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Maximal and singular integral operators and their commutators in the vanishing generalized weighted Morrey spaces with variable exponent

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Abstract. *We consider the vanishing generalized weighted Morrey spaces $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$ with variable exponent $p(x)$ and a general function $\varphi(x,r)$ defining the Morrey-type norm. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the Hardy-Littlewood maximal operator and Calderón-Zygmund singular operators with standard kernel, in such spaces. We also prove the boundedness of the commutators of maximal operator and Calderón-Zygmund singular operators in the vanishing generalized weighted Morrey spaces with variable exponent.*

Keywords. Maximal operator, singular integral operators, generalized Morrey space, BMO space.

Mathematics Subject Classification (2010): 42B20,42B25, 42B35.

1 Introduction

The classical Morrey spaces were originally introduced by Morrey in [20] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [6, 8, 20]. Mizuhara [21] and Nakai [23] introduced generalized Morrey spaces. Later, Guliyev [8] defined the generalized Morrey spaces $M^{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [16] considered the weighted Morrey spaces $L_w^{p,\kappa}$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [9] gave a concept of generalized weighted Morrey space $M_w^{p,\varphi}$ which could be viewed as extension of both generalized Morrey space $M^{p,\varphi}$ and weighted Morrey space $L_w^{p,\kappa}$. In [9] the boundedness of the classical operators and its commutators in spaces $M_w^{p,\varphi}$ also was studied, see also [14].

Vanishing Morrey spaces $VM^{p,\varphi}(\mathbb{R}^n)$ are subspaces of functions in Morrey spaces which were introduced by Vitanza [24] satisfying the condition

$$\lim_{r\to 0} \sup_{\substack{x\in\mathbb{R}^n \\ 0<t<r}} t^{-\frac{\lambda}{p}} \|f\chi_{B(x,t)}\|_{L^{p(\cdot)}(B(x,t)}) = 0$$

and applied there to obtain a regularity result for elliptic partial differential equations. Also Ragusa [25] proved a sufficient condition for commutators of fractional integral operators to

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belong to vanishing Morrey spaces $VM^{p,\lambda}(\mathbb{R}^n)$. About commutator operators in vanishing Morrey spaces. The vanishing generalized Morrey spaces $VM^{p,\varphi}(\mathbb{R}^n)$ were introduced and studied by Samko in [26].

Since Kovácik and Rákosnik [17] introduced the variable exponent Lebesgue space and Sobolev space in higher dimensional Euclidean spaces, many mathematicians have been involved deeply in this field. As for the nonweighted and weighted variable exponent cases, we may refer to [3,5,7,27]. For the variable exponent cases in Morrey type spaces, we provide these articles from Kokilashvili [15], Guliyev [11–13], Mizuta [22].

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [5,27], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein.

We introduce the generalized variable exponent weighted Morrey spaces $\mathcal{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$. Within the frameworks of the spaces $\mathcal{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$, over unbounded sets $\Omega \subseteq \mathbb{R}^n$ we consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)| dy$$

and Calderón-Zygmund type singular operator

$$Tf(x) = \int_{\Omega} K(x,y) f(y) dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C \frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C \frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if } |x-y| > 2|x-\xi|.$$

Let

$$T^* f(x) = \sup_{\varepsilon>0} |T_{\varepsilon} f(x)|$$

be the maximal singular operator, where $T_{\varepsilon} f(x)$ is the usual truncation

$$T_{\varepsilon} f(x) = \int_{\{y \in \Omega : |x-y| \geq \varepsilon\}} K(x,y) f(y) dy.$$

We find the condition on the Morrey function $\varphi(x,r)$ for the boundedness of the maximal operator M and the singular integral operators T in vanishing generalized weighted Morrey space $\mathcal{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces. In Section 3 we deal with the maximal operator and its commutator. In Section 4 we treat Calderón-Zygmund singular operators and its commutators.

The main results are given in Theorems 3.2, 3.4, 4.2, 4.4. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\varphi(x,r)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces

We refer to the book [3] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{2.1}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

For the basics on variable exponent Lebesgue spaces we refer to [28].
$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1,\infty)$;
$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2} \ \ x, y \in \Omega, \tag{2.2}$$

where $A = A(p) > 0$ does not depend on x, y;
$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \to \mathbb{R}^n$ satisfying the condition (2.2);
$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \le p_+ < \infty$;
for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \le \frac{A_\infty}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n. \tag{2.3}$$

where $p(\infty) = \lim_{x\to\infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [3], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\theta_p(x,r)}, \quad x \in \Omega, \ p \in \mathbb{P}^{log}_\infty(\Omega), \tag{2.4}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1, \\ \frac{n}{p(\infty)}, & r \ge 1 \end{cases}$.

By ω we always denote a weight, i.e. a positive, locally integrable function with domain Ω. The weighted Lebesgue space $L^{p(\cdot)}_\omega(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}_\omega(\Omega)} = \|f\omega\|_{L^{p(\cdot)}(\Omega)}.$$

Let us define the class $A_{p(\cdot)}(\Omega)$ (see [5], [18]) to consist of those weights ω for which

$$\sup_B |B|^{-1} \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,r))} < \infty.$$

Singular operators within the framework of the spaces with variable exponents were studied in [4]. From Theorem 4.8 and Remark 4.6 of [4] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy,$$

where $f_{\widetilde{B}(x,r)}(x) = |\widetilde{B}(x,r)|^{-1} \int_{\widetilde{B}(x,r)} f(y)dy$.

Definition 2.1 *We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO} = \sup_{x\in\Omega} M^\sharp f(x) = \sup_{x\in\Omega,\, r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy.$$

Definition 2.2 *We define the $BMO_{p(\cdot),\omega}(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO_{p(\cdot),\omega}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}}.$$

Theorem 2.1 *[19, Theorem 4.4] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and ω be a Lebesgue measurable function. If $\omega \in A_{p(\cdot)}(\Omega)$, then the norms $\|\cdot\|_{BMO_{p(\cdot),\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are non-negative measurable functions on $\Omega \times (0,\infty)$. We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

Definition 2.3 *Let $1 \le p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ and variable exponent generalized weighted Morrey space $\mathcal{M}^{p(\cdot),\varphi(\cdot)}_\omega(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms*

$$\|f\|_{\mathcal{M}^{p(\cdot),\varphi}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r) r^{\theta_p(x,r)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

$$\|f\|_{\mathcal{M}^{p(\cdot),\varphi}_\omega} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,r))},$$

respectively.

According to this definition, we recover the space $\mathcal{L}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\varphi(x,r) = r^{\theta_p(x,r)-\frac{\lambda(x)}{p(x)}}$:

$$\mathcal{L}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}_\omega^{p(\cdot),\varphi(\cdot)}(\Omega)\Bigg|_{\varphi(x,r)=r^{\theta_p(x,r)-\frac{\lambda(x)}{p(x)}}}.$$

Definition 2.4 *(Vanishing generalized weighted Morrey space) The vanishing generalized weighted Morrey space $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$ is defined as the space of functions $f \in \mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$ such that*

$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\chi_{\widetilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0}\frac{1}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\inf\limits_{x\in\Omega}\varphi(x,r)} = 0. \tag{2.5}$$

and

$$\sup_{0<r<\infty}\frac{1}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\inf\limits_{x\in\Omega}\varphi(x,r)} = 0. \tag{2.6}$$

which makes the spaces $V\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$ non-trivial, because bounded functions with compact support belong then to this space.

3 The maximal operator and its commutators in $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$

The following weighted local estimates were proved [10].

Theorem 3.1 *[10] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\|Mf\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,r))} \le C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\sup_{t\ge r}\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,t))}, \tag{3.1}$$

for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and r.

The following theorem is valid.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and the function $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the conditions*

$$C_\gamma := \sup_{t>\gamma}\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} < \infty \tag{3.2}$$

for every γ and

$$\sup_{t>r}\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \le C\varphi_2(x,r) \tag{3.3}$$

where C does not depend on $x \in \Omega$ and r. Then the operator M is bounded from the space $V\mathcal{M}_\omega^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}_\omega^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 3.1, so we only have to prove that if

$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}=0\Rightarrow$$
$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|Mf\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}=0 \tag{3.4}$$

otherwise.

To show that $\sup_{x\in\Omega}\frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|Mf\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}<\varepsilon$ for small r, we split the right-hand side of (3.1):

$$\sup_{x\in\Omega}\frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|Mf\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}\le C_0\left(I_{1,\gamma}(x,r)+I_{2,\gamma}(x,r)\right), \tag{3.5}$$

where $\gamma>0$ will be chosen as shown below (we may take $\gamma<1$),

$$I_{1,\gamma}(x,r):=\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\int_r^\gamma\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$
$$I_{2,\gamma}(x,r):=\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\int_\gamma^\infty\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

and it is supposed that $r<\gamma$. Now we choose any fixed $\gamma>0$ such that

$$\sup_{x\in\Omega}\frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}<\frac{\varepsilon}{2CC_0},\ \text{for all } 0<t<\gamma,$$

where C and C_0 are constants from (3.3) and (3.5), which is possible since $f\in V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)$. Then

$$\sup_{x\in\Omega}CI_{1,\gamma}(x,r)<\frac{\varepsilon}{2},\ 0<r<\gamma,$$

by (3.4).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.6). We have

$$I_{2,\gamma}(x,r)\le C_\gamma\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)},$$

where C_γ is the constant from (3.2). Then, by (2.6) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}<\frac{\varepsilon}{2CC_\gamma\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)}}$$

which completes the proof of (3.4).

In the case $\omega=1$ we get

Corollary 3.1 *Let $\Omega\subset\mathbb{R}^n$ be an open unbounded set, $p\in\mathbb{P}^{log}_\infty(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\sup_{t>r}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\theta_p(x,s)}}{t^{\theta_p(x,t)}}\le C\varphi_2(x,r),$$

where C does not depend on $x\in\Omega$ and r. Then the operator M is bounded from the space $V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

The commutator generated by M and a suitable function b is formally defined by

$$[M,b]f = M(bf) - bM(f).$$

Given a measurable function b the maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x)-b(y)||f(y)|dy$$

for all $x \in \mathbb{R}^n$.

Operators M_b and $[M,b]$ essentially differ from each other. For example, M_b is a positive and sublinear operator, but $[M,b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator M_b controls $[M,b]$.

Lemma 3.1 *[1] Let b be any non-negative locally integrable function. Then*

$$|[M,b]f(x)| \le M_b(f)(x), \;\; x \in \mathbb{R}^n$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n)$.

The following weighted local estimates were proved [10].

Theorem 3.3 *[10] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$. Then*

$$\begin{aligned} &\|M_b f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,r))} \\ &\le C\|b\|_* \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \sup_{t\ge r} \left(1+\ln\frac{t}{r}\right) \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,t))} \end{aligned} \tag{3.6}$$

for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and r.

The following theorem is valid.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$ and the function $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the conditions*

$$C_\delta := \sup_{r>\delta} \left(1+\ln\frac{r}{\delta}\right) \frac{\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} < \infty \tag{3.7}$$

for every δ and

$$\sup_{t>r} \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \le C\varphi_2(x,r), \tag{3.8}$$

where C does not depend on $x \in \Omega$ and r. Then the operator M_b is bounded from the space $V\mathcal{M}_\omega^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}_\omega^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 3.3, so we only have to prove that if

$$\begin{aligned} &\lim_{r\to 0} \sup_{x\in\Omega} \frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\chi_{\widetilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)} = 0 \Rightarrow \\ &\lim_{r\to 0} \sup_{x\in\Omega} \frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|M_b f\chi_{\widetilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)} = 0 \end{aligned} \tag{3.9}$$

otherwise.

To show that $\sup\limits_{x\in\Omega} \frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|M_b f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_{\omega}(\Omega)} < \varepsilon$ for small r, we split the right-hand side of (3.6):

$$\sup_{x\in\Omega} \frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|M_b f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_{\omega}(\Omega)} \le C_0 \left(I_{1,\delta}(x,r) + I_{2,\delta}(x,r)\right), \quad (3.10)$$

where $\delta > 0$ will be chosen as shown below (we may take $\delta < 1$),

$$I_{1,\delta}(x,r) := \|b\|_* \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \int_r^{\delta} \left(1 + \ln\frac{s}{r}\right) \|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

$$I_{2,\delta}(x,r) := \|b\|_* \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \int_{\delta}^{\infty} \left(1 + \ln\frac{s}{r}\right) \|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

and it is supposed that $r < \delta$. Now we choose any fixed $\delta > 0$ such that

$$\sup_{x\in\Omega} \frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_{\omega}(\Omega)} < \frac{\varepsilon}{2CC_0\|b\|_*}, \text{ for all } 0 < r < \delta,$$

where C and C_0 are constants from (3.8) and (3.10), which is possible since $f \in V\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)$. Then

$$\sup_{x\in\Omega} CI_{1,\delta}(x,r) < \frac{\varepsilon}{2}, \ 0 < r < \delta,$$

by (3.9).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.6). We have

$$I_{2,\delta}(x,r) \le C_\delta \frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|b\|_* \|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)},$$

where C_δ is the constant from (3.7). Then, by (2.6) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} < \frac{\varepsilon}{2CC_\delta\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)}}$$

which completes the proof of (3.9).

4 Singular integral operators and its commutators in $V\mathcal{M}^{p(\cdot),\varphi}_{\omega}(\Omega)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [2] states that the commutator operator $[b,K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order.

The following weighted local estimates were proved [10].

Theorem 4.1 *[10] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $f \in L^{p(\cdot)}_\omega(\Omega)$. Then*

$$\|Tf\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,r))} \le C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \int_r^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}, \tag{4.1}$$

where C does not depend on f, $x \in \Omega$ and r.

The following theorem is valid.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $\varphi_1(x,t)$ and $\varphi_2(x,r)$ fulfill satisfy the conditions*

$$C_\gamma := \int_\gamma^\infty \frac{\operatorname*{ess\,inf}_{s<t<\infty} \varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}} \frac{ds}{s} < \infty \tag{4.2}$$

for every γ and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{s<t<\infty} \varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}} \frac{ds}{s} \le C\varphi_2(x,r), \tag{4.3}$$

where C does not depend on $x \in \Omega$ and t. Then the singular integral operators T and T^* are bounded from the space $V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)$ to the space $V\mathcal{M}^{p(\cdot),\varphi_2}_\omega(\Omega)$.

Proof. The norm inequalities follow from Theorem 4.1, so we only have to prove that if

$$\begin{aligned} &\lim_{r\to 0} \sup_{x\in\Omega} \frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)} = 0 \Rightarrow \\ &\lim_{r\to 0} \sup_{x\in\Omega} \frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|Tf\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)} = 0 \end{aligned} \tag{4.4}$$

otherwise.

To show that $\sup_{x\in\Omega} \frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|Tf\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)} < \varepsilon$ for small r, we split the right-hand side of (4.1):

$$\sup_{x\in\Omega} \frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|Tf\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)} \le C_0 \left(I_{1,\gamma}(x,r) + I_{2,\gamma}(x,r)\right), \tag{4.5}$$

where $\gamma > 0$ will be chosen as shown below (we may take $\gamma < 1$),

$$I_{1,\gamma}(x,r) := \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \int_r^\gamma \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

$$I_{2,\gamma}(x,r) := \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \int_\gamma^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

and it is supposed that $r < \gamma$. Now we choose any fixed $\gamma > 0$ such that

$$\sup_{x\in\Omega} \frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)} < \frac{\varepsilon}{2CC_0}, \text{ for all } 0 < r < \gamma,$$

where C and C_0 are constants from (4.3) and (4.5), which is possible since $f \in V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$. Then

$$\sup_{x\in\Omega} CI_{1,\gamma}(x,r) < \frac{\varepsilon}{2},\ 0<r<\gamma,$$

by (4.4).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.6). We have

$$I_{2,\gamma}(x,r) \le C_\gamma \frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\|_{V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)},$$

where C_γ is the constant from (4.2). Then, by (2.6) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} < \frac{\varepsilon}{2CC_\gamma \|f\|_{V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)}}$$

which completes the proof of (4.4).

The boundedness of the operator T^* is bounded from the space $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_2}(\Omega)$ follows from the known estimate

$$T^*f(x) \lesssim M(Tf)(x) + Mf(x),$$

from Theorem 3.2 and the boundedness of the operator T is bounded from the space $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_2}(\Omega)$.

The following weighted local estimates were proved [10].

Theorem 4.3 *[10] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $b \in BMO(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\|[b,T]f\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,r))} \le C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \int_r^\infty \left(1+\ln\frac{s}{r}\right) \|f\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{4.6}$$

for every $f \in L_{\omega}^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and r.

The following theorem is valid.

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the conditions*

$$C_{\delta_0} := \int_{\delta_0}^\infty \left(1+\ln\frac{s}{\delta_0}\right) \frac{\operatorname*{ess\,inf}_{s<t<\infty} \varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}} \frac{ds}{s} < \infty \tag{4.7}$$

for every δ_0 and

$$\int_r^\infty \left(1+\ln\frac{s}{r}\right) \frac{\operatorname*{ess\,inf}_{s<t<\infty} \varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}} \frac{ds}{s} \le C\varphi_2(x,r). \tag{4.8}$$

Then the operator $[b,T]$ is bounded from the space $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}_{\omega}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 4.3, so we only have to prove that if

$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}=0\Rightarrow$$
$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|[b,T]f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}=0 \tag{4.9}$$

otherwise.

To show that $\sup\limits_{x\in\Omega}\frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|[b,T]f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}<\varepsilon$ for small r, we split the right-hand side of (4.6):

$$\sup_{x\in\Omega}\frac{1}{\varphi_2(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|[b,T]f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}\le C_0\left(I_{1,\delta_0}(x,r)+I_{2,\delta_0}(x,r)\right), \tag{4.10}$$

where $\delta_0>0$ will be chosen as shown below (we may take $\delta_0<1$),

$$I_{1,\delta_0}(x,r):=\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\int_r^{\delta_0}\left(1+\ln\frac{s}{r}\right)\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

$$I_{2,\delta_0}(x,r):=\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\int_{\delta_0}^{\infty}\left(1+\ln\frac{s}{r}\right)\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

and it is supposed that $r<\delta_0$. Now we choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\Omega}\frac{1}{\varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_\omega(\Omega)}<\frac{\varepsilon}{2CC_0\|b\|_*},\ \text{ for all } 0<r<\delta_0,$$

where C and C_0 are constants from (4.8) and (4.10), which is possible since $f\in V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)$. Then

$$\sup_{x\in\Omega}CI_{1,\delta_0}(x,r)<\frac{\varepsilon}{2},\ \ 0<r<\delta_0,$$

by (4.9).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.6). We have

$$I_{2,\delta_0}(x,r)\le C_{\delta_0}\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)},$$

where C_{δ_0} is the constant from (4.7). Then, by (2.6) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}<\frac{\varepsilon}{2CC_\delta\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_\omega(\Omega)}}$$

which completes the proof of (4.9).

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