φ, the set*

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \, \textit{for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

$L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 2.3 *The weak Orlicz space*

$$WL_\Phi(\mathbb{R}^n) = \{ f \in L_1^{\rm loc}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < \infty \}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf \left\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1 \right\}.$$

We note that $\|f\|_{WL_\Phi} \le \|f\|_{L_\Phi}$,

$$\sup_{t>0} \Phi(t)m(\Omega,\ f,\ t) = \sup_{t>0} t\, m(\Omega,\ f,\ \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega,\ \Phi(|f|),\ t)$$

and

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi(\Omega)}}\Big)dx \le 1, \qquad \sup_{t>0} \Phi(t)m\Big(\Omega,\ \frac{f}{\|f\|_{WL_\Phi(\Omega)}},\ t\Big) \le 1, \tag{2.3}$$

where $\|f\|_{L_\Phi(\Omega)} = \|f\chi_\Omega\|_{L_\Phi}$ and $\|f\|_{WL_\Phi(\Omega)} = \|f\chi_\Omega\|_{WL_\Phi}$.

The following analogue of the Hölder's inequality is well known (see, for example, [14]).

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f, g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)|dx \le 2\|f\|_{L_\Phi(\Omega)}\|g\|_{L_{\widetilde{\Phi}}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 2.1 *Let Φ be a Young function and $\mathcal{E}$ be a parabolic balls in $\mathbb{R}^n$. Then*

$$\|\chi_{\mathcal{E}}\|_{L_\Phi} = \|\chi_{\mathcal{E}}\|_{WL_\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 2.1, Lemma 2.1 and (2.1) we get the following estimate.

Lemma 2.2 *For a Young function Φ and for the parabolic balls $\mathcal{E} = \mathcal{E}(x,r)$ the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)|dy \leq 2|\mathcal{E}|\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)\|f\|_{L_\Phi(\mathcal{E})}.$$

3 Parabolic maximal function and its commutators in Orlicz spaces

In [1] the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was obtained, see also [9].

Theorem 3.1 *[1] Let Φ any Young function. Then the parabolic maximal operator M^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}^n)$.*

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|}\int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|dy < \infty,$$

where $b_{\mathcal{E}(x,r)} = \frac{1}{|\mathcal{E}(x,r)|}\int_{\mathcal{E}(x,r)} b(y)dy$. We will need the following property of BMO-functions:

$$\left|b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,t)}\right| \leq C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{3.1}$$

where C does not depend on b, x, r and t. We refer for instance to [10] and [11] for details on this space and properties.

Lemma 3.1 *[2] Let $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\left(r^{-\gamma}\right)\left\|b(\cdot) - b_{\mathcal{E}(x,r)}\right\|_{L_\Phi(\mathcal{E}(x,r))}.$$

The commutators generated by $b \in L_1^{\rm loc}(\mathbb{R}^n)$ and the parabolic maximal operator M^P is defined by

$$M_b^P(f)(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1}\int_{\mathcal{E}(x,t)} |b(x) - b(y)||f(y)|dy.$$

The known boundedness statements for the parabolic maximal commutator operator M_b^P on Orlicz spaces run as follows, see [8, Corollary 2.3].

Theorem 3.2 *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{R}^n)$. Then the operator M_b^P is bounded on $L_\Phi(\mathbb{R}^n)$ and the inequality*

$$\|M_b^P f\|_{L_\Phi} \leq C_0\|b\|_*\|f\|_{L_\Phi} \tag{3.2}$$

holds with constant C_0 independent of f.

The parabolic generalized Orlicz-Morrey spaces and the weak parabolic generalized Orlicz-Morrey spaces are defined as follows.

Definition 3.1 *Let $\varphi(r)$ be a positive measurable function on $(0,\infty)$ and Φ any Young function. We denote by $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized Orlicz-Morrey space, the space of all functions $f \in L_{\Phi}^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{\Phi,\varphi,P}} \equiv \|f\|_{M_{\Phi,\varphi,P}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}(x,r)|)^{-1})\|f\|_{L_{\Phi}(\mathcal{E}(x,r))},$$

where $L_{\Phi}^{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L_{\Phi}(\mathbb{R}^n)$ for all ellipsoids $\mathcal{E} \subset \mathbb{R}^n$.

Also by $WM_{\Phi,\varphi,P}(\mathbb{R}^n)$ we denote the weak parabolic generalized Orlicz-Morrey space of all functions $f \in WL_{\Phi}^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\varphi,P}} \equiv \|f\|_{WM_{\Phi,\varphi,P}(\mathbb{R}^n)}$$

$$= \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}(x,r)|)^{-1})\|f\|_{WL_{\Phi}(\mathcal{E}(x,r))} < \infty,$$

where $WL_{\Phi}^{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in WL_{\Phi}(\mathbb{R}^n)$ for all parabolic balls $\mathcal{E} \subset \mathbb{R}^n$.

Remark 3.1 Thanks to (2.2) we have

$$\Phi^{-1}(|\mathcal{E}(x,r)|)^{-1}) \approx \Phi^{-1}(r^{-\gamma}).$$

Therefore we can also write

$$\|f\|_{M_{\Phi,\varphi,P}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(r^{-\gamma})\|f\|_{L_{\Phi}(\mathcal{E}(x,r))},$$

and

$$\|f\|_{WM_{\Phi,\varphi,P}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(r^{-\gamma})\|f\|_{WL_{\Phi}(\mathcal{E}(x,r))},$$

respectively.

According to this definition, we recover the parabolic generalized Morrey space $M_{p,\varphi,P}(\mathbb{R}^n)$ and weak parabolic generalized Morrey space $WM_{p,\varphi,P}(\mathbb{R}^n)$ under the choice $\Phi(r) = r^p$, $1 \le p < \infty$. If $\Phi(r) = r^p$, $1 \le p < \infty$ and $\varphi(r) = r^{\frac{\lambda-\gamma}{p}}$, $0 \le \lambda \le \gamma$, then $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ and $WM_{\Phi,\varphi,P}(\mathbb{R}^n)$ coincide with $M_{p,\lambda,P}(\mathbb{R}^n)$ and $WM_{p,\lambda,P}(\mathbb{R}^n)$, respectively and if $\varphi(r) = \Phi^{-1}(r^{-\gamma})$, then $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ and $WM_{\Phi,\varphi,P}(\mathbb{R}^n)$ coincide with the $L_{\Phi}(\mathbb{R}^n)$ and $WL_{\Phi}(\mathbb{R}^n)$, respectively.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_{\Phi}$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-\gamma})}$ is almost increasing.

Lemma 3.2 *Let $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$. If $\varphi \in \mathcal{G}_{\Phi}$, then there exist $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi,P}} \le \frac{C}{\varphi(r_0)}.$$

Proof. Let $\mathcal{E} = \mathcal{E}(x,r)$ denote an arbitrary ellipsoid in $\mathbb{R}^n$. By the definition and Lemma 2.1, it is easy to see that

$$\|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi,P}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}|^{-1}) \frac{1}{\Phi^{-1}(|\mathcal{E}\cap\mathcal{E}_0|^{-1})}$$

$$\geq \varphi(r_0)^{-1}\Phi^{-1}(|\mathcal{E}_0|^{-1}) \frac{1}{\Phi^{-1}(|\mathcal{E}_0\cap\mathcal{E}_0|^{-1})} = \frac{1}{\varphi(r_0)}.$$

Now if $r \leq r_0$, then $\varphi(r_0) \leq C\varphi(r)$ and

$$\varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}|^{-1})\|\chi_{\mathcal{E}_0}\|_{L_\Phi(\mathcal{E})} \leq \frac{1}{\varphi(r)} \leq \frac{C}{\varphi(r_0)}.$$

On the other hand if $r \geq r_0$, then $\frac{\varphi(r_0)}{\Phi^{-1}(|\mathcal{E}_0|^{-1})} \leq C\frac{\varphi(r)}{\Phi^{-1}(|\mathcal{E}|^{-1})}$ and

$$\varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}|^{-1})\|\chi_{\mathcal{E}_0}\|_{L_\Phi(\mathcal{E})} \leq \frac{C}{\varphi(r_0)}.$$

This completes the proof.

4 Parabolic maximal operator and its commutators in parabolic generalized Orlicz-Morrey spaces

The following local estimates for parabolic maximal operator M^P in Orlicz spaces are valid.

Lemma 4.1 *Let $\Phi \in \mathcal{Y}$, $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$ and $\mathcal{E} = \mathcal{E}(x,r)$. Then*

$$\|M^P f\|_{L_\Phi(\mathcal{E})} \lesssim \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big)\, \|f\|_{L_\Phi(\mathcal{E}(x,t))} \tag{4.1}$$

for any Young function $\Phi \in \nabla_2$ and

$$\|M^P f\|_{WL_\Phi(\mathcal{E})} \lesssim \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big)\, \|f\|_{L_\Phi(\mathcal{E}(x,t))} \tag{4.2}$$

for any Young function Φ.

Proof. Let $\Phi \in \nabla_2$. We put $f = f_1 + f_2$, where $f_1 = f\chi_{\mathcal{E}(x,2kr)}$ and $f_2 = f\chi_{\complement_{\mathcal{E}(x,2kr)}}$, where k is the constant from the triangle inequality.

Estimation of $M^P f_1$: By Theorem 3.1 we have

$$\|M^P f_1\|_{L_\Phi(\mathcal{E})} \leq \|M^P f_1\|_{L_\Phi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.$$

By using the monotonicity of the functions $\|f\|_{L_\Phi(\mathcal{E}(x,t))}$, $\Phi^{-1}\big(t\big)$ with respect to t and (2.2) we get,

$$\begin{aligned} &\frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Phi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))} \\ &\quad \geq \frac{\|f\|_{L_\Phi(\mathcal{E}(x,2kr))}}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Phi^{-1}\big(t^{-\gamma}\big) \gtrsim \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}. \end{aligned} \tag{4.3}$$

Consequently we have

$$\|M^P f_1\|_{L_\Phi(\mathcal{E})} \lesssim \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \tag{4.4}$$

Estimation of $M^P f_2$: Let y be an arbitrary point from $\mathcal{E}$. If $\mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr)) \neq \emptyset$, then $t > r$. Indeed, if $z \in \mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr))$, then $t > \rho(y-z) \geq \frac{1}{k}\rho(x-z) - \rho(x-y) > 2r - r = r$.

On the other hand, $\mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr)) \subset \mathcal{E}(x,2kt)$. Indeed, if $z \in \mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr))$, then we get $\rho(x-z) \leq k\rho(y-z) + k\rho(x-y) < kt + kr < 2kt$.

Therefore,

$$\begin{aligned} M^P f_2(y) &= \sup_{t>0} \frac{1}{|\mathcal{E}(y,t)|} \int_{\mathcal{E}(y,t)\cap {}^{\complement}(\mathcal{E}(x,2kr))} |f(z)|dz \\ &\leq \sup_{t>r} \frac{1}{|\mathcal{E}(y,t)|} \int_{\mathcal{E}(x,2kt)} |f(z)|dz \\ &\leq \sup_{t>r} \frac{C}{|\mathcal{E}(y,2kt)|} \int_{\mathcal{E}(x,2kt)} |f(z)|dz \\ &= \sup_{t>2kr} \frac{C}{|\mathcal{E}(y,t)|} \int_{\mathcal{E}(x,t)} |f(z)|dz. \end{aligned}$$

Hence by Lemma 2.2

$$\begin{aligned} M^P f_2(y) &\lesssim \sup_{t>2kr} \frac{|\mathcal{E}(x,t)|}{|\mathcal{E}(y,t)|} \Phi^{-1}\big(|\mathcal{E}(x,t)|^{-1}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \\ &\lesssim \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))}. \end{aligned} \tag{4.5}$$

Thus the function $M^P f_2(y)$, with fixed x and r, is dominated by the expression not depending on y. Then we integrate the obtained estimate for $M^P f_2(y)$ in y over $\mathcal{E}$, we get

$$\|M^P f_2\|_{L_\Phi(\mathcal{E})} \lesssim \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x,t))} \tag{4.6}$$

Gathering the estimates (4.4) and (4.6) we arrive at (4.1).

Let now Φ be an arbitrary Young function. It is obvious that

$$\|M^P f\|_{WL_\Phi(\mathcal{E})} \leq \|M^P f_1\|_{WL_\Phi(\mathcal{E})} + \|M^P f_2\|_{WL_\Phi(\mathcal{E})}.$$

By the boundedness of the operator M^P from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$, provided by Theorem 3.1, we have

$$\|M^P f_1\|_{WL_\Phi(\mathcal{E})} \lesssim \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.$$

By using (4.3), (4.5) and Lemma 2.1 we arrive at (4.2).

Theorem 4.1 *Let $\Phi \in \mathcal{Y}$, the functions φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \leq C\,\varphi_2(r), \tag{4.7}$$

where C does not depend on r. Then the operator M^P is bounded from $M_{\Phi,\varphi_1,P}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2,P}(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$, the operator M^P is bounded from $M_{\Phi,\varphi_1,P}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2,P}(\mathbb{R}^n)$.

Proof. Note that

$$\Big(\operatorname*{ess\,inf}_{x\in A} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in A} \frac{1}{f(x)}$$

is true for any real-valued nonnegative function f and measurable on A and the fact that $\|f\|_{L_\Phi(\mathcal{E}(x,t))}$ is a nondecreasing function of t

$$\frac{\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\operatorname*{ess\,inf}\limits_{0<t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)}} = \operatorname*{ess\,sup}_{0<t<s<\infty} \frac{\Phi^{-1}\big(s^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\varphi_1(s)}$$
$$\le \sup_{x\in\mathbb{R}^n, r>0} \frac{\Phi^{-1}\big(s^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,s))}}{\varphi_1(s)} = \|f\|_{M_{\Phi,\varphi_1,P}}.$$

Since (φ_1,φ_2) and Φ satisfy the condition (4.7),

$$\begin{aligned}
&\sup_{r<t<\infty} \|f\|_{L_\Phi(\mathcal{E}(x,t))}\Phi^{-1}\big(t^{-\gamma}\big) \\
&\le \sup_{r<t<\infty} \frac{\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\operatorname*{ess\,inf}\limits_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)}} \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)}\Phi^{-1}\big(t^{-\gamma}\big) \\
&\le C\|f\|_{M_{\Phi,\varphi_1,P}} \sup_{r<t<\infty} \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)}\Big)\Phi^{-1}\big(t^{-\gamma}\big) \\
&\le C\varphi_2(r)\|f\|_{M_{\Phi,\varphi_1,P}} \qquad (4.8)
\end{aligned}$$

Then by (4.1) and (4.8)

$$\|M^P f\|_{M_{\Phi,\varphi_2,P}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(r)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}$$
$$= \|f\|_{M_{\Phi,\varphi_1,P}}.$$

The estimate $\|M^P f\|_{WM_{\Phi,\varphi_2,P}} \lesssim \|f\|_{M_{\Phi,\varphi_1,P}}$ can be proved similarly by the help of local estimate (4.2).

Remark 4.1 Note that Theorem 4.1 in the isotropic case $P = I$ was proved in [4].

Lemma 4.2 *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(\mathbb{R}^n)$, then the inequality*

$$\|M_b^P f\|_{L_\Phi(\mathcal{E}(x_0,r))} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x_0,t))}$$

holds for any ball $\mathcal{E}(x_0,r)$ and for all $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For $\mathcal{E} = \mathcal{E}(x_0,r)$, write $f = f_1 + f_2$ with $f_1 = f\chi_{2k\mathcal{E}}$ and $f_2 = f\chi_{\complement_{(2k\mathcal{E})}}$, where k is the constant from the triangle inequality, so that

$$\big\|M_b^P f\big\|_{L^\Phi(\mathcal{E})} \le \big\|M_b^P f_1\big\|_{L^\Phi(\mathcal{E})} + \big\|M_b^P f_2\big\|_{L^\Phi(\mathcal{E})}.$$

By the boundedness of the operator M_b^P in the space $L_\Phi(\mathbb{R}^n)$ provided by Theorem 3.2, we obtain

$$\|M_b^P f_1\|_{L^\Phi(\mathcal{E})} \le \|M_b^P f_1\|_{L^\Phi(\mathbb{R}^n)} \lesssim \|b\|_*\, \|f_1\|_{L^\Phi(\mathbb{R}^n)} = \|b\|_*\, \|f\|_{L^\Phi(2\mathcal{E})}. \qquad (4.9)$$

As we proceed in the proof of Lemma 4.1, we have for $x \in \mathcal{E}$

$$M_b^P(f_2)(x) \lesssim \sup_{t>2kr} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |b(y)-b(x)||f(y)|dy.$$

Then

$$\|M_b^P f_2\|_{L^\Phi(\mathcal{E})} \lesssim \left\| \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |b(y)-b(\cdot)||f(y)|dy \right\|_{L^\Phi(\mathcal{E})}$$

$$\lesssim J_1 + J_2 = \left\| \sup_{t>2r} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}}||f(y)|dy \right\|_{L^\Phi(\mathcal{E})}$$

$$+ \left\| \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |b(\cdot)-b_{\mathcal{E}}||f(y)|dy \right\|_{L^\Phi(\mathcal{E})}.$$

For the term J_1 by Lemma 2.1 we obtain

$$J_1 \approx \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}}||f(y)|dy$$

and split it as follows:

$$J_1 \lesssim \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}(x_0,t)}||f(y)|dy$$

$$+ \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \int_{\mathcal{E}(x_0,t)} |f(y)|dy.$$

Applying Hölder's inequality, by Lemmas 2.2 and 3.1 and (3.1) we get

$$J_1 \lesssim \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} \left\|b(\cdot)-b_{\mathcal{E}(x_0,t)}\right\|_{L_{\widetilde{\Phi}}(\mathcal{E}(x_0,t))} \|f\|_{L^\Phi(\mathcal{E}(x_0,t))}$$

$$+ \frac{1}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}||\mathcal{E}(x_0,t)| \Phi^{-1}\left(t^{-\gamma}\right) \|f\|_{L^\Phi(\mathcal{E}(x_0,t))}$$

$$\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2r} \Phi^{-1}\left(t^{-\gamma}\right)\left(1+\ln\frac{t}{r}\right)\|f\|_{L^\Phi(\mathcal{E}(x_0,t))}.$$

For J_2 we obtain

$$J_2 \approx \|b(\cdot)-b_B\|_{L^\Phi(\mathcal{E})} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,t)|} \int_{\mathcal{E}(x_0,t)} |f(y)|dy$$

$$\lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big)\|f\|_{L^\Phi(\mathcal{E}(x_0,t))}$$

gathering the estimates for J_1 and J_2, we get

$$\|M_b^P f_2\|_{L^\Phi(\mathcal{E})} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-\gamma}\big)} \sup_{t>r} \Phi^{-1}\big(t^{-\gamma}\big)\left(1+\ln\frac{t}{r}\right)\|f\|_{L^\Phi(\mathcal{E}(x_0,t))}. \tag{4.10}$$

By using (4.3) we unite (4.10) with (4.9), which completes the proof.

Theorem 4.2 *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(\mathbb{R}^n)$ and the functions φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \le C\, \varphi_2(r), \tag{4.11}$$

where C does not depend on r. Then the operator M_b^P is bounded from $M_{\Phi,\varphi_1,P}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2,P}(\mathbb{R}^n)$.

Proof. The proof is similar to the proof of Theorem 4.1 thanks to Lemma 4.2.

Remark 4.2 Note that Theorem 4.2 in the isotropic case $P = I$ was proved in [5].

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